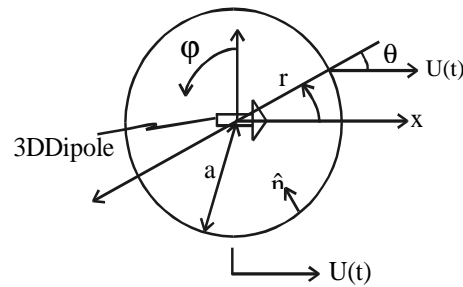


2.20 - Marine Hydrodynamics Lecture 12

3.14 Unsteady Motion - Added Mass

D'Alembert: ideal, irrotational, unbounded, steady.

Example Force on a sphere accelerating ($U = U(t)$, unsteady) in an unbounded fluid that is at rest at infinity.



K.B.C on sphere: $\left. \frac{\partial \phi}{\partial r} \right|_{r=a} = U(t) \cos \theta$

Solution: Simply a 3D dipole (no stream)

$$\phi = -U(t) \frac{a^3}{2r^2} \cos \theta$$

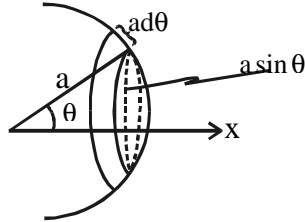
Check: $\left. \frac{\partial \phi}{\partial r} \right|_{r=a} = U(t) \cos \theta$

Hydrodynamic force:

$$F_x = -\rho \iint_B \left(\frac{\partial \phi}{\partial t} + \frac{1}{2} |\nabla \phi|^2 \right) n_x dS$$

On $r = a$,

$$\begin{aligned} \frac{\partial \phi}{\partial t} \Big|_{r=a} &= -\dot{U} \frac{a^3}{2r^2} \cos \theta \Big|_{r=a} = -\frac{1}{2} \dot{U} a \cos \theta \\ \nabla \phi \Big|_{r=a} &= \left(\frac{\partial \phi}{\partial r}, \frac{1}{r} \frac{\partial \phi}{\partial \theta}, \frac{1}{r \sin \theta} \frac{\partial \phi}{\partial \varphi} \right) = \left(U \cos \theta, \frac{1}{2} U \sin \theta, 0 \right) \\ |\nabla \phi|^2 \Big|_{r=a} &= U^2 \cos^2 \theta + \frac{1}{4} U^2 \sin^2 \theta; \hat{n} = -\hat{e}_r, n_x = -\cos \theta \\ \iint_B dS &= \int_0^\pi (a d\theta) (2\pi a \sin \theta) \end{aligned}$$



Finally,

$$\begin{aligned}
 F_x &= (-\rho) 2\pi a^2 \int_0^\pi d\theta (\sin \theta) \underbrace{\left(-\cos \theta \right)}_{n_x} \left[\underbrace{-\frac{1}{2} \dot{U} a \cos \theta}_{\frac{\partial \phi}{\partial t}} + \frac{1}{2} \underbrace{\left(U^2 \cos^2 \theta + \frac{1}{4} U^2 \sin^2 \theta \right)}_{|\nabla \phi|^2} \right] \\
 F_x &= -\dot{U} (\rho a^3) \pi \underbrace{\int_0^\pi d\theta \sin \theta \cos^2 \theta}_{2/3} + (\rho U^2) \pi a^2 \underbrace{\int_0^\pi d\theta \sin \theta \cos \theta \left(\cos^2 \theta + \frac{1}{4} \sin^2 \theta \right)}_{= 0, \text{ D'Alembert revisited}} \\
 \underbrace{F_x}_{\text{Hydrodynamic Force}} &= - \underbrace{\dot{U}(t)}_{\text{Acceleration}} \left[\underbrace{\rho}_{\text{Fluid Density}} \underbrace{\frac{2}{3} \pi a^3}_{\text{Volume} = 1/2 V_{\text{sphere}}} \right]
 \end{aligned}$$

