2.20 – Marine Hydrodynamics, Fall 2008
Lecture 22

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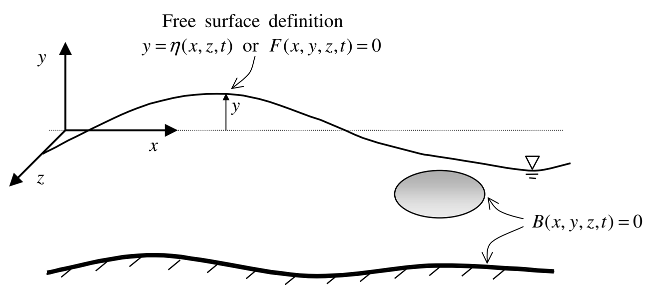
Chapter 6 - Water Waves

6.1 Exact (Nonlinear) Governing Equations for Surface Gravity Waves, Assuming Potential Flow



Unknown variables

Velocity field: $\vec{v}\,(x,y,z,t) = \nabla\phi\,(x,y,z,t)$

Position of free surface: $y = \eta\,(x,z,t)$ or $F\,(x,y,z,t) = 0$

Pressure field: $p\,(x,y,z,t)$

Governing equations

Continuity: $\nabla^2\phi = 0 \quad y < \eta$ or $F < 0$

Bernoulli for P-Flow: $\frac{\partial\phi}{\partial t} + \frac{1}{2}\left|\nabla\phi\right|^2 + \frac{p-p_a}{\rho} + gy = 0; \quad y < \eta$ or $F < 0$

Far way, no disturbance: $\partial\phi/\partial t, \; \nabla\phi \rightarrow 0$ and $p = \underbrace{p_a}_{\text{atmospheric}} - \underbrace{\rho g y}_{\text{hydrostatic}}$

Boundary Conditions

1. On an impervious boundary $B(x, y, z, t) = 0$, we have KBC:

$$\vec{v} \cdot \hat{n} = \nabla\phi \cdot \hat{n} = \frac{\partial \phi}{\partial n} = \vec{U}(\vec{x}, t) \cdot \hat{n}(\vec{x}, t) = U_n \text{ on } B = 0$$

Alternatively: a particle P on B remains on B, i.e., B is a material surface. For example if P is on B at $t = t_0$, P stays on B for all t.

$$B(\vec{x}_P, t_0) = 0, \text{ then } B(\vec{x}_P(t), t) = 0 \text{ for all } \ t,$$

so that, following P B is always 0.

$$\therefore \frac{DB}{Dt} = \frac{\partial B}{\partial t} + (\nabla\phi \cdot \nabla)\, B = 0 \text{ on } B = 0$$

For example, for a flat bottom at $y = -h \Rightarrow B = y + h = 0 \Rightarrow$

$$\frac{DB}{Dt} = \left(\frac{\partial \phi}{\partial y}\right) \Bigg(\underbrace{\frac{\partial}{\partial y}(y+h)}_{=1}\Bigg) = 0 \Rightarrow \frac{\partial \phi}{\partial y} = 0 \text{ on } B = y + h = 0$$

2. On the free surface, $y = \eta$ or $F = y - \eta(x, z, t) = 0$ we have KBC and DBC.

KBC: free surface is a material surface, no normal velocity relative to the free surface. A particle on the free surface remains on the free surface for all times.

$$\frac{DF}{Dt} = 0 = \frac{D}{Dt}(y - \eta) = \underbrace{\frac{\partial \phi}{\partial y}}_{\substack{\text{vertical}\\ \text{velocity}}} - \frac{\partial \eta}{\partial t} - \frac{\partial \phi}{\partial x} \underbrace{\frac{\partial \eta}{\partial x}}_{\substack{\text{slope}\\ \text{of f.s.}}} - \frac{\partial \phi}{\partial z} \underbrace{\frac{\partial \eta}{\partial z}}_{\substack{\text{slope}\\ \text{of f.s.}}} \text{ on } y = \underbrace{\eta}_{\substack{\text{still}\\ \text{unknown}}}$$

