

Op-Amp Implementation of Analog Filters using the State-Variable Approach.

1 Introduction

Practical realizations of analog filters are usually based on factoring the transfer function into cascaded second-order sections, each based on a complex conjugate pole-pair or a pair of real poles, and a first-order section if the order is odd. Any zeros in the system may be distributed among the second- and first-order sections. Each first- and second-order section is then implemented by an active filter and connected in series. For example the third-order Butterworth high-pass filter

$$H(s) = \frac{s^3}{s^3 + 40s^2 + 800s + 8000}$$

would be implemented as

$$H(s) = \frac{s^2}{s^2 + 20s + 400} \times \frac{s}{s + 20}$$

as shown in Fig. 1. The design of each low-order block can be handled independently.

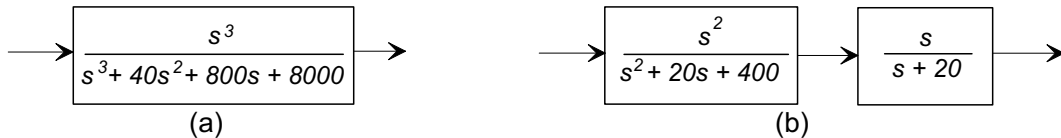


Figure 1: A third-order Butterworth filter (a) as a single third-order section, and (b) as a second-order and first-order section cascaded.

2 State-variable active filters

The *state-variable* filter design method is based on the block diagram representation used in the so-called *phase-variable* description of linear systems that uses the outputs of a chain of cascaded integrators as state variables. Consider a second-order filter block with a transfer function

$$H(s) = \frac{Y(s)}{U(s)} = \frac{b_2s^2 + b_1s + b_0}{s^2 + a_1s + a_0} \tag{1}$$

and split $H(s)$ into two sub-blocks representing the denominator and numerator by introducing an intermediate variable x and rewriting

$$H_1(s) = \frac{X(s)}{U(s)} = \frac{1}{s^2 + a_1s + a_0} \quad (2)$$

$$H_2(s) = \frac{Y(s)}{X(s)} = b_2s^2 + b_1s + b_0 \quad (3)$$

so that $H(s) = H_2(s)H_1(s)$.

The differential equations corresponding to Eqs. (2) and (3) are

$$\frac{d^2x}{dt^2} + a_1\frac{dx}{dt} + a_0x = u \quad (4)$$

and

$$y = b_2\frac{d^2x}{dt^2} + b_1\frac{dx}{dt} + b_0x. \quad (5)$$

Rewrite Eq. (4) explicitly in terms of the highest derivative

$$\frac{d^2x}{dt^2} = -a_1\frac{dx}{dt} - a_0x + u. \quad (6)$$

Consider a pair of cascaded analog integrators with the output defined as $x(t)$, as shown in Fig. 2, so that the derivatives of $x(t)$ appear as inputs to the integrators. Note that Eq. (6)

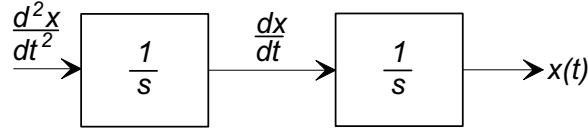


Figure 2: Cascaded integrators with output $x(t)$.

gives an explicit expression for the input to the first block in terms of the outputs of the two integrators and the system input, and therefore generates the block diagram for $H_1(s)$ (Eq. (5)) shown in Fig. 3.

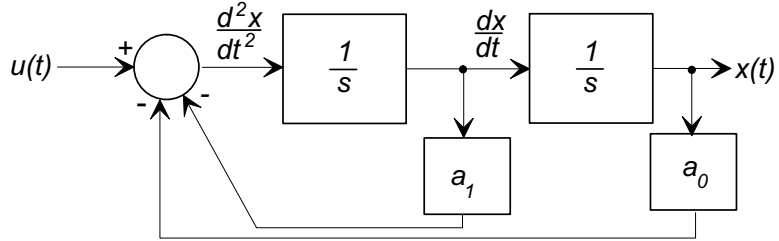


Figure 3: State variable realization of $H_1(s) = X(s)/U(s)$.