Thus the **Hydrodynamic Force** on a sphere of radius a moving with velocity $U(t)$ in an unbounded fluid of density ρ is given by

$$\boxed{F_x = -\dot{U}(t) \left[\rho \frac{2}{3} \pi a^3 \right]}$$

Comments:

- If $\dot{U} = 0 \rightarrow F_x = 0$, i.e., steady translation \rightarrow no force (D'Alembert's Condition ok).
- $F_x \propto \dot{U}$ with a $(-)$ sign, i.e., the fluid tends to 'resist' the acceleration.
- $[\dots]$ has the units of (**fluid**) mass $\equiv m_a$
- Equation of Motion for a body of mass M that moves with velocity U :

$$\underbrace{M}_{\text{Body mass}} \dot{U} = \Sigma F = \underbrace{F_H}_{\int \int_S p \hat{n} dS} + \underbrace{F_B}_{\text{All other forces on body}} = \left(-\dot{U} \underbrace{m_a}_{\text{Fluid mass}} \right) + F_B \Leftrightarrow$$

$$\boxed{(M + m_a) \dot{U} = F_B}$$

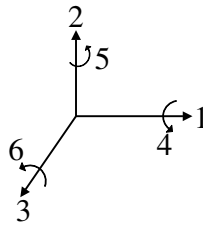
i.e., the presence of fluid around the body acts as an **added** or **virtual** mass to the body.

3.15 General 6 Degrees of Freedom Motions

3.15.1 Notation Review

(3D) U_1, U_2, U_3 : Translational velocities

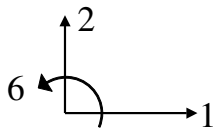
$U_4 \equiv \Omega_1, U_5 \equiv \Omega_2, U_6 \equiv \Omega_3$: Rotational velocities



(2D) U_1, U_2 : Translational velocities

$U_6 \equiv \Omega_3$: Rotational velocity

$U_3 = U_4 = U_5 = 0$



3.15.2 Added Mass Tensor (matrix)

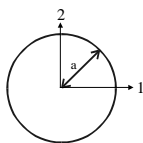
$$m_{ij} ; i, j = 1, 2, 3, 4, 5, 6$$

m_{ij} : associated with force on body in i direction due to unit acceleration in j direction. For example, for a sphere:

$$m_{11} = m_{22} = m_{33} = \frac{1}{2}\rho V = (m_A) \text{ all other } m_{ij} = 0$$

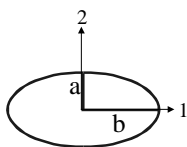
3.15.3 Added Masses of Simple 2D Geometries

- Circle



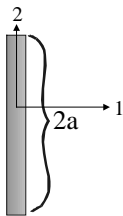
$$m_{11} = m_{22} = \rho V = \rho \pi a^2$$

- Ellipse



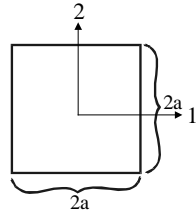
$$m_{11} = \rho \pi a^2, m_{22} = \rho \pi b^2$$

- Plate



$$m_{11} = \rho \pi a^2, m_{22} = 0$$

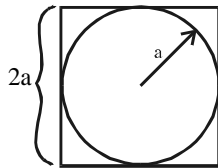
- Square



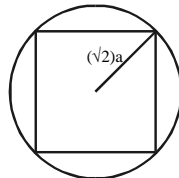
$$m_{11} = m_{22} \approx 4.754\rho a^2$$

A reasonable approximation to estimate the added mass of a 2D body is to use the displaced mass (ρV) of an ‘equivalent cylinder’ of the same lateral dimension or one that ‘rounds off’ the body. For example, consider a square and approximate with an

- (a) inscribed circle: $m_A = \rho\pi a^2 = 3.14\rho a^2$.



- (b) circumscribed circle: $m_A = \rho\pi (\sqrt{2}a)^2 = 6.28\rho a^2$.