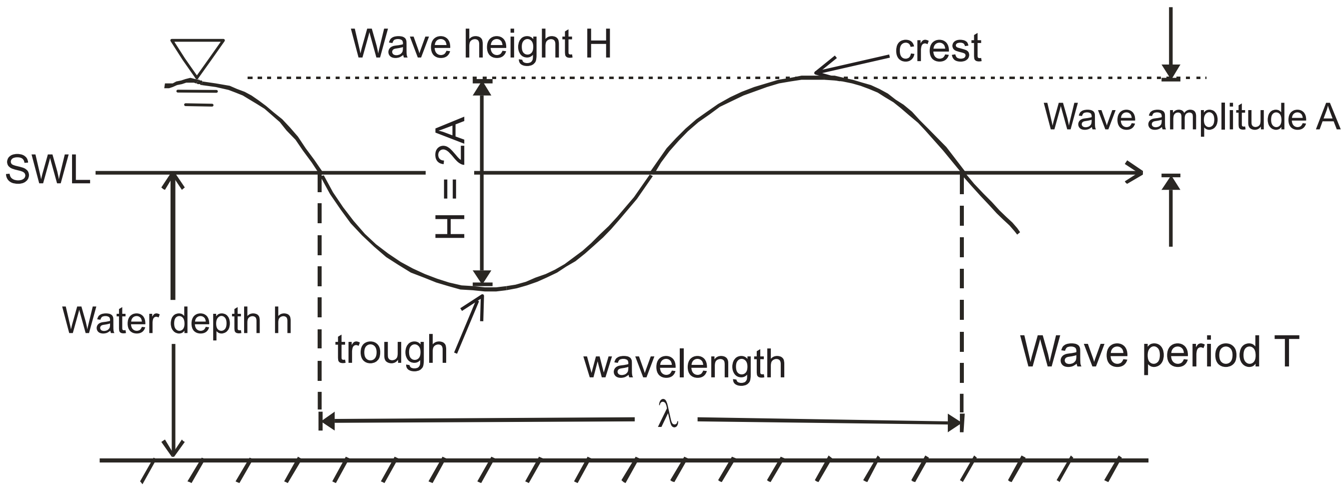
DBC: $p = p_a$ on $y = \eta$ or $F = 0$. Apply Bernoulli equation at $y = \eta$:

$$\frac{\partial \phi}{\partial t} + \underbrace{\frac{1}{2}\left|\nabla\phi\right|^2}_{\text{non-linear term}} + g \underbrace{\eta}_{\text{still unknown}} = 0 \text{ on } y = \eta$$

6.2 Linearized (Airy) Wave Theory

Assume small wave amplitude compared to wavelength, i.e., small free surface slope

$$\frac{A}{\lambda} << 1$$



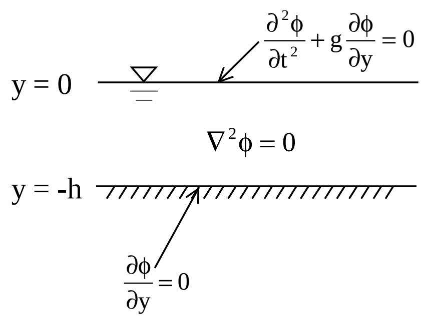
Consequently

$$\frac{\phi}{\lambda^2/T}, \frac{\eta}{\lambda} << 1$$

We keep only linear terms in ϕ, η.

$$\text{For example: } ()|_{y=\eta} = \underbrace{()_{y=0}}_{\text{keep}} + \underbrace{\eta \frac{\partial}{\partial y} ()|_{y=0}}_{\text{discard}} + \ldots \text{ Taylor series}$$

6.2.1 **BVP** In this paragraph we state the Boundary Value Problem for linear (Airy) waves.



