Outline:

- Indicial notation
- Vector calculus
- Taylor Series
- Divergence Theorem
- Leibniz Integral Rule
- ODEs & PDEs
- (Coordinate Systems)

Einstein's Indicial Notation

Index notation

- Free index : appears only <u>once</u> in each term of an expression
- Dummy index : appears <u>twice</u> in each term of an expression

Scalar vs. Vector vs. Tensor

- Scalar : No index (1 term)
  $\boxed{\alpha}$ Only a magnitude
- Vector : One free index (3 terms in 3D)
  $\boxed{a_i}$ Magnitude & Direction
- Tensor : Two free indices (9 terms in 3D)
  $\boxed{b_{ij}}$

Examples

Scalars	Vectors	Tensors
Temperature	Position	Stress
Pressure	Velocity	Deformation
Density	Acceleration	Momentum flux
Energy	Force	
	Momentum	

- Einstein's Indicial Notation

- Range convention: Whenever a subscript appears only once in a term, the subscript takes all the values

$x_i \rightarrow x_1, x_2, x_3$

free index i → the name of a free index MUST remain the same throughout the calculation

$v_i \neq v_j$

$$x_{ij} \rightarrow \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}$$

- Summation convention: repeated indices are summed over and no index can be repeated more than twice

$$a_i b_i = \sum_{i=1}^{3} a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3$$

$$a_{ij} b_i c_j = \sum_{i=1}^{3} \sum_{j=1}^{3} a_{ij} b_i c_j = \sum_{i=1}^{3} b_i \sum_{j=1}^{3} a_{ij} c_j$$

$$= \sum_{i=1}^{3} b_i \left(a_{i1} c_1 + a_{i2} c_2 + a_{i3} c_3 \right)$$

$$= b_1 (a_{11} c_1 + a_{12} c_2 + a_{13} c_3) +$$
$$+ \ b_2 (a_{21} c_1 + a_{22} c_2 + a_{23} c_3) +$$
$$b_3 (a_{31} c_1 + a_{32} c_2 + a_{33} c_3)$$

- Comma convention: $a_{i,i}$ indicates partial differentiation with respect to each coordinate

e.g. $a_{i,i} = \dfrac{\partial a_i}{\partial x_i}$

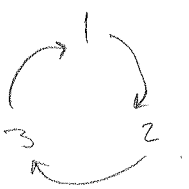
- Kronecker delta: δ_{ij}

$$\delta_{ij} = \begin{cases} 0 \; , & i \neq j \\ 1 \; , & i = j \end{cases} \qquad \delta_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

e.g. $\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 1 + 1 + 1 = 3$

$\delta_{ij} a_j = a_i$

- Levi-Civita permutation symbol: ε_{ijk}
  - used for cross products



$$\varepsilon_{ijk} = \begin{cases} 1 \; , & \text{clockwise} \\ 0 \; , & \text{otherwise (e.g. repeated indices)} \\ -1 \; , & \text{anti-clockwise} \end{cases}$$

$$\boxed{\vec{a} \times \vec{b} = \varepsilon_{ijk} a_j b_k}$$

- very useful rule: $\varepsilon_{ijk} \varepsilon_{imn} = \delta_{jm} \delta_{kn} - \delta_{jn} \delta_{km}$

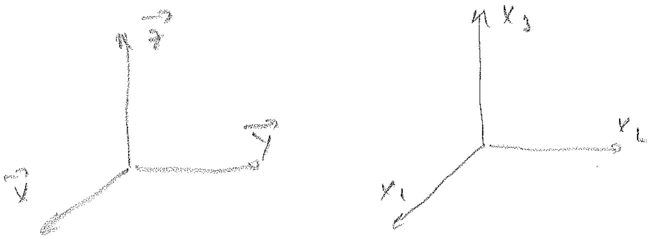
e.g. $\operatorname{curl} \vec{v} = \vec{\nabla} \times \vec{v} = \varepsilon_{ijk} \dfrac{\partial v_k}{\partial x_j}$

- Vector Calculus

  - Reference: H.M. Schey: "Div, Grad, Curl and all that"

- The Del / Nabla operator $\vec{\nabla}$ ($\partial/\partial x_i$ in indicial notation)

  - in 3D space



$$\boxed{\vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}} \quad \text{or} \quad \frac{\partial}{\partial x_i} \text{ in indicial notation}$$

- Four important uses of Del operator

1) Gradient - gives the magnitude and the direction of the maximum increase of a field

   - has the effect of increasing the order of the field by 1 (e.g. scalar field → vector field)

   - e.g. for a scalar field $f(x, y, z)$

$$\text{grad } f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k} = \vec{\nabla} f \quad \leftarrow \text{vector field}$$

or

$$\text{grad } f = \frac{\partial f}{\partial x_i} \quad \leftarrow (1 \text{ free index} = \text{vector})$$