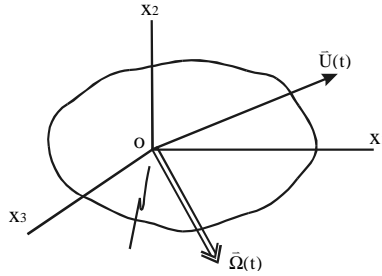


Arithmetic mean of (a) + (b) $\approx 4.71\rho a^2$.

3.15.4 Generalized Forces and Moments

In this paragraph we are looking at the most general case where forces and moments are induced on rigid body moving with 6 DoF motions, in an unbounded fluid that is at rest at infinity.

Body fixed reference frame, i.e., $OX_1X_2X_3$ is **fixed** on the body.



$$\vec{U}(t) = (U_1, U_2, U_3), \text{ translational velocity}$$

$$\vec{\Omega}(t) = (\Omega_1, \Omega_2, \Omega_3) \equiv (U_4, U_5, U_6), \text{ rotational velocity with respect to } O$$

Consider a body with a 6 DoF motion $(\vec{U}, \vec{\Omega})$, and a fixed reference frame $OX_1X_2X_3$. Then the hydrodynamic forces and moments with respect to O are given by the following relations (JNN §4.13)

- Forces

$$F_j = -\underset{1.}{\dot{U}_i} m_{ji} - E_{jkl} \underset{2.}{U_i} \Omega_k m_{li} \quad \text{with} \quad i = 1, 2, 3, 4, 5, 6$$

$$\text{and} \quad j, k, l = 1, 2, 3$$

- Moments

$$M_j = -\underset{3.}{\dot{U}_i} m_{j+3,i} - E_{jkl} \underset{2.}{U_i} \Omega_k m_{l+3,i} - E_{jkl} \underset{3.}{U_k} U_i m_{li} \quad \text{with} \quad i = 1, 2, 3, 4, 5, 6$$

$$\text{and} \quad j, k, l = 1, 2, 3$$

Einstein's Σ notation applies.

$$E_{jkl} = \text{'alternating tensor'} = \begin{cases} 0 & \text{if any } j, k, l \text{ are equal} \\ 1 & \text{if } j, k, l \text{ are in cyclic order, i.e.,} \\ & (1, 2, 3), (2, 3, 1), \text{ or } (3, 1, 2) \\ -1 & \text{if } j, k, l \text{ are not in cyclic order i.e.,} \\ & (1, 3, 2), (2, 1, 3), (3, 2, 1) \end{cases}$$

Note:

(a) if $\Omega_k \equiv 0$, $F_j = -\dot{U}_i m_{ji}$ (as expected by definition of m_{ij}).

Also if $\dot{U}_i \equiv 0$, then $F_j = 0$ for any U_i , no force in steady translation.

(b) $B_l \sim U_i m_{li}$ 'added momentum' due to rotation of axes.

Then all the terms marked as 2. are proportional to $\sim \vec{\Omega} \times \vec{B}$ where \vec{B} is linear momentum (momentum from i coordinate into new x_j direction).

(c) If $\Omega_k \equiv 0$: $M_j = -\dot{U}_i m_{j+3,i} - \underbrace{E_{jkl} U_k U_i m_{li}}_{\text{even with } \dot{U}=0, M_j \neq 0 \text{ due to this term}}.$

Moment on a body due to pure steady translation – 'Munk' moment.

3.15.5 Example Generalized motions, forces and moments.

A certain body has non-zero added mass coefficients only on the diagonal, i.e. $m_{ij} = \delta_{ij}$. For a body motion given by $U_1 = t$, $U_2 = -t$, and all other U_i , $\Omega_i = 0$, the forces and moments on the body in terms of m_i are:

$$F_1 = \underline{\hspace{2cm}}, F_2 = \underline{\hspace{2cm}}, F_3 = \underline{\hspace{2cm}}, M_1 = \underline{\hspace{2cm}}, M_2 = \underline{\hspace{2cm}}, M_3 = \underline{\hspace{2cm}}$$