	Finite depth $h = const$	**Infinite depth**
GE:	$\nabla^2\phi = 0, \quad -h < y < 0$	$\nabla^2\phi = 0, \quad y < 0$
BKBC:	$\frac{\partial\phi}{\partial y} = 0, \quad y = -h$	$\nabla\phi \to 0, \quad y \to -\infty$
FSKBC:	$\frac{\partial\phi}{\partial y} = \frac{\partial\eta}{\partial t}, \quad y = 0$	$\to \frac{\partial^2\phi}{\partial t^2} + g\frac{\partial\phi}{\partial y} = 0$
FSDBK:	$\frac{\partial\phi}{\partial t} + g\eta = 0, \quad y = 0$	

Introducing the notation {} for infinite depth we can rewrite the BVP:

Constant finite depth h — **{Infinite depth}**

$$\nabla^2\phi = 0, \quad -h < y < 0 \qquad \left\{\nabla^2\phi = 0, \quad y < 0\right\} \tag{1}$$

$$\frac{\partial\phi}{\partial y} = 0, \quad y = -h \qquad \{\nabla\phi \to 0, \quad y \to -\infty\} \tag{2}$$

$$\frac{\partial^2\phi}{\partial t^2} + g\frac{\partial\phi}{\partial y} = 0, \quad y = 0 \qquad \left\{\frac{\partial^2\phi}{\partial t^2} + g\frac{\partial\phi}{\partial y} = 0, \quad y = 0\right\} \tag{3}$$

Given ϕ **calculate:**

$$\eta(x,t) = -\frac{1}{g}\left.\frac{\partial\phi}{\partial t}\right|_{y=0} \qquad \left\{\eta(x,t) = -\frac{1}{g}\left.\frac{\partial\phi}{\partial t}\right|_{y=0}\right\} \tag{4}$$

$$p - p_a = \underbrace{-\rho\frac{\partial\phi}{\partial t}}_{\text{dynamic}} - \underbrace{\rho g y}_{\text{hydrostatic}} \qquad \left\{p - p_a = \underbrace{-\rho\frac{\partial\phi}{\partial t}}_{\text{dynamic}} - \underbrace{\rho g y}_{\text{hydrostatic}}\right\} \tag{5}$$

6.2.2 **Solution** Solution of 2D periodic plane progressive waves, applying separation of variables.

We seek solutions to Equation (1) of the form $e^{i\omega t}$ with respect to time. Using the KBC (2), after some algebra we find ϕ. Upon substitution in Equation (4) we can also obtain η.

$$\phi = \frac{gA}{\omega} \sin(kx - \omega t) \frac{\cosh k(y+h)}{\cosh kh} \qquad \left\{ \phi = \frac{gA}{\omega} \sin(kx - \omega t)\, e^{ky} \right\}$$

$$\eta \underbrace{=}_{\text{using (4)}} A \cos(kx - \omega t) \qquad \left\{ \eta \underbrace{=}_{\text{using(4)}} A \cos(kx - \omega t) \right\}$$

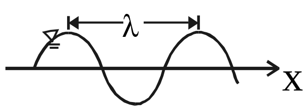
where A is the wave amplitude $A = H/2$.

Exercise Verify that the obtained values for ϕ and η satisfy Equations (1), (2), and (4).

6.2.3 **Review on plane progressive waves**

(a) At $t = 0$ (say), $\eta = A \cos kx \rightarrow$ periodic in x with **wavelength**: $\lambda = 2\pi/k$

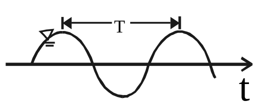
Units of λ : $[L]$



k = wavenumber = 2π/λ [L^{-1}]

(b) At $x = 0$ (say), $\eta = A \cos \omega t \rightarrow$ periodic in t with **period:** $T = 2\pi/\omega$

Units of T : $[T]$



ω = frequency = 2π/T [T^{-1}], e.g. rad/sec

(c) $\eta = A \cos\left[k\left(x - \frac{\omega}{k} t\right)\right]$

Units of $\frac{\omega}{k}$: $\left[\frac{L}{T}\right]$

Following a point with velocity $\frac{\omega}{k}$, i.e., $x_p = \left(\frac{\omega}{k}\right) t + const$, the phase of η does not change, i.e., $\frac{\omega}{k} = \frac{\lambda}{T} \equiv V_p \equiv$ phase velocity.