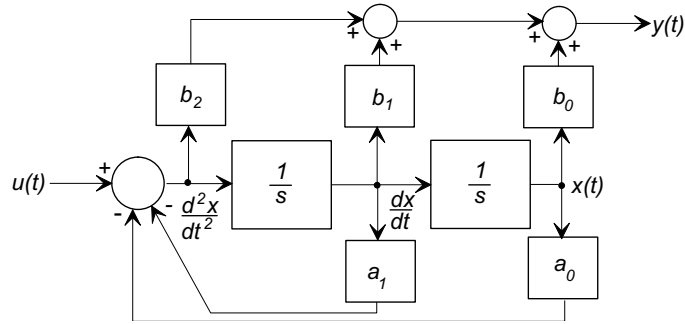


Figure 4: Full second-order state variable realization.

Equation (5) shows that the output $y(t)$ is a weighted sum of $x(t)$ and its derivatives, leading to the complete second-order state variable filter block shown in Fig. 4.

This basic structure may be used to realize the four basic filter types by appropriate choice of the numerator. Figure 5 shows how the output may be selected to achieve the following transfer functions:

$$H_{lp}(s) = \frac{Y_1(s)}{U(s)} = \frac{a_0}{s^2 + a_1s + a_0} \quad \text{a unity gain low-pass filter} \quad (7)$$

$$H_{bp}(s) = \frac{Y_2(s)}{U(s)} = \frac{a_1s}{s^2 + a_1s + a_0} \quad \text{a unity gain band-pass filter} \quad (8)$$

$$H_{hp}(s) = \frac{Y_3(s)}{U(s)} = \frac{s^2}{s^2 + a_1s + a_0} \quad \text{a unity gain high-pass filter} \quad (9)$$

$$H_{bs}(s) = \frac{Y_4(s)}{U(s)} = \frac{s^2 + a_0}{s^2 + a_1s + a_0} \quad \text{a unity gain band-stop filter} \quad (10)$$

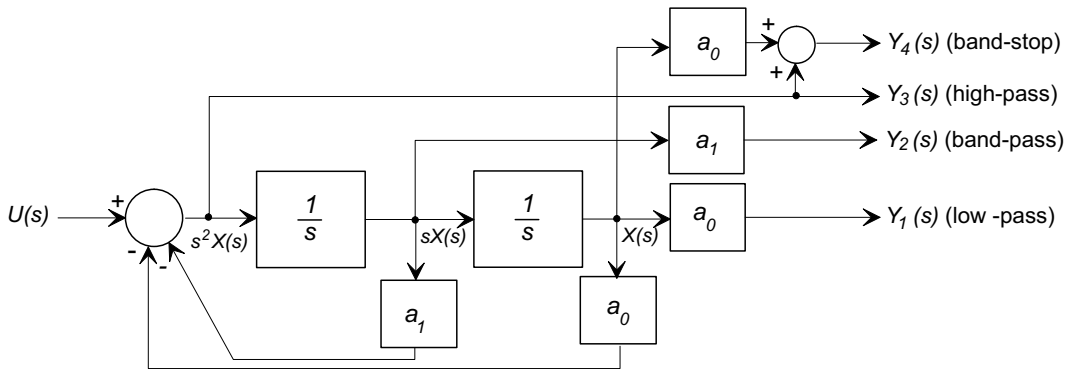


Figure 5: State variable implementation of various filter types.

2.1 Op-amp Based State-Variable Filters

Electronic implementation of the block diagram structure of Fig. 5 involves weighted summation and integration. These two operations can be achieved by the two op-amp circuits shown in Fig. 6. For the summer in Fig. 6a the output is

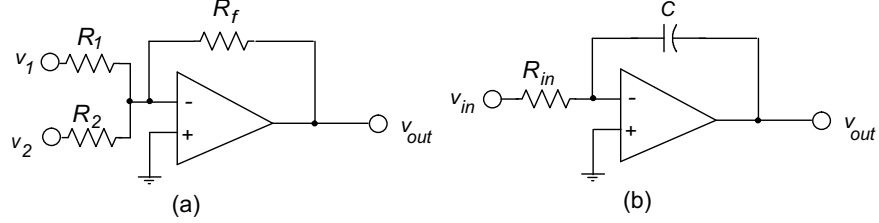


Figure 6: Elementary op-amp circuits: (a) a summer, and (b) an integrator.

$$v_{out} = - \left(\frac{R_f}{R_1} v_1 + \frac{R_f}{R_2} v_2 \right)$$

and for the integrator in Fig. 6b

$$v_{out}(t) = - \frac{1}{R_{in}C} \int_0^t v_{in}(t) dt.$$

and we note

1. Common op-amp summing and integrating circuits involve a sign inversion.
2. Op-amp integrators implicitly have a non-unity gain (unless $R_{in}C = 1$).