Solution:

$$m_{ij} = \delta_{ij}$$

$$U_1 = t \quad U_2 = -t \quad U_i = 0 \quad i = 3, 4, 5, 6 \quad \Omega_k = 0 \quad k = 1, 2, 3$$

$$\dot{U}_1 = 1 \quad \dot{U}_2 = -1 \quad \dot{U}_i = 0 \quad i = 3, 4, 5, 6$$

Use the relations from (JNN §4.13):

$$\begin{aligned} F_j &= -\dot{U}_i m_{ij} - E_{jkl} U_i \Omega_k m_{il} \xrightarrow{\Omega_k=0} \\ F_j &= -\dot{U}_i m_{ij} \end{aligned}$$

$$\begin{aligned} M_j &= -\dot{U}_i m_{i(j+3)} - E_{jkl} U_i \Omega_k m_{i(l+3)} - E_{jkl} U_k U_i m_{li} \xrightarrow{\Omega_k=0} \\ M_j &= -\dot{U}_i m_{i(j+3)} - E_{jkl} U_k U_i m_{li} \end{aligned}$$

where $i = 1, 2, 3, 4, 5, 6$ and $j, k, l = 1, 2, 3$

For F_1, F_2, F_3 use the previous relationship for F_j with $j = 1, 2, 3$ respectively:

$$\begin{aligned} F_1 &= -\underbrace{\dot{U}_1}_{=1} m_{11} - \underbrace{\dot{U}_2}_{=0} m_{21} - \underbrace{\dot{U}_3}_{=0} m_{31} - \underbrace{\dot{U}_4}_{=0} m_{41} - \underbrace{\dot{U}_5}_{=0} m_{51} - \underbrace{\dot{U}_6}_{=0} m_{61} \rightarrow \boxed{F_1 = -m_{11}} \\ F_2 &\stackrel{\text{Check}}{=} -\underbrace{\dot{U}_2}_{=-1} m_{22} \rightarrow \boxed{F_2 = m_{22}} \\ F_3 &\stackrel{\text{Check}}{=} -\underbrace{\dot{U}_3}_{=0} m_{33} \rightarrow \boxed{F_3 = 0} \end{aligned}$$

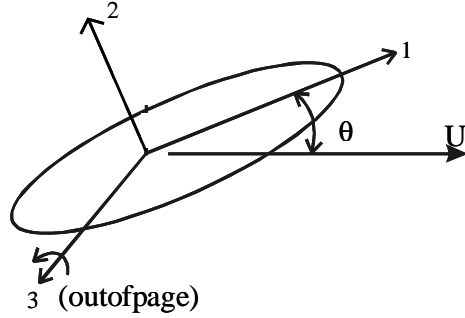
For M_1, M_2, M_3 use the previous relationship for M_j with $j = 1, 2, 3$ respectively:

$$\begin{aligned}
M_1 &= -\dot{U}_i m_{i(1+3)} - E_{1kl} U_k U_i m_{li} \\
&= -\dot{U}_i m_{i4} - E_{1kl} U_k U_i m_{li} \\
&= -\dot{U}_1 \underbrace{m_{14}}_{=0} - \dot{U}_2 \underbrace{m_{24}}_{=0} - \dot{U}_3 \underbrace{m_{34}}_{=0} - \underbrace{\dot{U}_4}_{=0} m_{44} - \dot{U}_5 \underbrace{m_{54}}_{=0} - \dot{U}_6 \underbrace{m_{64}}_{=0} \\
&\quad - E_{123} U_2 (U_1 \underbrace{m_{13}}_{=0} + U_2 \underbrace{m_{23}}_{=0} + \underbrace{U_3}_{=0} m_{33} + U_4 \underbrace{m_{43}}_{=0} + U_5 \underbrace{m_{53}}_{=0} + U_6 \underbrace{m_{63}}_{=0}) \\
&\quad - E_{132} \underbrace{U_3}_{=0} (U_1 \underbrace{m_{12}}_{=0} + \underbrace{U_2}_{=-1} m_{22} + U_3 \underbrace{m_{32}}_{=0} + U_4 \underbrace{m_{42}}_{=0} + U_5 \underbrace{m_{52}}_{=0} + U_6 \underbrace{m_{62}}_{=0}) \rightarrow \boxed{M_1 = 0}
\end{aligned}$$

