MASSACHUSETTS INSTITUTE OF TECHNOLOGY
DEPARTMENT OF MECHANICAL ENGINEERING
2.151 Advanced System Dynamics and Control

Computing the Matrix Exponential
The Cayley-Hamilton Method [1]

The matrix exponential $e^{\mathbf{A}t}$ forms the basis for the homogeneous (unforced) and the forced response of LTI systems. We consider here a method of determining $e^{\mathbf{A}t}$ based on the the *Cayley-Hamiton theorem.*

Consider a square matrix $\mathbf{A}$ with dimension n and with a characteristic polynomial

$$\Delta(s) = |s\mathbf{I} - \mathbf{A}| = s^n + c_{n-1}s^{n-1} + \ldots + c_0,$$

and define a corresponding matrix polynomial, formed by substituting $\mathbf{A}$ for s above

$$\Delta(\mathbf{A}) = \mathbf{A}^n + c_{n-1}\mathbf{A}^{n-1} + \ldots + c_0\mathbf{I}$$

where $\mathbf{I}$ is the identity matrix. The Cayley-Hamilton theorem states that *every matrix satisfies its own characteristic equation*, that is

$$\Delta(\mathbf{A}) \equiv [\mathbf{0}]$$

where $[\mathbf{0}]$ is the null matrix. (Note that the normal characteristic equation $\Delta(s) = 0$ is satisfied only at the eigenvalues $(\lambda_1, \ldots, \lambda_n)$).

1 The Use of the Cayley-Hamilton Theorem to Reduce the Order of a Polynomial in A

Consider a square matrix $\mathbf{A}$ and a polynomial in s, for example $P(s)$. Let $\Delta(s)$ be the characteristic polynomial of $\mathbf{A}$. Then write $P(s)$in the form

$$P(s) = Q(s)\Delta(s) + R(s)$$

where $Q(s)$ is found by long division, and the remainder polynomial $R(s)$ is of degree $(n-1)$ or less. At the eigenvalues $s = \lambda_i, i = 1, \ldots, n$ by definition $\Delta(s) = 0$, so that

$$P(\lambda_i) = R(\lambda_i). \tag{1}$$

Now consider the corresponding matrix polynomial $P(\mathbf{A})$:

$$P(\mathbf{A}) = Q(\mathbf{A})\Delta(\mathbf{A}) + R(\mathbf{A})$$

But Cayley-Hamilton states that $\Delta(\mathbf{A}) \equiv [\mathbf{0}]$, therefore

$$P(\mathbf{A}) = R(\mathbf{A}). \tag{2}$$

where the coefficients of $R(\mathbf{A})$ may determined from Eq. (1), or by long division.

[1] D. Rowell 10/16/04

■ **Example**

Reduce the order of $P(\mathbf{A}) = \mathbf{A}^4 + 3\mathbf{A}^3 + 2\mathbf{A}^2 + \mathbf{A} + \mathbf{I}$ for the matrix

$$\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$$

Solution:

$$\begin{aligned}
\Delta(s) &= |s\mathbf{I} - \mathbf{A}| = s^2 - 5s + 5 \\
\frac{P(s)}{\Delta(s)} &= \frac{s^4 + 3s^3 + 2s^2 + s + 1}{s^2 - 5s + 5} \\
&= s^2 + 8s + 37 \quad + \quad \frac{146s - 184}{s^2 - 5s + 5} \\
P(s) &= (s^2 + 8s + 37)\Delta(s) \quad + \quad 146s - 184
\end{aligned}$$

or

$$R(s) = 146s - 184.$$

Then for the given $\mathbf{A}$, $P(\mathbf{A}) = R(\mathbf{A})$, or

$$\begin{aligned}
P(\mathbf{A}) &= \mathbf{A}^4 + 3\mathbf{A}^3 + 2\mathbf{A}^2 + \mathbf{A} + \mathbf{I} \\
&= 146\mathbf{A} - 184.
\end{aligned}$$

Summary: A matrix polynomial, of a matrix $\mathbf{A}$ of degree n, can always be expressed as a polynomial of degree $(n-1)$ or less.

2 The Use of Cayley-Hamilton to Determine Analytic Functions of a Matrix

Assume that a scalar function $f(s)$ is analytic in a region of the complex plane. Then in that region $f(s)$ may be expressed as a polynomial

$$f(s) = \sum_{k=0}^{\infty} \beta_k s^k.$$

Let $\mathbf{A}$ be a square matrix of dimension n, with characteristic polynomial $\Delta(s)$ and eigenvalues λ_i. Then as above $f(s)$ may be written

$$f(s) = \Delta(s)Q(s) + R(s)$$

where $R(s)$ is of degree $(n-1)$ or less. In particular, for $s = \lambda_i$

$$\begin{aligned}
f(\lambda_i) &= R(\lambda_i) \\
&= \sum_{k=0}^{n-1} \alpha_k \lambda_i^k
\end{aligned} \tag{3}$$

Since the $\lambda_i, i = 1 \dots n$ are known, Eq. (3) defines a set of simultaneous linear equations that will generate the coefficients $\alpha_0, \dots, \alpha_{n-1}$.

The matrix function $f(\mathbf{A}$ is defined to have the same series expansion as $f(s)$, that is

$$\begin{aligned} f(\mathbf{A}) &= \sum_{k=0}^{\infty} \beta_k s^k \\ &= \Delta(\mathbf{A}) \sum_{k=0}^{\infty} \beta_k s^k \quad + \quad R(\mathbf{A}) \\ &= R(\mathbf{A}) \end{aligned}$$

by Cayley-Hamilton, since $\Delta(\mathbf{A}) \equiv [\mathbf{0}]$. then

$$f(\mathbf{A}) = \sum_{k=0}^{n-1} \alpha_k \mathbf{A}^k \tag{4}$$

where the α_i's may be found from Eq. (3).

