Department of Mechanical Engineering

2.14 ANALYSIS AND DESIGN OF FEEDBACK CONTROL SYSTEMS

Fall Term 2003

Problem Set 2 Solution

Problem 1: (Nise, Ch. 2, Pr. 1)

Derive the Laplace transform for the following time functions:

Apply the definition of the Laplace transform: $F(s) = L\{f(t)\} = \int_{0^-}^{\infty} f(t)e^{-st}dt$

a) *u(t)*

$$L\{u(t)\} = \int_{0^-}^{\infty} u(t)e^{-st}dt = \int_{0}^{\infty} e^{-st}dt = -\left.\frac{e^{-st}}{s}\right|_0^{\infty} = \frac{1}{s}$$

b) *tu(t)* (use integration by parts: $u = t, dv = e^{-st}dt$)

$$L\{tu(t)\} = \int_{0^-}^{\infty} tu(t)e^{-st}dt = \int_{0}^{\infty} te^{-st}dt = -\left.\frac{te^{-st}}{s}\right|_0^{\infty} + \int_0^{\infty}\frac{e^{-st}dt}{s} = -\left.\frac{te^{-st}}{s}\right|_0^{\infty} - \left.\frac{e^{-st}}{s^2}\right|_0^{\infty} = \frac{1}{s^2}$$

c) *sinωt u(t)* (use Euler's formula)

$$L\{\sin\omega t\, u(t)\} = \int_{0^-}^{\infty} \sin\omega t\, u(t)e^{-st}dt = \int_0^{\infty}\frac{e^{j\omega t} - e^{-j\omega t}}{2j}e^{-st}dt = -\frac{1}{2j}\left|\frac{e^{-(s-j\omega)t}}{s-j\omega} - \frac{e^{-(s+j\omega)t}}{s+j\omega}\right|_0^{\omega} =$$

$$L\{\sin\omega t\, u(t)\} = \frac{1}{2j}\left(\frac{1}{s-j\omega} - \frac{1}{s+j\omega}\right) = \frac{\omega}{s^2+\omega^2}$$

d) *cosωt u(t)*

$$L\{\cos\omega t\, u(t)\} = \int_{0^-}^{\infty} \cos\omega t\, u(t)e^{-st}dt = \int_0^{\infty}\frac{e^{j\omega t} + e^{-j\omega t}}{2}e^{-st}dt = -\frac{1}{2}\left|\frac{e^{-(s-j\omega)t}}{s-j\omega} + \frac{e^{-(s+j\omega)t}}{s+j\omega}\right|_0^{\omega} =$$

$$L\{\sin\omega t\, u(t)\} = \frac{1}{2}\left(\frac{1}{s-j\omega} + \frac{1}{s+j\omega}\right) = \frac{s}{s^2+\omega^2}$$

Problem 2: (Nise, Ch. 2, Pr. 2)

Derive the Laplace transforms for the following function using Table 2.1 and 2.2 of the textbook.

a) $e^{-at}\sin\omega t\,u(t)$

$$L\{e^{-at}\sin\omega t\,u(t)\} = L(\sin\omega t\,u(t))\Big|_{s=s+a} = \frac{\omega}{(s+a)^2+\omega^2}$$

b) $e^{-at}\cos\omega t\,u(t)$

$$L\{e^{-at}\cos\omega t\,u(t)\} = L(\cos\omega t\,u(t))\Big|_{s=s+a} = \frac{s+a}{(s+a)^2+\omega^2}$$

c) $t^3u(t)$ (from table 2.1)

$$L\{t^3u(t)\} = \frac{6}{s^4}$$

Problem 3: (Nise, Ch 2, Pr 7)

Given the following differential equation:

$$\frac{d^3y}{dt^3}+3\frac{d^2y}{dt^2}+5\frac{dy}{dt}+y=\frac{d^3x}{dt^3}+4\frac{d^2x}{dt^2}+6\frac{dx}{dt}+8x$$

taking the Laplace transform,

$$s^3Y(s)+3s^2Y(s)+5sY(s)+Y(s)=s^3X(s)+4s^2X(s)+6sX(s)+8X(s) \;\Rightarrow$$

$$\frac{Y(s)}{X(s)}=\frac{s^3+4s^2+6s+8}{s^3+3s^2+5s+1}$$

Problem 4: (Nise, Ch 2, Pr 9)

Given the system described by the transfer function

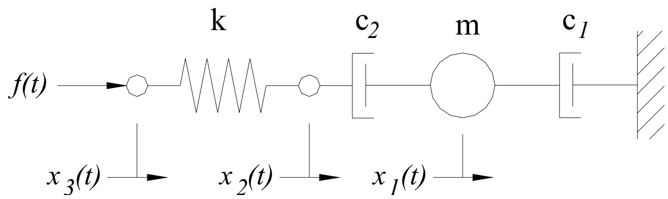
$$\frac{C(s)}{R(s)}=\frac{s^5+2s^4+4s^3+s^2+3}{s^6+7s^5+3s^4+2s^3+s^2+3}$$

The differential equation is then given by,

$$\frac{d^6c}{dt^6}+7\frac{d^5c}{dt^5}+3\frac{d^4c}{dt^4}+2\frac{d^3c}{dt^3}+\frac{d^2c}{dt^2}+3c=\frac{d^5r}{dt^5}+2\frac{d^4r}{dt^4}+4\frac{d^3r}{dt^3}+\frac{d^2r}{dt^2}+3r$$

