

2.14 Quiz 1 Review – Warning: This does not necessarily cover “all you need to know”, and some topics covered will not necessarily show up on the quiz. It’s just Katie’s quiz review, for what it’s worth. Some entire sections/pages you are probably best-off simply ignoring for this first quiz – but it’s up to you to some extent to filter through the information to judge what is actually fair game (or most likely to be asked) on Quiz 1.

Also, **Katie is not responsible for any misinformation, typos, etc** – so you should not use any of the equations or methods you see here unless you have carefully checked yourself to see that they make sense! (It will not help you on the quiz to say, “but Katie’s review sheet said that was right...”, because the probability I will manage to type this all up without a single error of some sort is close to zero.) ☺ Ditto for comments made at the quiz review.

The format of this review packet is to intercut random discussion/digression on various topics with specific questions for you to try to answer to test your comprehension.

Topics covered in this review:

- A) Time response (step, impulse, etc. 1st- and 2nd-order, IVT and FVT)
- B) Frequency response (Bode plots)
- C) System modeling (block diagrams, closing a loop, multiple inputs)
- D) Stability – effect of gain of pole locations, effect of zero
- E) Dominant poles
- F) Linearizing a system (constant-coefficient differential equations)
- G) Effect of zero(s) on time response (IVT)

A) Time response:

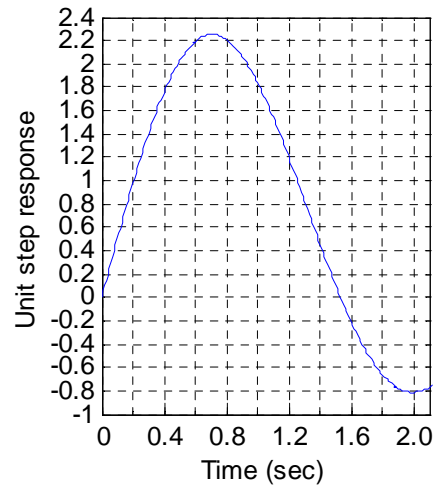
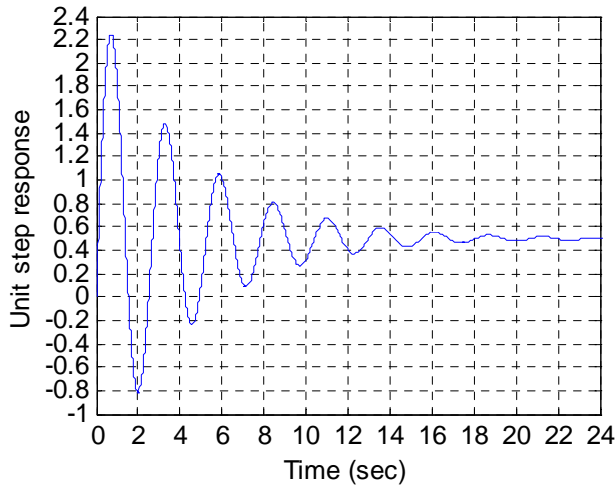
Given a transfer function $T(s)=Y(s)/X(s)$, the initial and final value theorems tell us:

$$\text{IVT: } y(t = 0) = \lim_{s \rightarrow \infty} (s \cdot Y(s)) = \lim_{s \rightarrow \infty} (s \cdot X(s) \cdot T(s))$$

$$\text{FVT: } y(t = \infty) = \lim_{s \rightarrow 0} (s \cdot Y(s)) = \lim_{s \rightarrow 0} (s \cdot X(s) \cdot T(s))$$

For a unit step response, $X(s) = \frac{1}{s}$. For an unit impulse, $X(s) = 1$.

- Second-order step response:



Above is the step response for 2nd-order system. (The figure at right is just a blow-up of the one at left, near $t=0$.) Note that the slope is clearly not zero at $t=0$. (What does this imply about the numerator....?) The rules laid out in Nise (and other sources) are typically for analyzing or plotting a second-order with no zeros and with a “static gain” of one (so that the final value of the unit step response is 1):

$$T_{2a}(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

We can adapt the same rules for a more general case where:

$$T_{2b}(s) = \frac{as^2 + bs + c}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

The poles are the same in both cases, so we can use any location where the slope is zero as a “surrogate” for “ $t=0$ ”. A second issue is correctly accounting for any y-axis scaling issues. On the plots above, we can pick the approximate point $t=2$ sec, $y=-.8$ as a location where the slope is zero. The final value of the response is 0.5, so we can define a height $H=(.5-(-.8))=1.3$ from this surrogate starting point to the final value.

1) What are the POLES for the step response shown? What are ω_d and σ ? ($\sigma = \zeta\omega_n$)

2) What are ω_n and ζ ?

Assume the TF (transfer function) takes the form shown for $T_{2b}(s)$. (In fact, it does.)

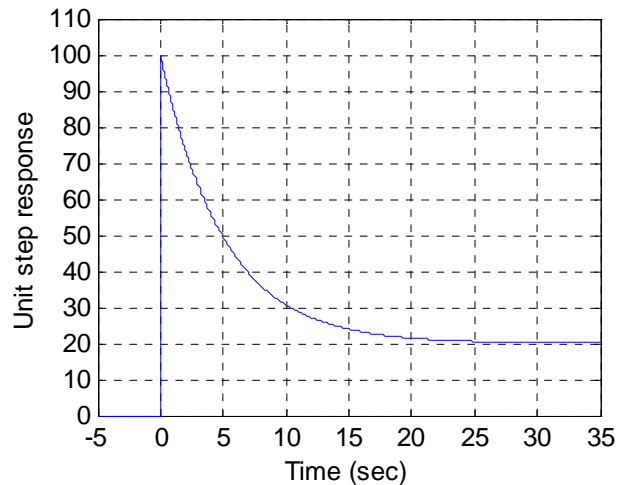
3) Use the FVT to find c .

