

**2.20 - Marine Hydrodynamics**  
**Lecture 5**

**Chapter 2 - Similitude** (Keyword: **EQUAL RATIOS**)

Similitude: Similarity of behavior for different systems with equal similarity parameters.

Prototype            ↔            Model  
(real world)                    (physical/ analytical/ numerical ... experiments)

<b>Similitude</b>	<b>Similarity Parameters (SP's)</b>
Geometric Similitude	Length ratios, angles
Kinematic Similitude	Displacement ratios, velocity ratios
Dynamic Similitude	Force ratios, stress ratios, pressure ratios
⋮	
Internal Constitution Similitude	$\rho, \nu$
Boundary Condition Similitude	
⋮	

For similitude we require that the similarity parameters SP's (eg. **angles**, length **ratios**, velocity **ratios**, etc) are equal for the model and the real world.

Example 1 Two similar triangles have equal **angles** or equal **length ratios**. In this case the two triangles have *geometric similitude*.

Example 2 For the flow around a model ship to be similar to the flow around the prototype ship, both model and prototype need to have equal **angles** and equal **length** and **force ratios**. In this case the model and the prototype have *geometric and dynamic similitude*.

## 2.1 Dimensional Analysis (DA) to Obtain SP's

### 2.1.1 Buckingham's $\pi$ theory

Reduce number of variables  $\rightarrow$  derive dimensionally homogeneous relationships.

1. Specify (all) the (say  $N$ ) relevant variables (dependent or independent):  $x_1, x_2, \dots, x_N$   
e.g. time, force, fluid density, distance...  
We want to relate the  $x_i$ 's to each other  $\mathcal{I}(x_1, x_2, \dots, x_N) = 0$
2. Identify (all) the (say  $P$ ) relevant basic physical units ("dimensions")  
e.g. M,L,T ( $P = 3$ ) [temperature, charge, ...].
3. Let  $\pi = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_N^{\alpha_N}$  be a dimensionless quantity formed from the  $x_i$ 's. Suppose

$$x_i = C_i M^{m_i} L^{l_i} T^{t_i}, i = 1, 2, \dots, N$$

where the  $C_i$  are dimensionless constants. For example, if  $x_1 = KE = \frac{1}{2}MV^2 = \frac{1}{2}M^1L^2T^{-2}$  (kinetic energy), we have that  $C_1 = \frac{1}{2}, m_1 = 1, l_1 = 2, t_1 = -2$ . Then

$$\pi = (C_1^{\alpha_1} C_2^{\alpha_2} \dots C_N^{\alpha_N}) M^{\alpha_1 m_1 + \alpha_2 m_2 + \dots + \alpha_N m_N} L^{\alpha_1 l_1 + \alpha_2 l_2 + \dots + \alpha_N l_N} T^{\alpha_1 t_1 + \alpha_2 t_2 + \dots + \alpha_N t_N}$$

For  $\pi$  to be dimensionless, we require

$$P \left\{ \begin{array}{l} \overbrace{\alpha_i m_i = 0}^N \\ \alpha_i l_i = 0 \\ \alpha_i t_i = 0 \\ \underbrace{\hspace{10em}}_{\Sigma \text{ notation}} \end{array} \right\} \text{ a } P \times N \text{ system of Linear Equations} \quad (1)$$

Since (1) is homogeneous, it always has a trivial solution,

$$\alpha_i \equiv 0, i = 1, 2, \dots, N \text{ (i.e. } \pi \text{ is constant)}$$

There are 2 possibilities:

- (a) (1) has no nontrivial solution (only solution is  $\pi = \text{constant}$ , i.e. independent of  $x_i$ 's), which implies that the  $N$  variable  $x_i, i = 1, 2, \dots, N$  are Dimensionally Independent (DI), i.e. they are 'unrelated' and 'irrelevant' to the problem.
- (b) (1) has  $J$  ( $J > 0$ ) nontrivial solutions,  $\pi_1, \pi_2, \dots, \pi_J$ . In general,  $J < N$ , in fact,  $J = N - K$  where  $K$  is the rank or 'dimension' of the system of equations (1).