Thus the defined analytic function of a matrix $\mathbf{A}$ *of dimension* n *may be expressed as a polynomial of degree* $(n-1)$ *or less.*

■ Example

Find $\sin(\mathbf{A})$ where

$$\mathbf{A} = \begin{bmatrix} -3 & 1 \\ 0 & -2 \end{bmatrix}.$$

Solution: For $\mathbf{A}$

$$\Delta(s) = |s\mathbf{I} - \mathbf{A}| = (s+2)(s+3)$$

giving $\lambda_1 = -3$ and $\lambda_1 = -2$. Since $n = 2$, $R(s)$ must be of degree 1 or less.

Let $R(s) = \alpha_0 + \alpha_1 s$. and from Eq. (3)

$$\begin{aligned} \sin(\lambda_1) &= \alpha_0 + \alpha_1 \lambda_1 \\ \sin(\lambda_2) &= \alpha_0 + \alpha_1 \lambda_2. \end{aligned}$$

Substituting for λ_1 and λ_2, and solving for α_0 and α_2 gives

$$\begin{aligned} \alpha_0 &= 3\sin(-2) - 2\sin(-3) \\ \alpha_1 &= \sin(-2) - \sin(-3). \end{aligned}$$

Substituting in Eq. (4) gives

$$\sin(\mathbf{A}) = (3\sin(-2) - 2\sin(-3))\mathbf{I} + (\sin(-2) - \sin(-3))\mathbf{A}$$

or

$$\sin \begin{bmatrix} -3 & 1 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} \sin(-3) & \sin(-2) - \sin(-3) \\ 0 & \sin(-2) \end{bmatrix}$$

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3 Computation of the Matrix Exponential $e^{\mathbf{A}t}$

The matrix exponential is simply one case of an analytic function as described above.

$$e^{\mathbf{A}t} = \sum_{k=0}^{n-1} \alpha_k \mathbf{A}^k \tag{5}$$

where the α_i's are determined from the set of equations given by the eigenvalues of $\mathbf{A}$.

$$e^{\lambda_i t} = \sum_{k=0}^{n-1} \alpha_k \lambda_i^k \tag{6}$$

■ Example

Find $e^{\mathbf{A}t}$ for

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}.$$

Solution: The characteristic equation is $s^2 + 3s + 2 = 0$, and the eigenvalues are $\lambda_1 = -1$, $\lambda_2 = -2$. From Eq. (5)

$$e^{\mathbf{A}t} = \alpha_0 \mathbf{I} + \alpha_1 \mathbf{A}$$

From Eq. (6), for $\lambda_1 = -1$ and $\lambda_2 = -2$

$$\begin{aligned} e^{-t} &= \alpha_0 - \alpha_1 \\ e^{-2t} &= \alpha_0 - 2\alpha_1, \end{aligned}$$

or $\alpha_0 = (2e^{-t} - e^{-t})$ and $\alpha_1 = (e^{-t} - e^{-2t})$. Then

$$\begin{aligned} e^{\mathbf{A}t} &= (2e^{-t} - e^{-t})\mathbf{I} + (e^{-t} - e^{-2t})\mathbf{A} \\ &= \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix} \end{aligned}$$

■ Example

Find $e^{\mathbf{A}t}$ for

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Solution: The characteristic equation is $s^2 + 1 = 0$, and the eigenvalues are $\lambda_1 = +j$, $\lambda_2 = -j$. From Eq. (5)

$$e^{\mathbf{A}t} = \alpha_0 \mathbf{I} + \alpha_1 \mathbf{A}$$

From Eq. (6), for $\lambda_1 = +j$ and $\lambda_2 = -j$

$$\begin{aligned} e^{jt} &= \cos(t) + j\sin(t) = \alpha_0 + \alpha_1 j \\ e^{-jt} &= \cos(t) - j\sin(t) = \alpha_0 - \alpha_1 j, \end{aligned}$$

or $\alpha_0 = \cos(t)$ and $\alpha_1 = \sin(t)$. Then

$$\begin{aligned} e^{\mathbf{A}t} &= \cos(t)\mathbf{I} + \sin(t)\mathbf{A} \\ &= \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} \end{aligned}$$

Note: If one or more of the eigenvalues is repeated ($\lambda_i = \lambda_j, i \neq j$, then Eqs. (6) will yield two or more identical equations, and therefore will not be a set of n independent equations.

For an eigenvalue of multiplicity m, the first $(m-1)$ derivatives of $\Delta(s)$ all vanish at the eigenvalues, therefore

$$\begin{aligned} f(\lambda_i) &= \sum_{k=0}^{(n-1)} \alpha_k \lambda_i^k = R(\lambda_i) \\ \left.\frac{df}{d\lambda}\right|_{\lambda=\lambda_i} &= \left.\frac{dR}{d\lambda}\right|_{\lambda=\lambda_i} \\ \vdots \qquad & \qquad \vdots \\ \left.\frac{d^{m-1}f}{d\lambda^{m-1}}\right|_{\lambda=\lambda_i} &= \left.\frac{d^{m-1}R}{d\lambda^{m-1}}\right|_{\lambda=\lambda_i} \end{aligned}$$

form a set of m linearly independent equations, which when combined with the others will yield the required set os n equations to solve for the α's.