4) Use the IVT to find a .

5) Find b and explain your method. (Hint, use the right-hand plot for more accuracy...)

- First-order step response:

Below is a first-order step response.



It is clear from the fact that the response “jumps” at $t=0$ that the transfer function contains a zero. We can write the following general form for the transfer function:

$$T(s) = \frac{as + b}{s + c}$$

1) What is the time constant, τ ?

2) What is c ?

3) Now use the FVT to find b .

4) Use the IVT to find a .

5) Plot the pole and zero on the s-plane.

B) Frequency Response, (specifically for us so far: Bode Plots)

It's easier for us to picture a time response – how does a system respond over TIME? A frequency response is a little less intuitive. Picturing sinusoids of increasing frequency on the x-axis is a little tough. A Bode plot is a common tool for displaying the frequency response of a system. (You can also use a Nyquist diagram, Nichols chart or Hall chart... but you'll see Bode plots most often, I suspect.) At any rate, Bode plots are all you are responsible for at this point (since that's all we've covered so far in class)!

Bode plots tend to intimidate many people at their first introduction. They are like the giant, walking Mickey Mouse you first encounter at Disney World as a 5-year-old: big and scary on first introduction, but years later, it seems so friend, simple and harmless, it is later easy to forget that first impression.

Below is a table outlining some basic rules for creating a Bode plot from a transfer function. My strategy is (usually):

1) **Start with a low-frequency asymptote.** (There are cases where it is actually more simple to start with the high-frequency asymptote, but as a starting point, we'll assume we're looking at low frequency when we start the plot...)

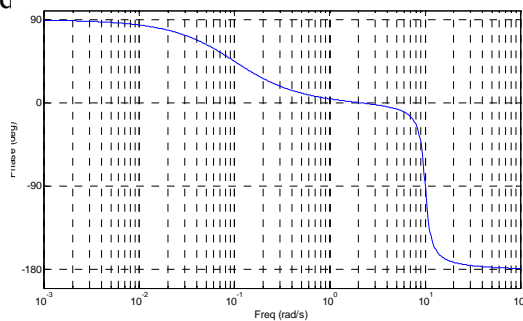
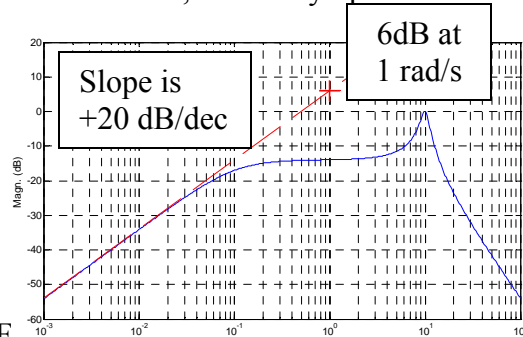
The low-frequency asymptote will have a slope of zero if and only if there are no poles or zeros at the origin. In general, the low frequency asymptote will have the form:

Low frequency asymptote: Ks^n , where $n = (\# \text{ zeros at origin}) - (\# \text{ poles at origin})$. Really, poles and zeros at the origin would cancel one another, so the asymptote has one of three possible forms:

- | |
|--|
| <ul style="list-style-type: none"> a) if no poles or zeros at the origin: K b) if n zeros at the origin: Ks^n c) if n poles at the origin: $\frac{K}{s^n}$ |
|--|

To get K , drop all but the lowest-order term in s in both the numerator and denominator of the transfer function. If you are trying to get the TF from a plot, extend the low-frequency asymptote to find what value it has at 1 rad/s. (i.e. magnitude at $s=1$, but don't forget really $s = j\omega$, so this is $s = j$ and involves some phase as well as magnitude.) At right, the low frequency asymptote is “ $2s$ ”, since 6dB is 2 and the slope is +1 (or 20 dB/dec).

$$T(s) = 2s \cdot \frac{(100)}{(s^2 + 2s + 100)} \cdot \frac{(0.1)}{(s + 0.1)}$$



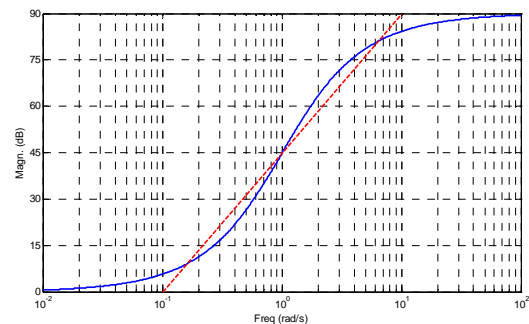
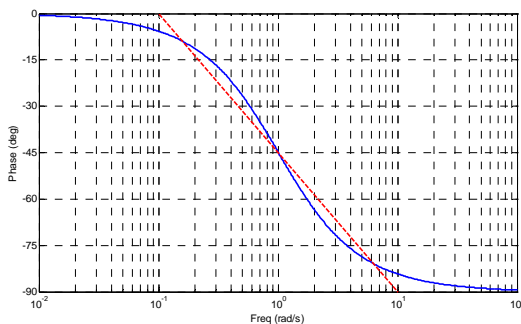
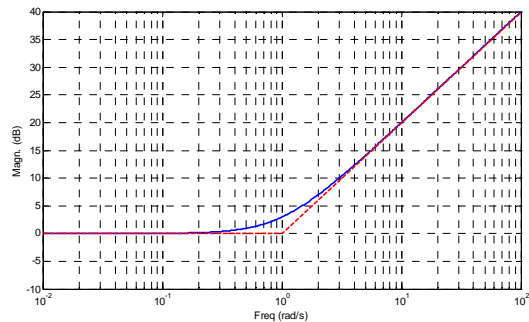
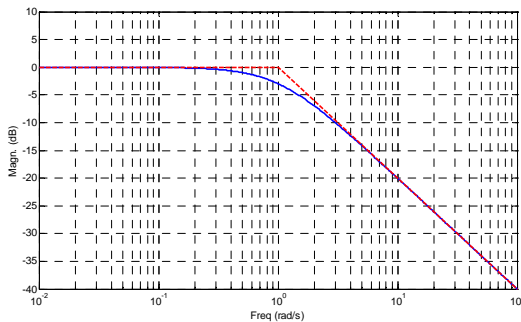
The idea, shown in the formatting of $T(s)$ at the bottom of the last page, is to write $T(s)$ as a low-frequency asymptote Ks^n multiplied times each of a series of expressions for each pole, zero, pole-pair or zero-pair. These other expressions are written in such a way that plugging in “ $s=0$ ” will give a magnitude of 1, so that they do not change the magnitude of the plot from Ks^n in the low-frequency regime.