2.2 A Three Op-amp State Variable Filter Circuit

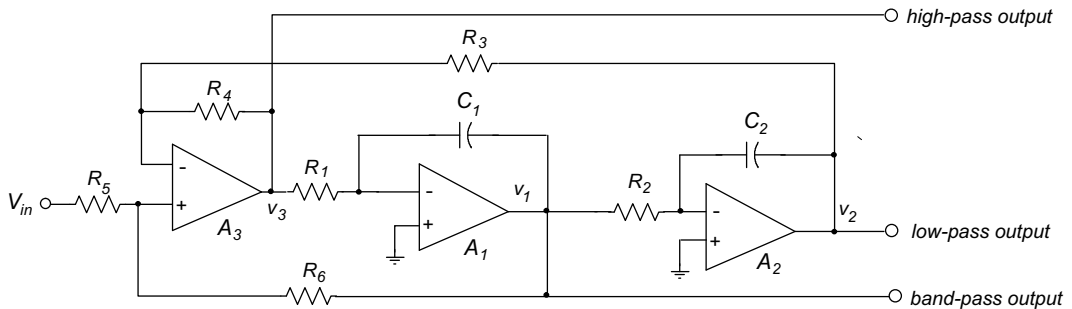


Figure 7: A three op-amp implementation of a second-order state-variable filter.

Figure 7 shows a common implementation of the second-order state-variable filter using three op-amps. Amplifiers A_1 and A_2 are integrators with transfer functions

$$H_1(s) = - \left(\frac{1}{R_1 C_1} \right) \frac{1}{s} \quad \text{and} \quad H_2(s) = - \left(\frac{1}{R_2 C_2} \right) \frac{1}{s}.$$

Let $\tau_1 = R_1C_1$ and $\tau_2 = R_2C_2$. Because of the gain factors in the integrators and the sign inversions we have

$$v_1(t) = -\tau_2 \frac{dv_2}{dt} \quad \text{and} \quad v_3(t) = \tau_1 \tau_2 \frac{d^2v_2}{dt^2}. \quad (11)$$

Amplifier A_3 is the summer. However, because of the sign inversions in the op-amp circuits we cannot use the elementary summer of Fig. 6a. Applying Kirchoff's Current Law at the non-inverting and inverting inputs of A_3 gives

$$\frac{V_{in} - v_+}{R_5} + \frac{v_1 - v_+}{R_6} = 0 \quad \text{and} \quad \frac{v_3 - v_-}{R_4} + \frac{v_2 - v_-}{R_1} = 0. \quad (12)$$

Using the infinite gain approximation for the op-amp, we set $v_- = v_+$ and

$$\frac{R_3}{R_3 + R_4}v_3 - \frac{R_5}{R_5 + R_6}v_1 + \frac{R_4}{R_3 + R_4}v_2 = \frac{R_6}{R_5 + R_6}V_{in},$$

and substituting for v_1 and v_3 from Eq. (11) we generate a differential equation in v_2

$$\frac{d^2v_2}{dt^2} + \left(\frac{1 + R_4/R_3}{\tau_1(1 + R_6/R_5)} \right) \frac{dv_2}{dt} + \left(\frac{R_4}{R_3} \frac{1}{\tau_1\tau_2} \right) v_2 = \left(\frac{1 + R_4/R_3}{\tau_1\tau_2(1 + R_5/R_6)} \right) V_{in} \quad (13)$$

which corresponds to a low-pass transfer function with

$$H(s) = \frac{K_{lp}a_0}{s^2 + a_1s + a_0} \quad (14)$$

where

$$\begin{aligned} a_0 &= \left(\frac{R_4}{R_3} \right) \frac{1}{\tau_1\tau_2} \\ a_1 &= \left(\frac{1 + R_4/R_3}{1 + R_6/R_5} \right) \frac{1}{\tau_1} \\ K_{lp} &= \frac{1 + R_3/R_4}{1 + R_5/R_6} \end{aligned}$$