### 2.1.2 Model Law

Instead of relating the  $N$   $x_i$ 's by  $\mathcal{I}(x_1, x_2, \dots, x_N) = 0$ , relate the  $J$   $\pi$ 's by

$$F(\pi_1, \pi_2, \dots, \pi_J) = 0, \text{ where } J = N - K < N$$

For similitude, we require

$$(\pi_{\text{model}})_j = (\pi_{\text{prototype}})_j \text{ where } j = 1, 2, \dots, J.$$

If 2 problems have all the same  $\pi_j$ 's, they have similitude (in the  $\pi_j$  senses), so  $\pi$ 's serve as similarity parameters.

Note:

- If  $\pi$  is dimensionless, so is  $\pi \times \text{const}$ ,  $\pi^{\text{const}}$ ,  $1/\pi$ , etc...
- If  $\pi_1, \pi_2$  are dimensionless, so is  $\pi_1 \times \pi_2$ ,  $\frac{\pi_1}{\pi_2}$ ,  $\pi_1^{\text{const}_1} \times \pi_2^{\text{const}_2}$ , etc...

In general, we want the set (not unique) of independent  $\pi_j$ 's, for e.g.,  $\pi_1, \pi_2, \pi_3$  or  $\pi_1, \pi_1 \times \pi_2, \pi_3$ , but not  $\pi_1, \pi_2, \pi_1 \times \pi_2$ .

**Example: Force on a smooth circular cylinder in steady, incompressible flow**  
Application of Buckingham's  $\pi$  Theory.

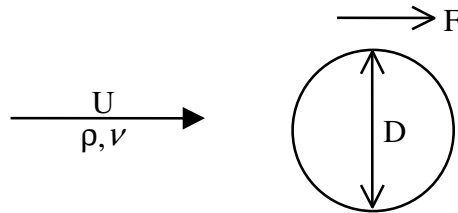


Figure 1: Force on a smooth circular cylinder in steady incompressible fluid (no gravity)

A Fluid Mechanician found that the relevant *dimensional* quantities required to evaluate the force  $F$  on the cylinder from the fluid are: the diameter of the cylinder  $D$ , the fluid velocity  $U$ , the fluid density  $\rho$  and the kinematic viscosity of the fluid  $\nu$ . Evaluate the *non-dimensional* independent parameters that describe this problem.

$$x_i : F, U, D, \rho, \nu \rightarrow N = 5$$

$$x_i = c_i M^{m_i} L^{l_i} T^{t_i} \rightarrow P = 3$$

		N = 5				
		F	U	D	$\rho$	$\nu$
P = 3	$m_i$	1	0	0	1	0
	$l_i$	1	1	1	-3	2
	$t_i$	-2	-1	0	0	-1

$$\pi = F^{\alpha_1} U^{\alpha_2} D^{\alpha_3} \rho^{\alpha_4} \nu^{\alpha_5}$$

For  $\pi$  to be non-dimensional, the set of equations

$$\alpha_i m_i = 0$$

$$\alpha_i l_i = 0$$

$$\alpha_i t_i = 0$$

has to be satisfied. The system of equations above after we substitute the values for the  $m_i$ 's,  $l_i$ 's and  $t_i$ 's assume the form:

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & -3 & 2 \\ -2 & -1 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The rank of this system is  $K = 3$ , so we have  $j = 2$  nontrivial solutions. Two families of solutions for  $\alpha_i$  for each fixed pair of  $(\alpha_4, \alpha_5)$ , exists a unique solution for  $(\alpha_1, \alpha_2, \alpha_3)$ . We consider the pairs  $(\alpha_4 = 1, \alpha_5 = 0)$  and  $(\alpha_4 = 0, \alpha_5 = 1)$ , all other cases are linear combinations of these two.

1. Pair  $\alpha_4 = 1$  and  $\alpha_5 = 0$ .

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ 2 \end{pmatrix}$$

which has solution

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$$

$$\therefore \pi_1 = F^{\alpha_1} U^{\alpha_2} D^{\alpha_3} \rho^{\alpha_4} \nu^{\alpha_5} = \frac{\rho U^2 D^2}{F}$$

Conventionally,  $\pi_1 \rightarrow 2\pi_1^{-1}$  and  $\therefore \pi_1 = \frac{F}{\frac{1}{2}\rho U^2 D^2} \equiv C_d$ , which is the Drag coefficient.