2) **At each pole or zero** (or complex pole pair or zero pair), change the slope of the magnitude plot by – or + 20 dB/dec (respectively). For LHP poles and zeros, the phase for each pole or zero changes by – or + 90 degrees (respectively). [For RHP (righthalf plane) poles and zeros, use the same rules for the change in slope of the magnitude, but now a RHP pole or zero changes the phase by + or – 90 degrees, respectively, which is just the opposite of the phase rules for lefthalf plane poles and zeros.]

For a single, real-valued pole or zero (or multiple real-valued poles and zeros at a particular breakpoint), you can look up the basic shape deviations for magnitude and phase compared with the straight-line asymptotes. I am not crazy about the rough-and-ready rule for phase change: “start one decade before and end one decade after”. But it’s not bad if you realize where/how the real plot will go compared to the asymptote. The real plot for magnitude is 3 dB away from the intersection of the asymptotes for a single pole or zero.

Pole at 1 rad/s: $\frac{1}{s+1}$

Zero at 1 rad/s: $s+1$



for each:	Change slope by:	If LHP, phase change is:	If RHP, phase change is:
Pole	-1 (or -20 dB/dec)	-90 degrees	+90 degrees
Zero	+1 (or +20dB/dec)	+90 degrees	-90 degrees

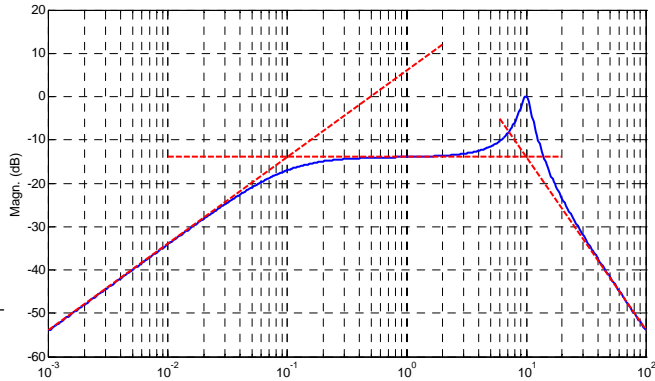
The expressions for each pole, zero, pole-pair or zero-pair will take the following forms:

A real-valued pole: $\frac{p}{s + p}$

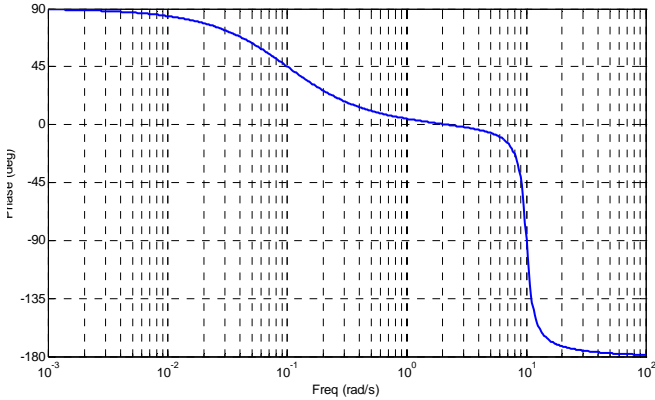
A real-valued zero: $\frac{s + z}{z}$

A complex pole-pair: $\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$

A complex zero-pair: $\frac{s^2 + 2\zeta\omega_z^2 + \omega_z^2}{\omega_z^2}$



For the same $T(s)$, shown again at right, there is a single real-valued pole at $s=-.1$, and there is a complex pole-pair at $\omega_n = 10$ rad/s. For the pole-pair, the magnitude at 10 rad/s is 0 dB, or 1, while the asymptotes intersect at -14 dB, which is $10^{(-14/20)} = 0.2$. The ratio of the value of the magnitude at ω_n to the value of the magnitude where the asymptotes intersect for the plot shown is:



$$\frac{1}{2\zeta} = \frac{1}{0.2} = 5, \text{ therefore: } \zeta = \frac{1}{2 \cdot 5} = 0.1$$

For the complex poles: $\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{10^2}{s^2 + 2 \cdot 0.1 \cdot 10 + 10^2} = \frac{100}{s^2 + 2 + 100}$