6.2.4 **Dispersion Relation**

So far, any ω, k combination is allowed. However, recall that we still have not made use of the FSBC Equation (3). Upon substitution of ϕ in Equation (3) we find that the following relation between h, k, and ω must hold:

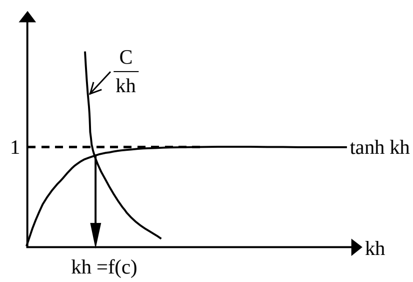
$$\frac{\partial^2\phi}{\partial t^2} + g\frac{\partial\phi}{\partial y} = 0 \quad \underset{\phi=\frac{gA}{\omega} sin(kx-\omega t)f(y)}{\underset{\uparrow}{\longrightarrow}} \quad -\omega^2 \cosh kh + gk \sinh kh = 0 \Rightarrow \omega^2 = gk \tanh kh$$

• This is the **Dispersion Relation**

$$\omega^2 = gk \tanh kh \qquad \left\{\omega^2 = gk\right\} \tag{6}$$

Given h, the Dispersion Relation (6) provides a **unique** relation between ω and k, i.e., $\omega = \omega(k; h)$ or $k = k(\omega; h)$.

• Proof



$$C \equiv \frac{\omega^2 h}{g} \underbrace{=}_{\text{from (6)}} (kh) \tanh (kh)$$

$$\frac{C}{kh} = \tanh kh$$

$\rightarrow$ obtain unique solution for k

• Comments

- *General* As $\omega \uparrow$ then $k \uparrow$, or equivalently as $T \uparrow$ then $\lambda \uparrow$.
- *Phase speed* $V_p \equiv \dfrac{\lambda}{T} = \dfrac{\omega}{k} = \sqrt{\dfrac{g}{k} \tanh kh} \qquad \left\{V_p = \sqrt{\dfrac{g}{k}}\right\}$

  Therefore as $T \uparrow$ or as $\lambda \uparrow$, then $V_p \uparrow$, i.e., longer waves are 'faster' in terms of phase speed.
- *Water depth effect* For waves the same k (or λ), at different water depths, as $h \uparrow$ then $V_p \uparrow$, i.e., for fixed k V_p is fastest in deep water.
- *Frequency dispersion* Observe that $V_p = V_p(k)$ or $V_p(\omega)$. This means that waves of different frequencies, have different phase speeds, i.e., frequency dispersion.

6.2.5 Solutions to the Dispersion Relation : $\omega^2 = gk \tanh kh$

Property of $\tanh kh$:

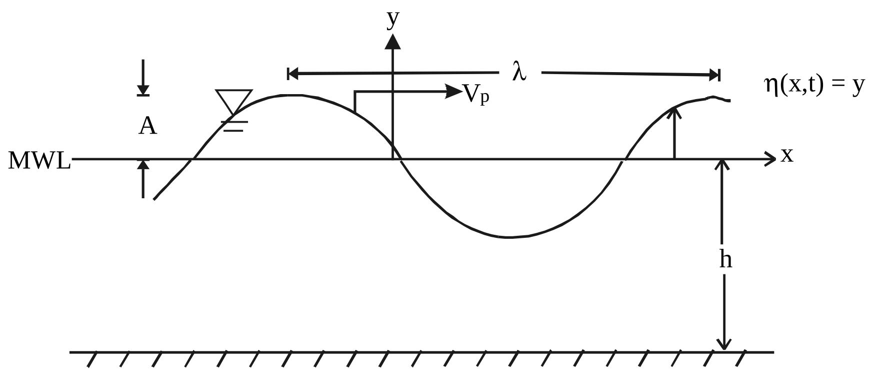
$$\tanh kh = \frac{\sinh kh}{\cosh kh} = \frac{1-e^{-2kh}}{1+e^{-2kh}} \cong \begin{cases} kh \text{ for } kh << 1. \text{ In practice } \overbrace{h < \lambda/20}^{\text{long waves shallow water}} \\ 1 \quad \text{for } kh >\sim 3. \text{ In practice } \underbrace{h > \frac{\lambda}{2}}_{\text{short waves deep water}} \end{cases}$$