$$\begin{aligned}
M_2 &= -\dot{U}_i m_{i5} - E_{2kl} U_k U_i m_{li} \\
&= \dot{U}_5 m_{55} - E_{231} U_3 U_i m_{1i} - E_{213} U_1 U_i m_{3i} \\
&= -E_{213} U_1 U_3 m_{33} \rightarrow \boxed{M_2 = 0}
\end{aligned}$$

$$\begin{aligned}
M_3 &= -\dot{U}_i m_{i6} - E_{3kl} U_k U_i m_{li} \\
&= \dot{U}_6 m_{66} - \underbrace{E_{312}}_{+1} U_1 U_i m_{2i} - \underbrace{E_{321}}_{-1} U_2 U_i m_{1i} \\
&= -\underbrace{U_1}_t \underbrace{U_2}_{-t} m_{22} + \underbrace{U_2}_{-t} \underbrace{U_1}_t m_{11} \rightarrow \boxed{M_3 = t^2(m_{22} - m_{11})}
\end{aligned}$$

3.15.6 Example Munk Moment on a 2D submarine in steady translation



$$\begin{aligned} U_1 &= U \cos \theta \\ U_2 &= -U \sin \theta \end{aligned}$$

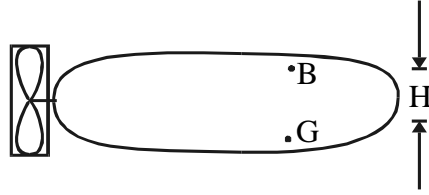
Consider steady translation motion: $\dot{U} = 0; \Omega_k = 0$. Then

$$M_3 = -E_{3kl} U_k U_l m_{li}$$

For a 2D body, $m_{3i} = m_{i3} = 0$, also $U_3 = 0, i, k, l = 1, 2$. This implies that:

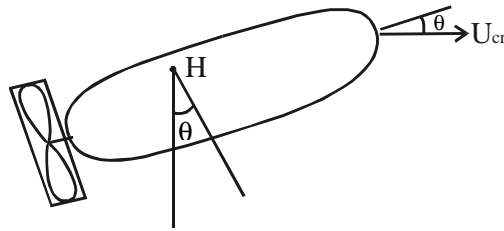
$$\begin{aligned} M_3 &= -\underbrace{E_{312}}_{=1} U_1 (U_1 m_{21} + U_2 m_{22}) - \underbrace{E_{321}}_{=-1} U_2 (U_1 m_{11} + U_2 m_{12}) \\ &= -U_1 U_2 (m_{22} - m_{11}) \\ &= U^2 \sin \theta \cos \theta \left(\underbrace{m_{22} - m_{11}}_{>0} \right) \end{aligned}$$

Therefore, $M_3 > 0$ for $0 < \theta < \pi/2$ ('Bow up'). Therefore, a submarine under forward motion is unstable in pitch (yaw). For example, a small bow-up tends to grow with time, and control surfaces are needed as shown in the following figure.



- Restoring moment $\approx (\rho g \nabla H) \sin \theta$.
- critical speed U_{cr} given by:

$$(\rho g \nabla) H \sin \theta \geq U_{cr}^2 \sin \theta \cos \theta (m_{22} - m_{11})$$



Usually $m_{22} \gg m_{11}$, $m_{22} \approx \rho \nabla$. For small θ , $\cos \theta \approx 1$. So, $U_{cr}^2 \leq gH$ or $F_{cr} \equiv \frac{U_{cr}}{\sqrt{gH}} \leq 1$. Otherwise, control fins are required.