Problem 5: (Nise, Ch. 2, Pr. 25)

Given the following system



The force balance equations at the three reference points are given by

$$\begin{cases} m\ddot{x}_1 = (\dot{x}_2 - \dot{x}_1)c_2 - \dot{x}_1 c_1 \\ (\dot{x}_2 - \dot{x}_1)c_2 = (x_3 - x_2)k \\ f = (x_3 - x_2)k \end{cases}$$

taking the Laplace transform and regrouping

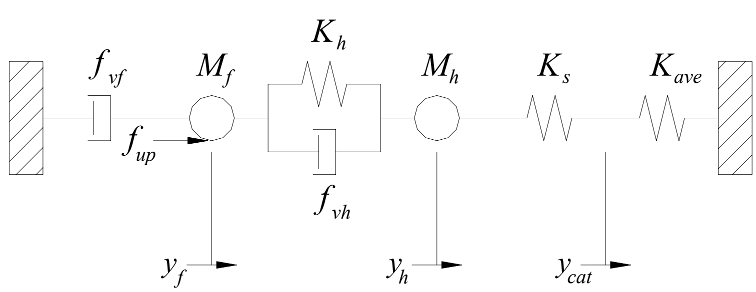
$$\begin{cases} (ms^2 + (c_1 + c_2)s)X_1 = c_2 s X_2 \\ (c_2 s + k)X_2 - c_2 s X_1 = kX_3 \\ F + kX_2 = kX_3 \end{cases} \Rightarrow \begin{cases} X_1 = \dfrac{c_2 s}{ms^2 + (c_1 + c_2)s} X_2 \\ c_2 s X_2 - c_2 s X_1 = F \end{cases} \Rightarrow c_2 s\left(1 - \dfrac{c_2 s}{ms^2 + (c_1 + c_2)s}\right) X_2 = F$$

$$\therefore \frac{X_2(s)}{F(s)} = \frac{ms + c_1 + c_2}{mc_2 s^2 + c_1 c_2 s}$$

Substituting the values k=2 N/m, m=10kg, c_1=2 Ns/m, and c_2=5 Ns/m we obtain

$$\frac{X_2}{F} = \frac{10s + 7}{50s^2 + 10s}$$

Problem 6: (Nise, Ch. 2, Pr. 55)



The force balance equations at the three nodes (masses and between the springs) are given by:

$$\begin{cases} M_f \ddot{y}_f = f_{up} - f_{vf}\dot{y}_f + f_{vh}(\dot{y}_h - \dot{y}_f) + K_h(y_h - y_f) \\ M_h \ddot{y}_h = K_s(y_{cat} - y_h) + f_{vh}(\dot{y}_f - \dot{y}_h) + K_h(y_f - y_h) \\ 0 = K_s(y_h - y_{cat}) - K_{ave} y_{cat} \end{cases}$$

Taking the Laplace transform the equations become,

$$\begin{cases} (M_f s^2 + (f_{vf} + f_{vh})s + K_h)Y_f = F_{up} + (f_{vh}s + K_h)Y_h \\ (M_h s^2 + f_{vh}s + K_h + K_s)Y_h = (f_{vh}s + K_h)Y_f + K_s Y_{cat} \\ (K_s + K_{ave})Y_{cat} = K_s Y_h \end{cases}$$

Rearranging in the equation matrix form **AY=F**

$$\begin{bmatrix} M_f s^2 + (f_{vf} + f_{vh})s + K_h & -(f_{vh}s + K_h) & 0 \\ -(f_{vh}s + K_h) & M_k s^2 + f_{vh}s + K_h + K_s & -K_s \\ 0 & -K_s & K_s + K_{ave} \end{bmatrix} \begin{bmatrix} Y_f \\ Y_h \\ Y_{cat} \end{bmatrix} = \begin{bmatrix} F_{up} \\ 0 \\ 0 \end{bmatrix}$$

a) The transfer function $G_1(s)=Y_{cat}(s)/F_{up}(s)$is found solving the equation system using Cramer's rule,

$$Y_{cat} = \frac{\begin{vmatrix} M_f s^2 + (f_{vf} + f_{vh})s + K_h & -(f_{vh}s + K_h) & F_{up} \\ -(f_{vh}s + K_h) & M_k s^2 + f_{vh}s + K_h + K_s & 0 \\ 0 & -K_s & 0 \end{vmatrix}}{\det \begin{vmatrix} M_f s^2 + (f_{vf} + f_{vh})s + K_h & -(f_{vh}s + K_h) & 0 \\ -(f_{vh}s + K_h) & M_k s^2 + f_{vh}s + K_h + K_s & -K_s \\ 0 & -K_s & K_s + K_{ave} \end{vmatrix}}$$

$$G_1(s) = \frac{Y_{cat}}{F_{up}} = \frac{K_s(f_{vh}s + K_h)}{\det(A)}$$

b) Similarly, the transfer function $G_2(s)=Y_h(s)/F_{up}(s)$ is,

$$G_2(s) = \frac{Y_h}{F_{up}} = \frac{(K_s + K_{ave})(f_{vh}s + K_h)}{\det(A)}$$

c) And the transfer function $G(s)=(Y_h(s)-Y_{cat}(s))/F_{up}(s)$ is given by:

$$G(s) = G_2(s) - G_1(s) = \frac{K_{ave}(f_{vh}s + K_h)}{\det(A)}$$