Shallow water waves or long waves	**Intermediate depth or wavelength**	**Deep water waves or short waves**
$kh << 1$ $\boxed{\sim h < \lambda/20}$	Need to solve $\omega^2 = gk \tanh kh$ given ω, h for k (given k, h for ω - easy!)	$kh >> 1$ $\boxed{\sim h > \lambda/2}$
$\omega^2 \cong gk \cdot kh \rightarrow \omega = \sqrt{gh}\ k$ $\lambda = \sqrt{gh}\ T$	(a) Use tables or graphs (e.g.JNN fig.6.3) $\omega^2 \quad = gk \tanh kh = gk_\infty$ $\Rightarrow \frac{k_\infty}{k} = \frac{\lambda}{\lambda_\infty} = \frac{V_p}{V_{p\infty}} = \tanh kh$ (b) Use numerical approximation (hand calculator, about 4 decimals) i. Calculate $C = \omega^2 h/g$ ii. If $C > 2$: "deeper" $\Rightarrow$ $kh \approx C(1 + 2e^{-2C} - 12e^{-4C} + \ldots)$ If $C < 2$: "shallower" $\Rightarrow$ $kh \approx \sqrt{C}(1 + 0.169C + 0.031C^2 + \ldots)$	$\omega^2 = gk$ $\lambda = \frac{g}{2\pi}T^2$ $\left(\lambda(\text{in ft.}) \approx 5.12T^2 \text{ (in sec.)}\right)$
No frequency dispersion $V_p = \sqrt{gh}$	**Frequency dispersion** $V_p = \sqrt{\frac{g}{k}\tanh kh}$	**Frequency dispersion** $V_p = \sqrt{\frac{g}{2\pi}\lambda}$

6.3 Characteristics of a Linear Plane Progressive Wave

$$k = \frac{2\pi}{\lambda}$$
$$\omega = \frac{2\pi}{T}$$
$$H = 2A$$

Define $U \equiv \omega A$



Linear Solution:

$$\eta = A\cos(kx - \omega t)\,; \quad \phi = \frac{Ag}{\omega}\frac{\cosh k(y+h)}{\cosh kh}\sin(kx-\omega t)\,, \text{where } \omega^2 = gk\tanh kh$$

6.3.1 **Velocity field**

Velocity on free surface $\vec{v}(x, y=0, t)$	
$u(x,0,t) \equiv U_o = A\omega \frac{1}{\tanh kh}\cos(kx-\omega t)$	$v(x,0,t) \equiv V_o = A\omega\sin(kx-\omega t) = \frac{\partial\eta}{\partial t}$
Velocity field $\vec{v}(x,y,t)$	
$u = \frac{\partial\phi}{\partial x} = \frac{Agk}{\omega}\frac{\cosh k(y+h)}{\cosh kh}\cos(kx-\omega t) = \underbrace{A\omega}_{U}\frac{\cosh k(y+h)}{\sinh kh}\cos(kx-\omega t) \Rightarrow$ $\frac{u}{U_o} = \frac{\cosh k(y+h)}{\cosh kh}\begin{cases}\sim e^{ky} & \text{deep water}\\ \sim 1 & \text{shallow water}\end{cases}$	$v = \frac{\partial\phi}{\partial y} = \frac{Agk}{\omega}\frac{\sinh k(y+h)}{\cosh kh}\sin(kx-\omega t) = \underbrace{A\omega}_{U}\frac{\sinh k(y+h)}{\sinh kh}\sin(kx-\omega t) \Rightarrow$ $\frac{v}{V_o} = \frac{\sinh k(y+h)}{\sinh kh}\begin{cases}\sim e^{ky} & \text{deep water}\\ \sim 1+\frac{y}{h} & \text{shallow water}\end{cases}$
• u is in phase with η	• v is out of phase with η