A Band-Pass Filter: Selection of the output as v_1 , the output of integrator A_1 , generates the transfer function

$$H_{bp}(s) = -\tau_1 s H_{lp}(s) = \frac{-K_{bp}a_1 s}{s^2 + a_1 s + a_0} \quad (15)$$

where

$$K_{bp} = \frac{R_6}{R_5}$$

A High-Pass Filter: Selection of the output as v_3 , the output of the summer A_3 , generates the transfer function

$$H_{hp}(s) = \tau_1 \tau_2 s^2 H_{lp}(s) = \frac{K_{hp} s^2}{s^2 + a_1 s + a_0} \quad (16)$$

where

$$K_{hp} = \frac{1 + R_4/R_3}{1 + R_5/R_6}$$

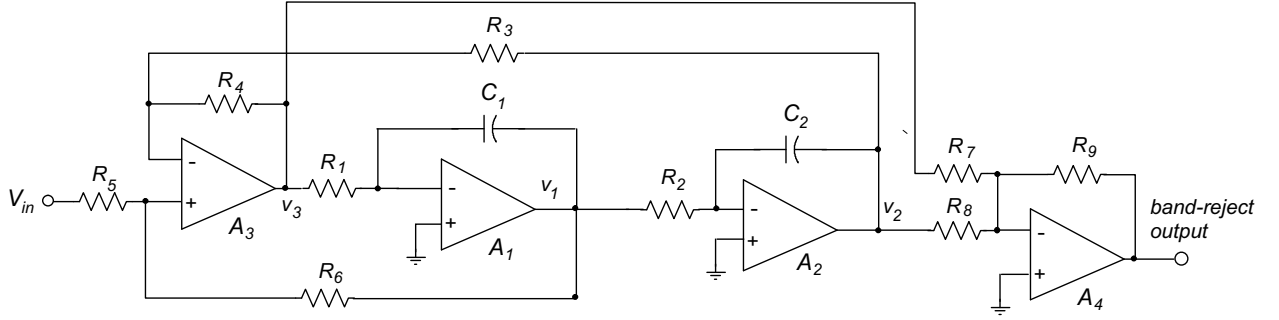


Figure 8: A band-stop second-order state-variable filter.

A Band-Stop Filter: A band-stop characteristic requires a pair of conjugate zeros on the imaginary axis as defined in Eq. (10). This may be done by including an additional summing amplifier A_4 as shown in Fig. 8. The output is

$$\begin{aligned} V_{out}(s) &= -\frac{R_9}{R_8}V_2(s) - \frac{R_9}{R_7}V_3(s) \\ &= -\left(\frac{R_9}{R_8} + \frac{R_9}{R_7}\tau_1\tau_2s^2\right)V_2(s) \end{aligned}$$

If $R_7 = R_8$ and $R_3 = R_4$, the filter transfer function simplifies to

$$H_{bs}(s) = \frac{V_o(s)}{V_{in}(s)} = \frac{V_o(s)}{V_2(s)} \frac{V_2(s)}{V_{in}(s)} = \frac{-K_{bs}(s^2 + a_0)}{s^2 + a_1s + a_0}$$

where

$$K_{bs} = \frac{2R_9}{(1 + R_5/R_6)R_8}.$$

2.3 A Simplified Two Op-amp Based State-variable Filter:

If the required filter does not require a high-pass action (that is, access to the output of the summer A_3) the summing operation may be included at the input of the first integrator, leading to a simplified circuit using only two op-amps shown in Fig. 9. With the infinite gain assumption for the op-amps, that is $V_- = V_+$, and with the assumption that no current flows in either input, we can apply Kirchoff's Current Law (KCL) at the node designated (a) in Fig. 10:

$$\begin{aligned} i_1 + i_f - i_3 &= 0 \\ (V_{in} - v_a)\frac{1}{R_1} + sC_1(v_1 - v_a) - v_a\frac{1}{R_3} &= 0 \end{aligned} \quad (17)$$