For the real-valued pole at $s=-.1$ rad/s: $\frac{0.1}{s + 0.1}$

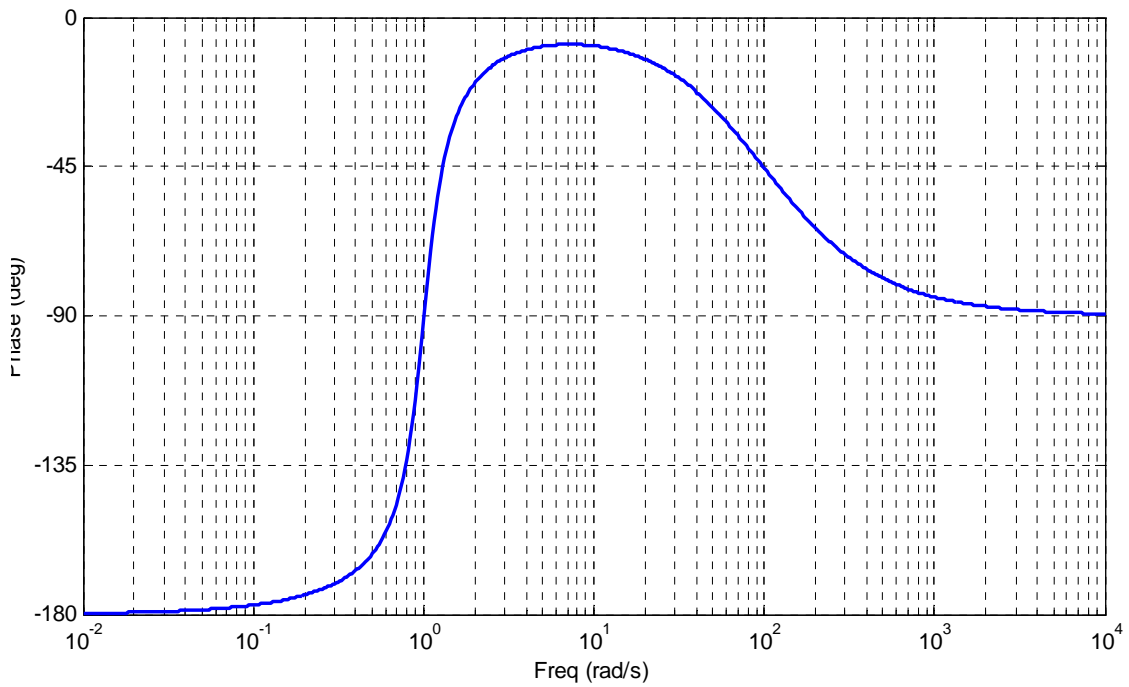
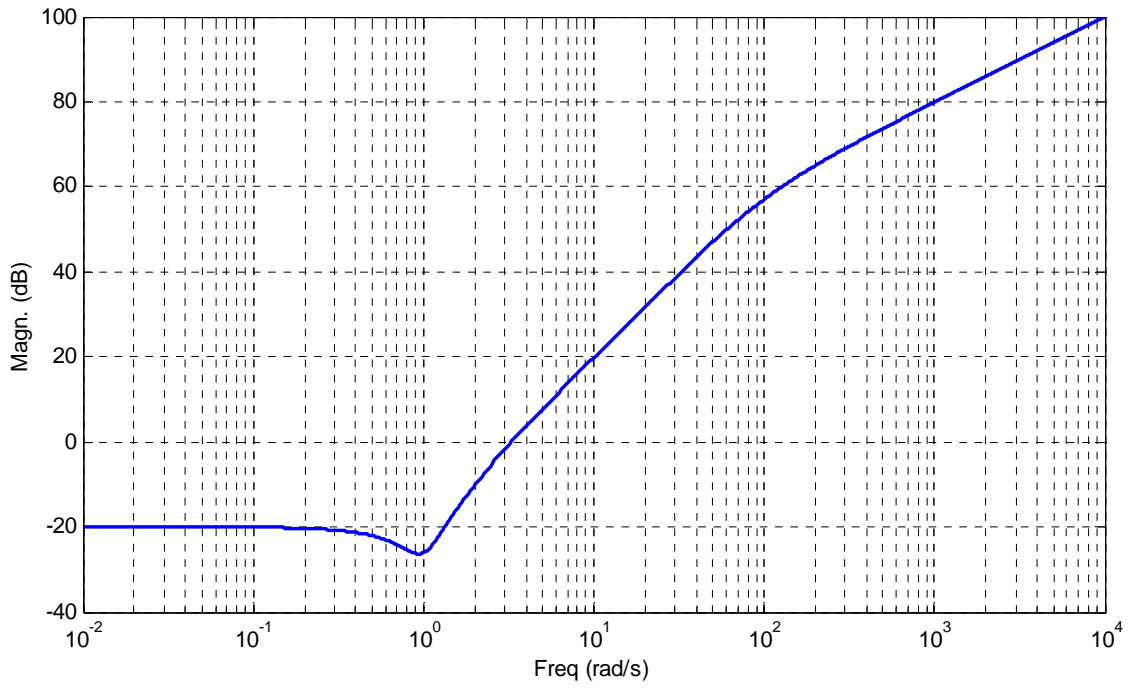
Combining, we can verify that $T(s) = 2s \cdot \frac{(100)}{(s^2 + 2s + 100)} \cdot \frac{(0.1)}{(s + 0.1)}$

3) Verify the phase everywhere! Specifically, check if $K>0$ or $K<0$.