<table>
<tr><th colspan="3">Velocity field $\vec{v}(x, y)$</th></tr>
<tr><th>Shallow water</th><th>Intermediate water</th><th>Deep water</th></tr>
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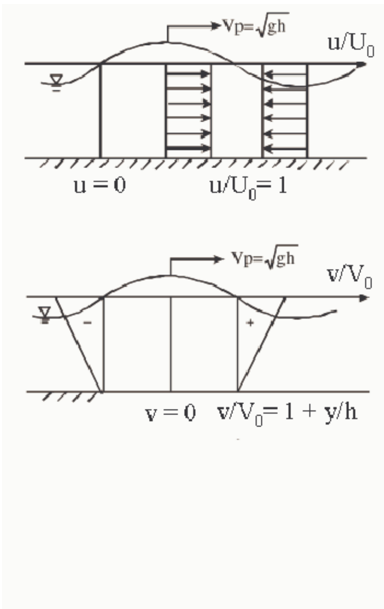
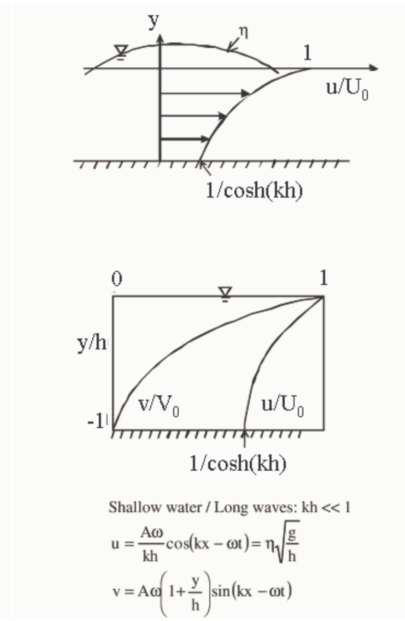
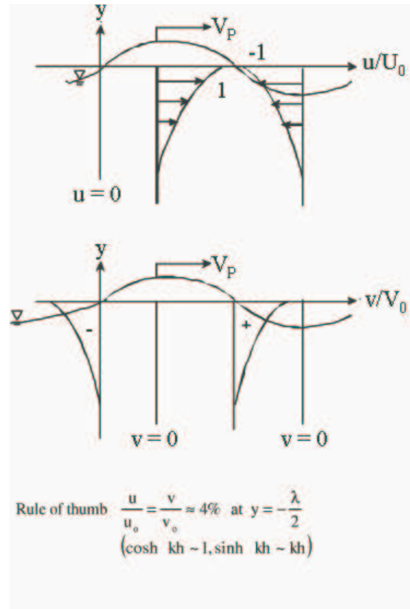
<td></td>
<td></td>
<td></td>
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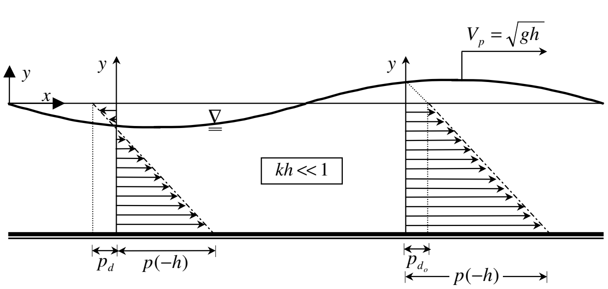
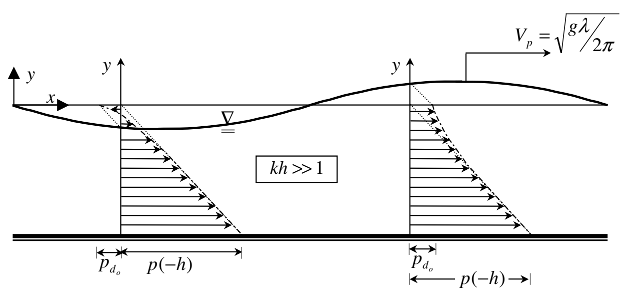
6.3.2 Pressure field

- Total pressure $p = p_d - \rho g y$.
- Dynamic pressure $p_d = -\rho \frac{\partial \phi}{\partial t}$.
- Dynamic pressure on free surface $p_d(x, y = 0, t) \equiv p_{d_o}$