Using assumption 2 above, $v_a = V_{out}$, and realizing that the second stage is a classical op-amp integrator with transfer function

$$\frac{V_{out}(s)}{v_1(s)} = -\frac{1}{R_2C_2s},$$

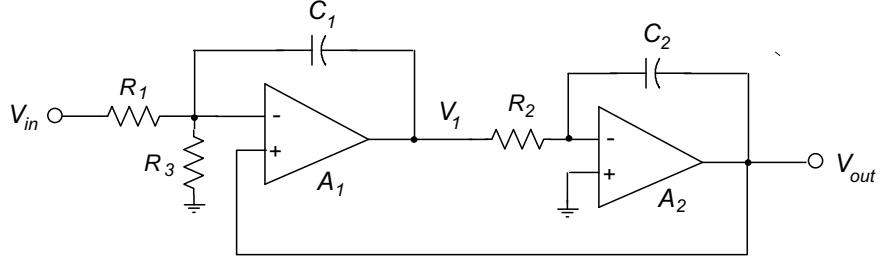


Figure 9: Two op-amp implementation of a “state-variable” second-order active low-pass filter.

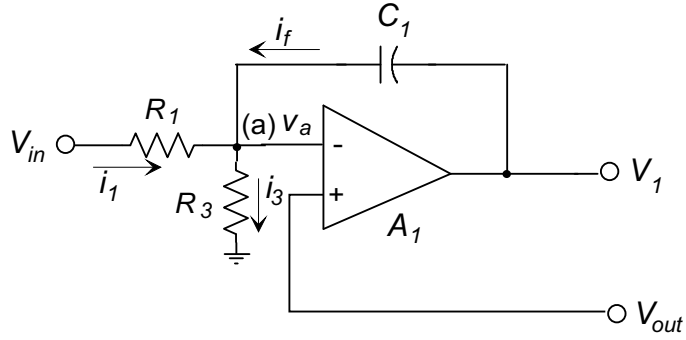


Figure 10: Feedback summation at the input of the first integrator.

Eq. (18) may be rewritten

$$(V_{in} - V_{out}) \frac{1}{R_1} + sC_1(-R_2C_2sV_{out} - V_{out}) - V_{out} \frac{1}{R_3} = 0 \quad (18)$$

which may be rearranged to give the second-order transfer function

$$\frac{V_{out}(s)}{V_{in}(s)} = \frac{1/\tau_1\tau_2}{s^2 + (1/\tau_2)s + (1 + R_1/R_3)/\tau_1\tau_2} \quad (19)$$

which is of the form

$$H_{lp}(s) = \frac{K_{lp}a_0}{s^2 + a_1s + a_0} \quad (20)$$

where

$$a_0 = (1 + R_1/R_3) \frac{1}{\tau_1\tau_2} \quad (21)$$

$$a_1 = \frac{1}{\tau_2} \quad (22)$$

$$K_{lp} = \frac{1}{1 + R_1/R_3} \quad (23)$$

2.4 First-Order Filter Sections:

Single pole low-pass filter sections with a transfer function of the form

$$H(s) = \frac{K\omega_0}{s + \omega_0}$$

may be implemented in either an inverting or non-inverting configuration as shown in Fig. 11. The inverting configuration (Fig. 11(a)) has transfer function

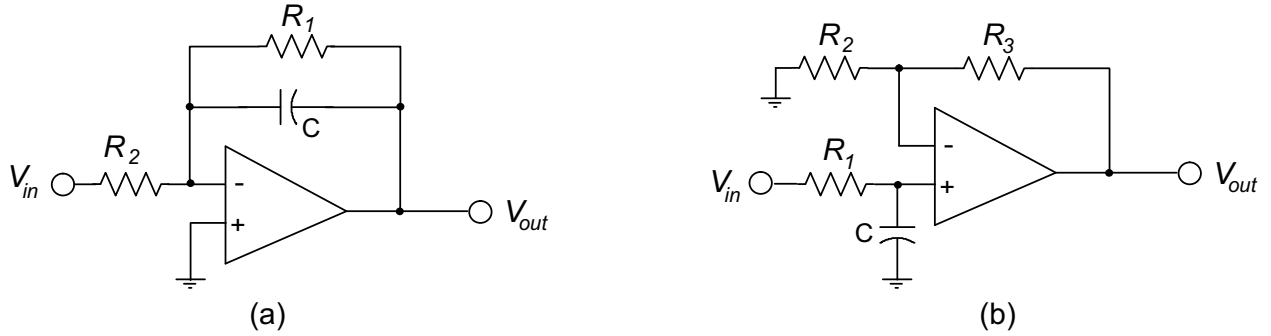


Figure 11: First-order low-pass filter sections (a) inverting, and (b) non-inverting.

$$\frac{V_{out}(s)}{V_{in}(s)} = -\frac{Z_f}{Z_{in}} = -\left(\frac{R_1}{R_2}\right) \frac{1/R_1 C}{s + 1/R_1 C}$$

where $\omega_0 = 1/R_1 C$ and $K = -R_1/R_2$.