For our example, if the low-frequency asymptote is indeed “ Ks ”, then plugging in $s = j\omega$ will yield a purely imaginary number. If $K>0$, the phase is +90 degrees. If $K<0$, then the phase is -90 degrees. If we multiplied $T(s)$ by -1, it should shift the phase EVERYWHERE by 180 degrees : $\sin(\phi) = -\sin(\phi \pm 180^\circ) \dots$

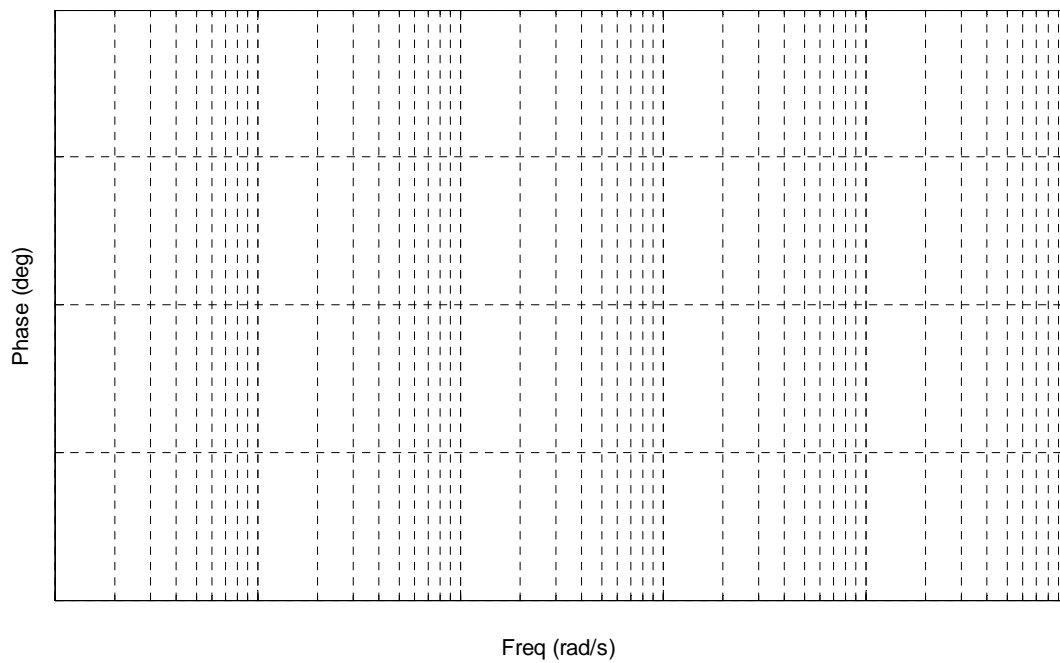
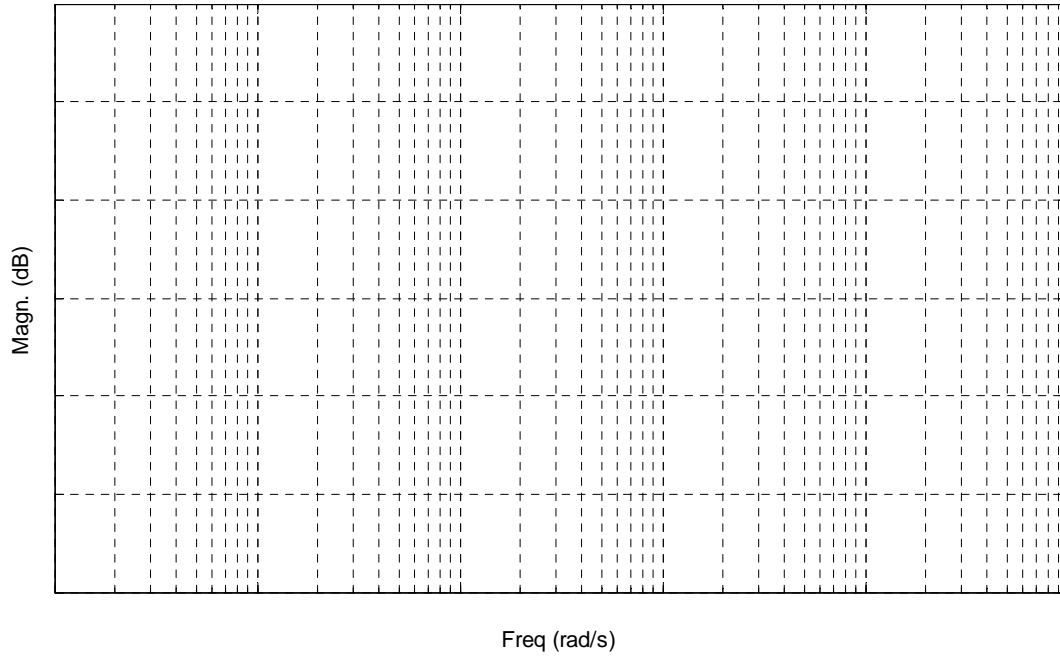
1) Find the transfer function corresponding to the Bode plot shown:

$$T(s) =$$



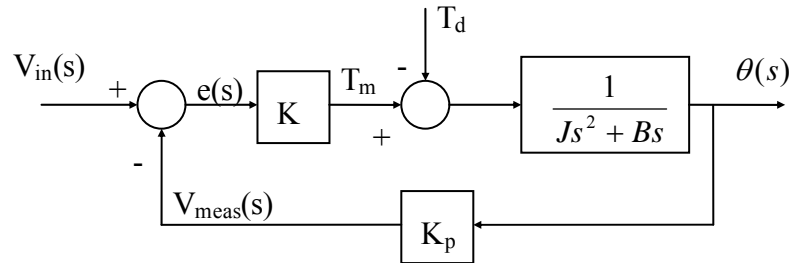
2) Draw a Bode plot for the following transfer function:

$$T(s) = \frac{100s^2 + 10s + 400}{s^3 + 100s^2}$$



C) System Modeling: Proportional (P) Controller for Angular Position of a Motor

Consider the control system below, which is designed to control the angular POSITION of a motor. (Recall that in lab 3, we controller angular VELOCITY.) This model includes two inputs. A command position, set by V_{in} , and a disturbance torque T_d , provided by a constant-force spring attached to the output shaft of the motor.