Pressure field		
Shallow water	**Intermediate water**	**Deep water**
$p_d = \rho g \eta$	$p_d = \rho g A \frac{\cosh k\,(y+h)}{\cosh kh} \cos\,(kx - \omega t)$ $= \rho g \frac{\cosh k\,(y+h)}{\cosh kh} \eta$	$p_d = \rho g e^{ky} \eta$
$\frac{p_d}{p_{d_o}}$ same picture as $\frac{u}{U_o}$		
$\frac{p_d(-h)}{p_{d_o}} = 1$ (no decay)	$\frac{p_d(-h)}{p_{d_o}} = \frac{1}{\cosh kh}$	$\frac{p_d\,(-h)}{p_{d_o}} = e^{-ky}$
$p = \underbrace{\rho g(\eta - y)}_{\text{"hydrostatic" approximation}}$		$p = \rho g \left(\eta e^{ky} - y\right)$



Pressure field in shallow water



Pressure field in deep water

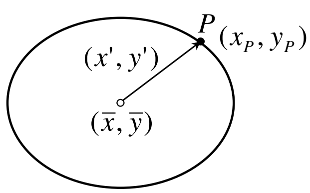
6.3.3 Particle Orbits ('Lagrangian' concept)

Let $x_p(t), y_p(t)$ denote the position of particle P at time t.

Let $(\bar{x}; \bar{y})$ denote the mean position of particle P.

The position P can be rewritten as $x_p(t) = \bar{x} + x'(t)$, $y_p(t) = \bar{y} + y'(t)$, where $(x'(t), y'(t))$ denotes the departure of P from the mean position.

In the same manner let $\vec{v} \equiv \vec{v}(\bar{x}, \bar{y}, t)$ denote the velocity at the mean position and $\vec{v}_p \equiv \vec{v}(x_p, y_p, t)$ denote the velocity at P.



$$\vec{v}_p = \vec{v}(\bar{x} + x', \bar{y} + y', t) \underset{\text{TSE}}{\Longrightarrow}$$

$$\vec{v}_p = \vec{v}\,(\bar{x}, \bar{y}, t) + \underbrace{\frac{\partial \vec{v}}{\partial x}\,(\bar{x}, \bar{y}, t)\, x' + \frac{\partial \vec{v}}{\partial y}\,(\bar{x}, \bar{y}, t)\, y' + \ldots}_{\text{ignore - linear theory}} \Rightarrow$$

$$\vec{v}_p \cong \vec{v}$$

To estimate the position of P, we need to evaluate $(x'(t), y'(t))$:

$$\begin{aligned} x' &= \int dt\; u\,(\bar{x}, \bar{y}, t) = \int dt\; \omega A \frac{\cosh k\,(\bar{y} + h)}{\sinh kh} \cos\,(k\bar{x} - \omega t) \Rightarrow \\ &= -A \frac{\cosh k\,(\bar{y} + h)}{\sinh kh} \sin\,(k\bar{x} - \omega t) \end{aligned}$$

$$\begin{aligned} y' &= \int dt\; v\,(\bar{x}, \bar{y}, t) = \int dt\; \omega A \frac{\sinh k\,(\bar{y} + h)}{\sinh kh} \sin\,(k\bar{x} - \omega t) \Rightarrow \\ &= A \frac{\sinh k\,(\bar{y} + h)}{\sinh kh} \cos\,(k\bar{x} - \omega t) \end{aligned}$$

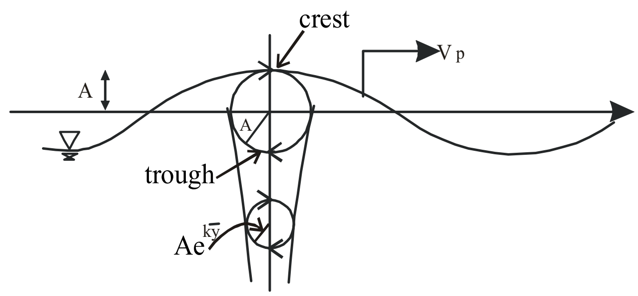
Check: On $\bar{y} = 0, y' = A\cos\,(k\bar{x} - \omega t) = \eta$, i.e., the vertical motion of a free surface particle (in linear theory) coincides with the vertical free surface motion.