The non-inverting configuration of Fig. 11b is a first-order R-C lag circuit buffered by a non-inverting (high input impedance) amplifier with a gain $K = 1 + R_3/R_2$. Its transfer function is

$$\frac{V_{out}(s)}{V_{in}(s)} = \left(1 + \frac{R_3}{R_2}\right) \frac{1/R_1 C}{s + 1/R_1 C}$$

2.5 Summary of Features of the State-variable Filters

- State-variable filters are capable of low-pass, band-pass, high-pass and band-stop functions.
- They are capable of realizing both overdamped and lightly damped pole pairs.
- They are relatively insensitive (compared to other designs) to variation in component values.
- They do not require a wide range of component values.
- The coefficients in the transfer function may be set independently.
- Other designs may require fewer op-amps.

3 Design Examples:

The first two examples involve all-pole low-pass filters, and are therefore suitable for the two op-amp circuit. We will use the following procedure to determine the component values. Given a filter with a unity-gain pole pair described by

$$H(s) = \frac{K_{lp}a_0}{s^2 + a_1s + a_0}$$

where a_0 , a_1 , and K_{lp} are as defined in Eqs. (22) through (24). The circuit components are chosen as follows:

- (a) We note that $a_1 = 1/\tau_2 = 1/R_2C_2$, and therefore choose a convenient value for C_2 and let $R_2 = 1/a_1C_2$.
- (b) We arbitrarily let $R_3 = R_1$, setting $K_{lp} = 0.5$.
- (c) With this condition $a_0 = 2/\tau_1\tau_2 = 2a_1/R_1C_1$, so we may choose a convenient value for C_1 and then determine $R_1 = 2a_1/(a_0C_1)$, which also defines R_3 .

The design is then complete.

3.1 Example 1

Implement a second-order Butterworth filter with a 3 dB cut-off frequency of 1000 rad/sec (159 Hz).

The transfer function of the Butterworth filter is

$$H(s) = \frac{10^6}{s^2 + 1414s + 10^6}$$

Following the above procedure

- (a) Let $C_2 = 0.47 \mu\text{F}$ (a common value). Then $R_2 = 10^6/(1414 \times 0.47) = 1504 \Omega$.
- (b) Let $C_1 = 0.47 \mu\text{F}$. Then $R_1 = R_3 = 2 \times 1414/(10^6 \times 0.47 \times 10^{-6}) = 6017 \Omega$

and the final filter is shown in Fig. 3. The common 741 op-amp has been specified in this case.

3.2 Example 2

Design a fifth-order Chebyshev Type 1 low-pass filter, with a cut-off frequency of 1000 rad/s, and allowing 1 dB of ripple in the pass-band.

The Matlab commands

```
[z,p,k] = cheby1(5,1,1000,'s') filter = zpkm(z,p,k)
```

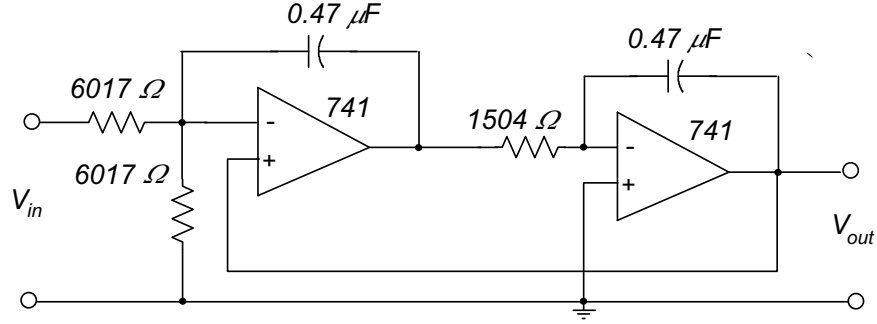


Figure 12: Second-order Butterworth design example.

generate the following filter transfer function

$$\begin{aligned}
 H(s) &= \frac{122828246505000}{(s^2 + 468.4s + 429300)(s^2 + 178.9s + 988300)(s + 289.5)} \\
 &= \frac{429300}{s^2 + 468.4s + 429300} \times \frac{988300}{s^2 + 178.9s + 988300} \times \frac{289.5}{s + 289.5}
 \end{aligned}$$

We implement the filter as two cascade second-order sections (each with a gain of $K_{lp} = 0.5$) as above, and a single first-order non-inverting section with a gain of 4. We will use the two op-amp circuit

Let all capacitors have a value of $0.47 \mu\text{F}$.