1) Find each of the following closed-loop transfer functions (CLTFs):

- $\frac{\theta(s)}{V_{in}(s)}$
- $\frac{V_{meas}(s)}{V_{in}(s)}$
- $\frac{e(s)}{V_{in}(s)}$
- $\frac{\theta(s)}{T_d(s)}$
- $\frac{V_{meas}(s)}{T_d(s)}$
- $\frac{e(s)}{T_d(s)}$

Now let: $J=0.01$ [kg-m²], $B=0.04$ [N-s/m], $K=2$ [Nm/volt], $K_p=0.2$ [volts/radian].

2) What are the closed-loop poles? What are ω_n and ζ ?

3) Why don't the closed-loop pole locations depend on T_d ?

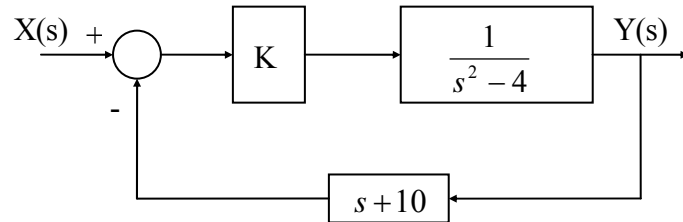
4) Say $V_{in} = 2$ volts. What are the steady-state values of V_{meas} and e for:

- $T_d = 0$
- $T_d = 0.2$ Nm
- $T_d = 3$ Nm

5) Say $V_{in} = 1$ volt. What are the steady-state values of V_{meas} and e for:

- $T_d = 0$
- $T_d = 0.2$ Nm
- $T_d = 3$ Nm

We can use a control loop to make such a plant stable. You are asked to analyze the stability and closed-loop characteristics of the closed-loop system shown below:



- 4) What is the CLTF (closed-loop transfer function) $Y(s)/X(s)$?

- 5) For what values/range of K will the closed-loop system be stable?

- 6) What value(s) of K will result in $\zeta = 0.2$?

- 7) What value(s) of ω_n correspond to this/these value(s) of K ?

- 8) What would the CLTF be if we put the “ $s+10$ ” block in the forward path instead of the feedback path, and had “unity feedback”?

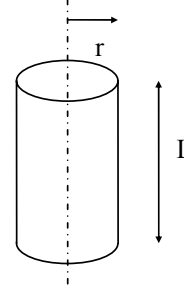
- 9) How do the pole(s) and zero(s) in part 8 compare with those in part 4?

- 10) Using your answer to part 9, which CL system would require the larger peak control effort for a unit step response compare: the one in part 4 or in part 8? Hint: Think about the effect(s) of zero(s) on a step response to answer this.

E) Dominant Poles

You are asked to study a proposed design for a gas spring. The spring consists of a metal cylinder filled with air. Your goal is to select values of r and L such that the natural frequency with a 10 kg payload will be close to 2 Hz.

1) What is the desired ω_n in rad/s?



Eventually (on the next page), we'll want to LINEARIZE the P-V relationship for the gas about some operating point. We know from the ideal gas law that $PV = nRT$. The first step is to decide if we can simplify the gas law using one of the following two assumptions:

- i) That the process is "isothermal". In this situation, the mechanical process happens slowly enough that the thermal process (heat transfer) keeps the air essentially at a constant temperature. $P_1V_1 = P_2V_2 = \text{constant}$.
- ii) That the process is "isentropic". That is, the mechanical process happens so quickly that virtually no heat transfer can take place, and entropy (s) remains constant. For isentropic gas expansion, $\frac{P_2}{P_1} = \left(\frac{V_1}{V_2}\right)^\gamma$, where $\gamma = 1.4$ for air.

We'll assume the metal walls conduct heat well, and that the only thermal resistance is due to convection from the air inside to the metal walls. *No heat transfer occurs through the top or bottom, just through the sides.* The differential equation for the heat transfer is:

$$C_{air} \frac{dT_{air}}{dt} + \frac{(T_{air} - T_{metal})}{R_{conv}}, \quad \text{where } \boxed{C_{air} = m_{air}c_p = \rho_{air}V_{air}c_p} \quad \text{and} \quad \boxed{R_{conv} = \frac{1}{hA_{side}}}$$

2) Derive an expression for the time constant of the thermal process in terms of the variables. (Do not plug numbers in! Also, any dependence on "L" should cancel out; as stated earlier, you are to assume all heat transfer only occurs through the SIDES.)

Now, given: $\rho_{air} = 1.3 \text{ (kg/m}^3\text{)}$, $c_p = 1000 \text{ (J/kg}\cdot\text{°C)}$, $h = 10 \text{ (W/°C}\cdot\text{m}^2\text{)}$

3) If $r=2 \text{ cm}$, where is the pole of the thermal process? Which assumption (isothermal or isentropic) is better? (Hint, compare to ω_n in part 1.)

F) **Linearizing a system** (Same gas-spring system in part E is further analyzed here.)

1) What value of k (N/m) is needed for $m_{load}=10$ kg to get an undamped natural frequency of 2 Hz (again, first converting from Hertz to rad/s to get ω_n !!!).

Let's assume the system is well-modeled as isothermal, and so: $P = (P_1 V_1) V^{-1}$

2) Write a linearized P-V relationship about the initial pressure (operating point) (P_1, V_1) .

Assume the system has a total pressure at rest of $P_1 = P_{atm} + \frac{m_{load} g}{A_{top}}$,

where $P_{atm}=100,000$ N/m²; $g=9.8$ m/s²

3) Now write a linearized expression for the F-x relationship. Your expression should be of the form $F = ax + b$ Note that $\boxed{P = \frac{F}{A}}$ and $\boxed{V = Ax}$.