2. Pair  $\alpha_4 = 0$  and  $\alpha_5 = 1$ .

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ -1 \end{pmatrix}$$

which has solution

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}$$

$$\therefore \pi_2 = F^{\alpha_1} U^{\alpha_2} D^{\alpha_3} \rho^{\alpha_4} \nu^{\alpha_5} = \frac{\nu}{UD}$$

Conventionally,  $\pi_2 \rightarrow \pi_2^{-1}$ ,  $\therefore \pi_2 = \frac{UD}{\nu} \equiv R_e$ , which is the Reynolds number.

Therefore, we can write the following equivalent expressions for the *non-dimensional* independent parameters that describe this problem:

$$\begin{array}{lll} F(\pi_1, \pi_2) = 0 & \text{or} & \pi_1 = f(\pi_2) \\ F(\mathcal{C}_d, \mathcal{R}_e) = 0 & \text{or} & \mathcal{C}_d = f(\mathcal{R}_e) \\ F\left(\frac{F}{1/2 \rho U^2 D^2}, \frac{UD}{\nu}\right) = 0 & \text{or} & \frac{F}{1/2 \rho U^2 D^2} = f\left(\frac{UD}{\nu}\right) \end{array}$$

## Appendix A

Dimensions of *some* fluid properties

Quantities		Dimensions ( $MLT$ )
Angle	$\theta$	none ( $M^0L^0T^0$ )
Length	$L$	$L$
Area	$A$	$L^2$
Volume	$\forall$	$L^3$
Time	$t$	$T$
Velocity	$V$	$LT^{-1}$
Acceleration	$\dot{V}$	$LT^{-2}$
Angular velocity	$\omega$	$T^{-1}$
Density	$\rho$	$ML^{-3}$
Momentum	$\mathcal{L}$	$MLT^{-1}$
Volume flow rate	$\mathcal{Q}$	$L^3T^{-1}$
Mass flow rate	$\mathcal{Q}$	$MT^{-1}$
Pressure	$p$	$ML^{-1}T^{-2}$
Stress	$\tau$	$ML^{-1}T^{-2}$
Surface tension	$\Sigma$	$MT^{-2}$
Force	$F$	$MLT^{-2}$
Moment	$M$	$ML^2T^{-2}$
Energy	$E$	$ML^2T^{-2}$
Power	$P$	$ML^2T^{-3}$
Dynamic viscosity	$\mu$	$ML^{-1}T^{-1}$
Kinematic viscosity	$\nu$	$L^2T^{-1}$

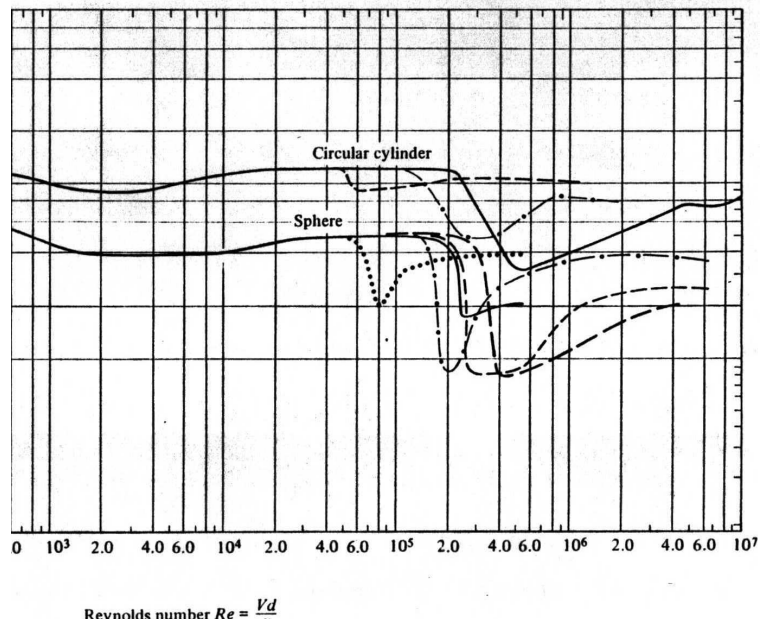


Figure 2: Drag Coefficient as a function of Reynolds number)