(1) For the first section $a = 468.4$, $b = 429300$:

$$\begin{aligned}
 R_2 &= \frac{1}{aC_2} = \frac{1}{468.4 \times 0.47 \times 10^{-6}} = 4,542 \Omega, \\
 R_1 &= R_3 = \frac{2a}{bC_1} = \frac{2 \times 468.4}{429,300 \times 0.47 \times 10^{-6}} = 4,642 \Omega
 \end{aligned}$$

(2) For the second section $a = 178.9$, $b = 988,300$:

$$\begin{aligned}
 R_2 &= \frac{1}{aC_2} = \frac{1}{178.9 \times 0.47 \times 10^{-6}} = 11,893 \Omega, \\
 R_1 &= R_3 = \frac{2a}{bC_1} = \frac{2 \times 178.9}{988,300 \times 0.47 \times 10^{-6}} = 770 \Omega
 \end{aligned}$$

(2) For the first-order section $K = 4$ $\omega_c = 289.5$:

$$\begin{aligned}
 R_1 &= \frac{1}{\omega_c C} = \frac{1}{289.5 \times 0.47 \times 10^{-6}} = 7,349 \Omega, \\
 \text{Let } R_2 &= 1,500 \Omega, \\
 R_3 &= (K - 1)R_2 = 3 \times 1,500 = 4,500 \Omega
 \end{aligned}$$

and the design is complete. The final circuit is shown in Fig. 13.

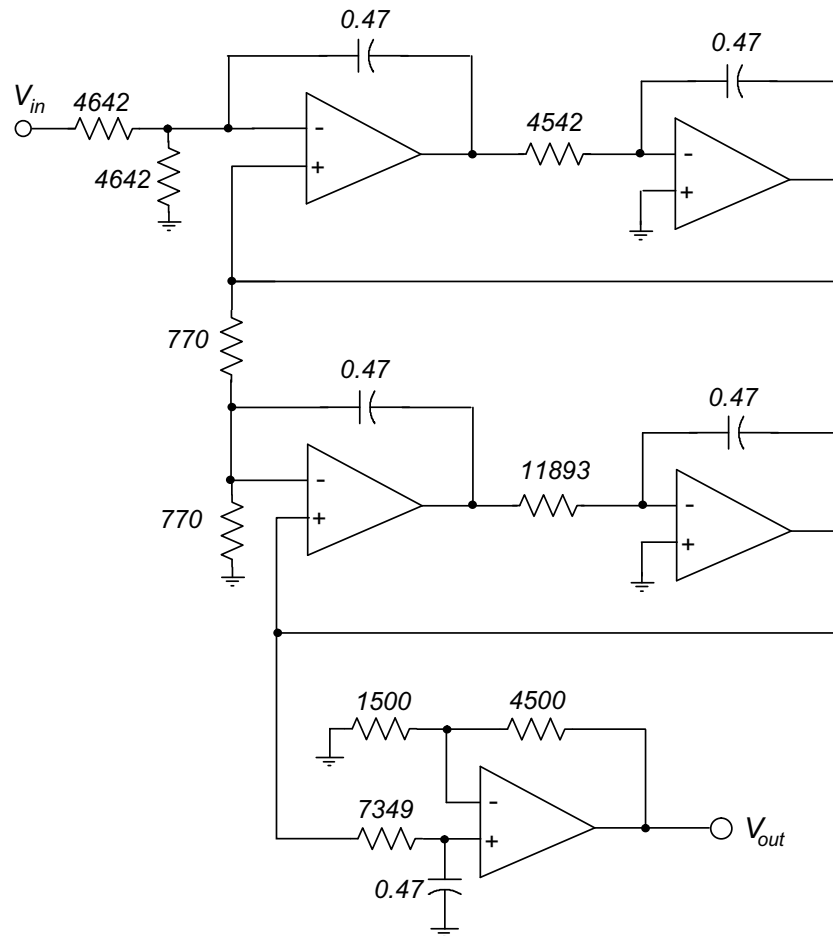


Figure 13: Fifth-order Chebyshev Type 1 low-pass design example. All resistor values are in ohms, all capacitor values are in microfarads.