4) From part F-3, what is the effective spring constant, k , of the linearized system?

5) Using the system parameters from the previous page ($r=0.02$ m, etc), what initial length $x_1=L=?$ will result in the desired spring constant (from part F-1, above).

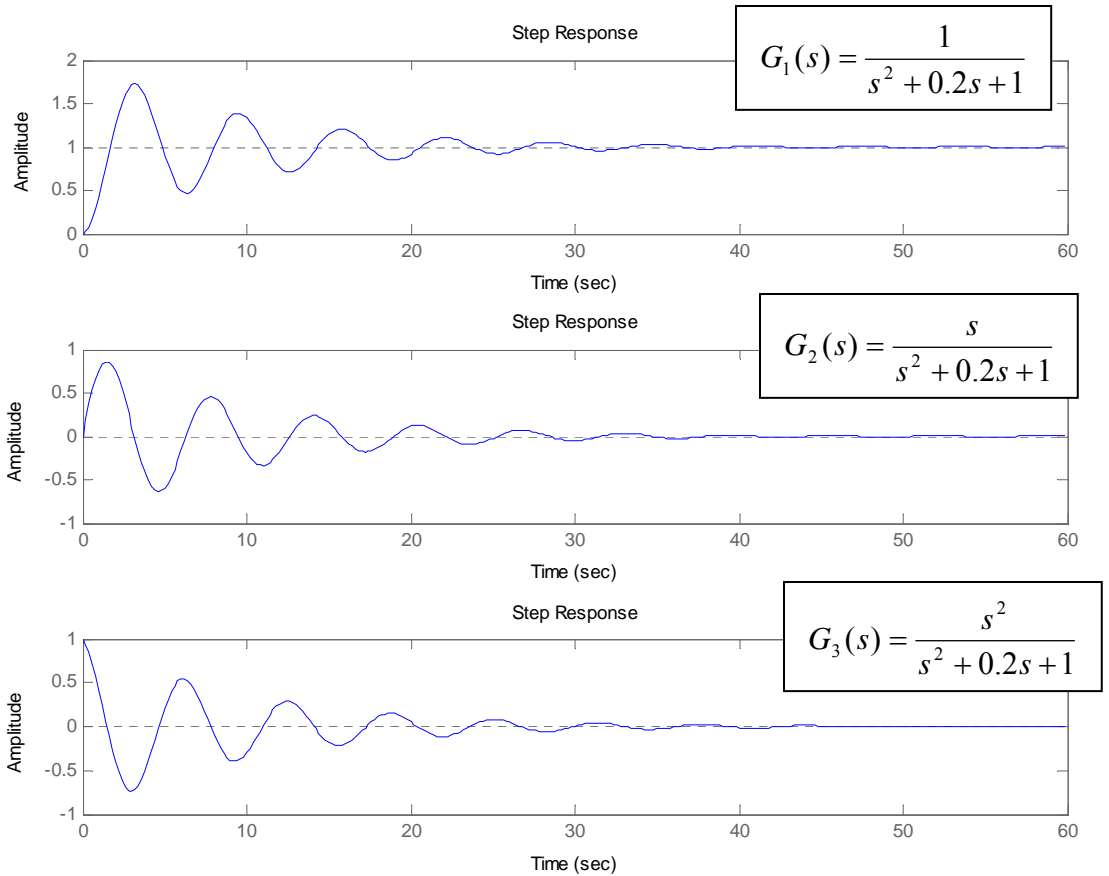
6) What would the actual spring constant be if the process is really isentropic rather than isothermal. That is, now assume $P = (P_1 V_1^{1.4}) V^{-1.4}$ and determine the ratio of

$$\frac{k_{isentropic}}{k_{isothermal}} =$$

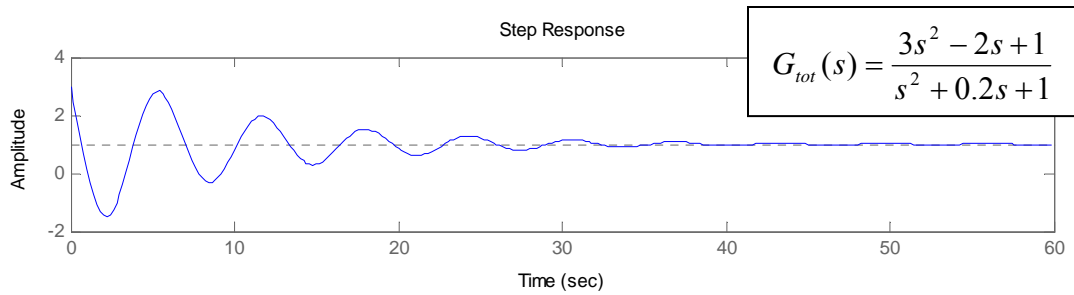
G) Effect of zeros

We can think of $\frac{as^2 + bs + c}{s^2 + 2\sigma s + \omega_n^2} = \frac{as^2}{s^2 + 2\sigma s + \omega_n^2} + \frac{bs}{s^2 + 2\sigma s + \omega_n^2} + \frac{c}{s^2 + 2\sigma s + \omega_n^2}$

Using both superposition and the fact that “s” in the Laplace domain represents a derivative, we can recognize that the total step response of the system will be a sum of steps response shown below, each scaled as appropriate (to get net coefficients a, b and c):



Let's say we are interested in $G_{tot}(s) = \frac{3s^2 - 2s + 1}{s^2 + 0.2s + 1}$. Adding scaled version of steps:



Let's look at the different equations corresponding to some of these TF's.

$$G_1(s) = \frac{Y(s)}{X(s)} = \frac{c}{s^2 + 2\sigma s + \omega_n^2} \rightarrow cx = \frac{d^2 y}{dt^2} + 2\sigma \frac{dy}{dt} + \omega_n^2 y$$

Above, since $x(t)$ is a unit step, the lefthand side of the diff eq is a constant for all $t > 0^+$. For the $G_2(s)$ and $G_3(s)$, the lefthand side would also be a constant. (Specifically, the lefthand side would be zero!) The characteristics of the response (frequency and decay envelope, for a 2nd-order system; tau for 1st-order) are determined by the POLES. The zeros affect the initial condition.