It can be shown that the particle motion satisfies

$$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1 \Leftrightarrow \frac{(x_p - \bar{x})^2}{a^2} + \frac{(y_p - \bar{y})^2}{b^2} = 1$$

where $a = A\dfrac{\cosh k\,(\bar{y} + h)}{\sinh kh}$ and $b = A\dfrac{\sinh k\,(\bar{y} + h)}{\sinh kh}$, i.e., the particle orbits form closed ellipses with horizontal and vertical axes a and b.

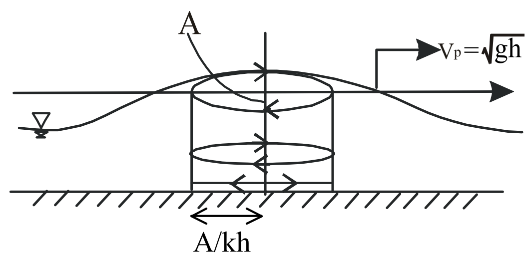
(a) deep water $kh >> 1$: $a = b = Ae^{k\bar{y}}$
circular orbits with radii $Ae^{k\bar{y}}$ decreasing exponentially with depth



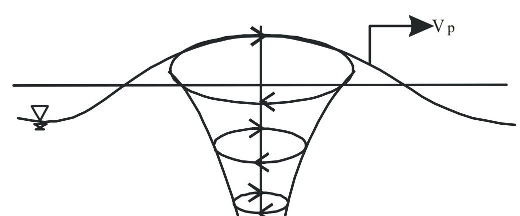
(b) shallow water $kh << 1$:

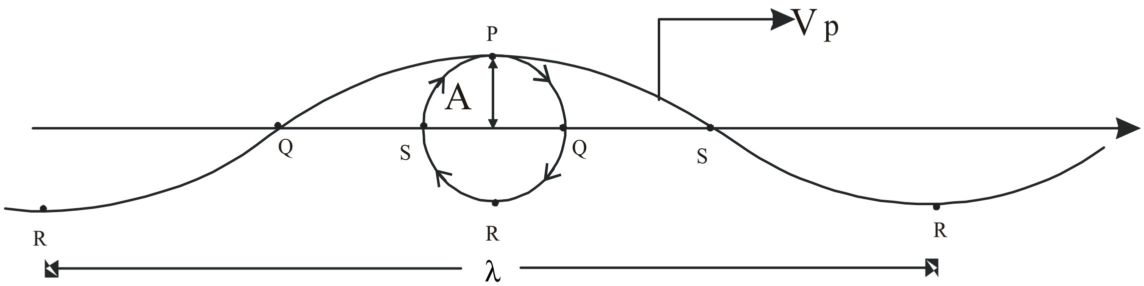
$a = \frac{A}{kh} =$ const. ; $b = A(1+\frac{y}{h})$

decreases linearly with depth



(c) Intermediate depth





6.3.4 **Summary of Plane Progressive Wave Characteristics**

$f(y)$	Deep water/ short waves $kh > \pi$ (say)	Shallow water/ long waves $kh << 1$
$\frac{\cosh k(y+h)}{\cosh kh} = f_1(y) \sim$ e.g. p_d	e^{ky}	1
$\frac{\cosh k(y+h)}{\sinh kh} = f_2(y) \sim$ e.g. u, a	e^{ky}	$\frac{1}{kh}$
$\frac{\sinh k(y+h)}{\sinh kh} = f_3(y) \sim$ e.g. v, b	e^{ky}	$1 + \frac{y}{h}$

$C(x) = \cos(kx - \omega t)$ (in phase with η)	$S(x) = \sin(kx - \omega t)$ (out of phase with η)
$\frac{\eta}{A} = C(x)$	
$\frac{u}{A\omega} = C(x)\, f_2(y)$	$\frac{v}{A\omega} = S(x)\, f_3(y)$
$\frac{p_d}{\rho g A} = C(x)\, f_1(y)$	
$\frac{y'}{A} = C(x)\, f_3(y)$	$\frac{x'}{A} = -S(x)\, f_2(y)$
$\frac{a}{A} = f_2(y)$	$\frac{b}{A} = f_3(y)$