3.3 Example 3

Design a second-order band-stop filter to reject 60 Hz interference, with a bandwidth of 20 Hz.

We start with a first-order low-pass prototype filter with a cut-off frequency of 1 rad/sec. (Note that the low-pass to band-stop transformation will generate a second-order filter)

$$H_{lp}(s) = \frac{1}{s + 1}.$$

The Matlab commands

```
[num,den]=lp2bs(1,[1 1],2*pi*60,2*pi*20) filter=tf(num,den)
```

generate the following filter transfer function

$$H(s) = \frac{s^2 + 142100}{s^2 + 125.7s + 142100}$$

We use the design equations described for band-stop filters in Section 2.3 with the circuit shown in Fig. 8, that is

$$H_{bs}(s) = \frac{-K_{bs}(s^2 + a_0)}{s^2 + a_1s + a_0}$$

where

$$\begin{aligned} a_0 &= \frac{1}{\tau_1\tau_2} \\ a_1 &= \left(\frac{2}{1 + R_6/R_5} \right) \frac{1}{\tau_1} \\ K_{bs} &= \frac{2R_9}{(1 + R_5/R_6)R_8} \end{aligned}$$

under the constraints that $R_3 = R_4$ and $R_7 = R_8$ in Fig. 8.

(a) Let $\tau_1 = \tau_2 = 1/\sqrt{a_0} = 1/\sqrt{142100} = 0.0027$. Then (arbitrarily) let $C_1 = C_2 = 0.47 \mu\text{F}$, so that $R_1 = R_2 = 0.0027/0.47 \times 10^{-6} = 5644 \Omega$.

(b) Since

$$\begin{aligned} a_1 &= \left(\frac{2}{1 + R_6/R_5} \right) \frac{1}{\tau_1}, \\ \frac{R_6}{R_5} &= \frac{2}{a_1\tau} - 1 = \frac{2}{125.7 \times 0.0027} - 1 = 5.0 \end{aligned}$$

We let $R_5 = 10000 \Omega$ and $R_6 = 50000 \Omega$.

(c) We let $R_3 = R_4 = R_7 = R_8 = 10000 \Omega$.

(d) We set $K_{bs} = 1$ so that $R_9 = 0.5K_{bs}(1 + R_5/R_6)R_8 = 6000 \Omega$.

Which completes the design, as shown in Fig. 14. Note that this filter inverts the signal, so if the application requires maintaining the sign of the input an extra op-amp inverter should be used.

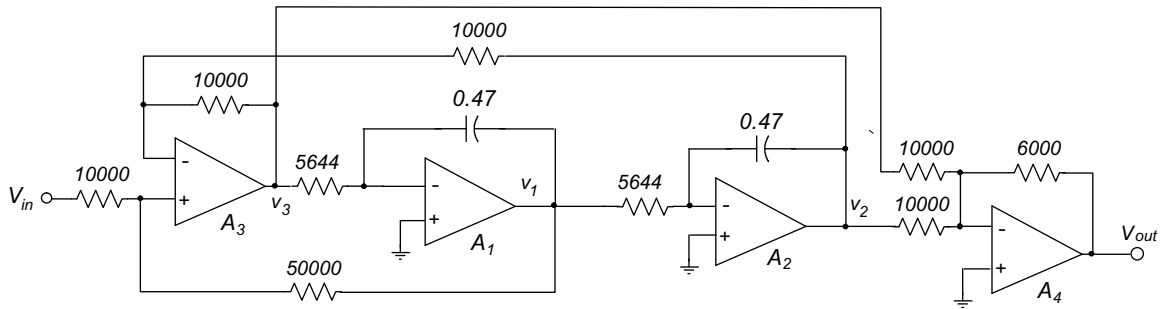


Figure 14: Second-order band-stop filter design example. All resistor values are in ohms, all capacitor values are in microfarads.