It is easy to see that the FVT theorem makes sense in the equation at top: when all derivatives of y have gone to zero (at steady state), we will be left with $y_{ss} = \frac{c}{\omega_n^2} x$,

which is just some constant times the magnitude of the step input, x . The initial value of both y and dy/dt are zero (using the initial value theorem). We'll show a more thorough explanation of why further on in this discussion. Now, for:

$$\frac{Y(s)}{X(s)} = \frac{as^2 + bs + c}{s^2 + 2\sigma s + \omega_n^2} \rightarrow a \frac{d^2 x}{dt^2} + b \frac{dx}{dt} + cx = \frac{d^2 y}{dt^2} + 2\sigma \frac{dy}{dt} + \omega_n^2 y \quad (\text{Eq. 1})$$

Here, we have the same final value. For all $t > 0^+$, x is a constant input, so its derivative terms (on the lefthand side) are zero in this time range. At steady state, we'll end up with the same relationship: $y_{ss} = \frac{c}{\omega_n^2} x$. To think more precisely about

the initial values of y and its derivatives, we can integrate the differential equation over the vanishing-small time interval in which the input x goes from zero to its final, step magnitude. Integrating once:

$$\int_0^{\Delta t} \left[a \frac{d^2 x}{dt^2} + b \frac{dx}{dt} + cx \right] = \int_0^{\Delta t} \left[\frac{d^2 y}{dt^2} + 2\sigma \frac{dy}{dt} + \omega_n^2 y \right]$$

multiplying through by dt^2 ,

$$\int_0^{\Delta t} \left[a(dx)(dt) + b(dx)(dt) + cx(dt^2) \right] = \int_0^{\Delta t} \left[(d^2 y) + 2\sigma(dy)(dt) + \omega_n^2 y(dt^2) \right]$$

$$a(dx) + b(x)(dt) + \frac{cx^2}{2}(dt^2) = (dy) + 2\sigma(y)(dt) + \frac{\omega_n^2 y^2}{2}(dt^2) \quad (\text{Eq. 2})$$

integrating again:

$$ax + b\left(\frac{x^2}{2}\right)(dt) + \frac{cx^3}{6}(dt^2) = y + 2\sigma\left(\frac{y^2}{2}\right)(dt) + \frac{\omega_n^2 y^3}{6}(dt^2) \quad (\text{Eq. 3})$$

In the limit (just after the "instantaneous" unit step) at $t = 0^+$, $dt \rightarrow 0$ in Eq. 3

(almost no time has elapsed), so we use Eq. 3 to find the initial value of y at $t = 0^+$ is:

$$y(0^+) = ax \quad (\text{Eq. 3a})$$

which agrees with the IVT (letting s go to infinity in the TF).

To find the initial VELOCITY of the output, $\frac{dy}{dt}(0^+)$, we can now use Eq. 2:

$$a(dx) + b(x)(dt) + \frac{cx^2}{2}(dt^2) = (dy) + 2\sigma(y)(dt) + \frac{\omega_n^2 y^2}{2}(dt^2) \quad (\text{Eq. 2})$$

Dividing by dt :

$$a\left(\frac{dx}{dt}\right) + b(x) + \frac{cx^2}{2}(dt) = \left(\frac{dy}{dt}\right) + 2\sigma(y) + \frac{\omega_n^2 y^2}{2}(dt) \quad (\text{Eq. 2a})$$

For all $t \geq 0^+$, dx/dt is zero (input x =constant), and dt is again vanishingly small in the limit at the initial time $t = 0^+$:

$$bx = \frac{dy}{dt}(0^+) + 2\sigma(y(0^+)) \quad (\text{Eq. 2b})$$

Plugging in from Eq. 3a:

$$\begin{aligned} \frac{dy}{dt}(0^+) &= bx - 2\sigma(y(t = 0^+)) = bx - 2\sigma ax \\ &= (b - 2\sigma a)x \end{aligned} \quad (\text{Eq. 2c})$$

Finally, we can use Eq. 1 to solve for $\frac{d^2 y}{dt^2}(0^+)$:

$$a \frac{d^2 x}{dt^2} + b \frac{dx}{dt} + cx = \frac{d^2 y}{dt^2} + 2\sigma \frac{dy}{dt} + \omega_n^2 y \quad (\text{Eq. 1a})$$

$$\begin{aligned} \frac{d^2 y}{dt^2}(0^+) &= cx - 2\sigma \frac{dy}{dt}(0^+) - \omega_n^2 y(0^+) \\ &= cx - 2\sigma(b - 2\sigma a)x - \omega_n^2 ax \end{aligned} \quad (\text{Eq. 1b})$$

There is a reason this is all thrown onto the back page on the “Quiz Review”! It’s more useful (like a lot of the material here) as a general “future reference” than for Quiz 1. It took me a while to figure out how to get to the result in “Eq. 2b” whenever it was I first went through it, b/c if you just multiply our initial $G(s)$ by s to get a TF for the velocity of $y(t)$ (dy/dt) as a function of x and then set s to infinity, you get “infinity” as the initial slope, which is only valid for EXACTLY $t=0$, but not for $t=0^+$. Don’t worry about it if you don’t quite follow what I’m talking about right now, though.

Good Luck!