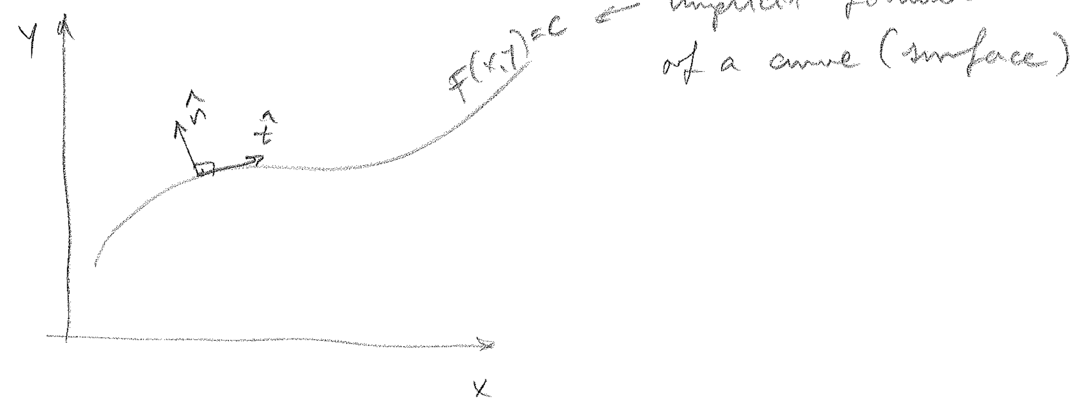
(- the gradient of a vector field is a tensor field (think of the above example operating component-wise)

$$\text{grad } \vec{v} = \frac{\partial v_i}{\partial x_j} \quad \leftarrow (2 \text{ free indices} = \text{tensor})$$
)

- with the knowledge of gradient, we can calculate the rate of change in any direction $\vec{s}$

$$\frac{\partial f}{\partial s} = \vec{\nabla} f \cdot \vec{s} = \frac{\partial f}{\partial x_i} s_i \qquad \left\| \frac{\partial f}{\partial s} \right\| \leq \| \vec{\nabla} f \|$$

- Normal to a curve or a surface



$$\hat{n} = \frac{\vec{\nabla} F}{\| \nabla F \|}$$ ← definition of the normal unit vector $\hat{n}$

- tangent vector $\hat{t}$ then has to satisfy the conditions

$\hat{n} \cdot \hat{t} = 0 \quad \rightarrow \quad \hat{n} \perp \hat{t}$

$\| \hat{t} \| = 1 \quad \rightarrow$ unit vector

2) <u>Divergence</u> – measures the net "expansion" of a field at a point

– has the effect of <u>decreasing</u> the order of the field by 1 (has to operate on at least vector fields)

– for a vector field $\vec{v}(x,y,z)$

$$\operatorname{div}\vec{v} = \vec{\nabla}\cdot\vec{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = \frac{\partial v_i}{\partial x_i}$$

– can also be defined as

$$\vec{\nabla}\cdot\vec{v} = \lim_{V\to 0}\frac{\oiint_S \vec{v}\cdot\vec{n}\,dA}{V} \leftarrow \text{flux across a surface}$$

$\therefore$ Divergence is the rate at which $\vec{v}$ leaves a given volume of space

3) <u>Curl</u> (rot in Europe) – describes the rotation or spin of a vector field

– for a vector field $\vec{v}(x,y,z)$

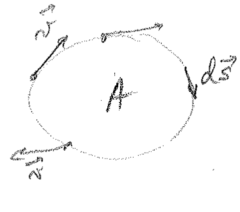
$$\operatorname{curl}\vec{v} = \vec{\nabla}\times\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ v_x & v_y & v_z \end{vmatrix} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{i} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{j} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{k}$$

or

$$\operatorname{curl}\vec{v} = \varepsilon_{ijk}\frac{\partial v_k}{\partial x_j}$$

– curl can also be defined as

$$(\vec{\nabla}\times\vec{v})\cdot\vec{n} = \lim_{A\to 0}\frac{\oint \vec{v}\,d\vec{s}}{A}$$



4) Laplacian (second derivative)

- Laplacian operator is a scalar operator that can be applied to scalar or vector fields

$$\Delta \equiv \vec{\nabla}^2 = \vec{\nabla}\cdot\vec{\nabla} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial x_i \partial x_i}$$

Laplacian

- scalar field $\phi(x,y,z)$ (e.g. potential flow later in course)

$$\Delta\phi = \nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

- vector field $\vec{v}(x,y,z)$ (e.g. in Navier-Stokes eq.)

$$\Delta\vec{v} = \nabla^2\vec{v} = \frac{\partial^2\vec{v}}{\partial x^2} + \frac{\partial^2\vec{v}}{\partial y^2} + \frac{\partial^2\vec{v}}{\partial z^2} = \frac{\partial^2 v_i}{\partial x_j \partial x_j}$$

- Five possible second derivatives

div (grad f) $= \vec{\nabla}\cdot(\vec{\nabla}f) = \Delta f$ - Laplacian

curl (grad f) $= \vec{\nabla}\times(\vec{\nabla}f) \equiv \vec{0}$ (identically)

grad (div $\vec{v}$) $= \vec{\nabla}(\vec{\nabla}\cdot\vec{v})$

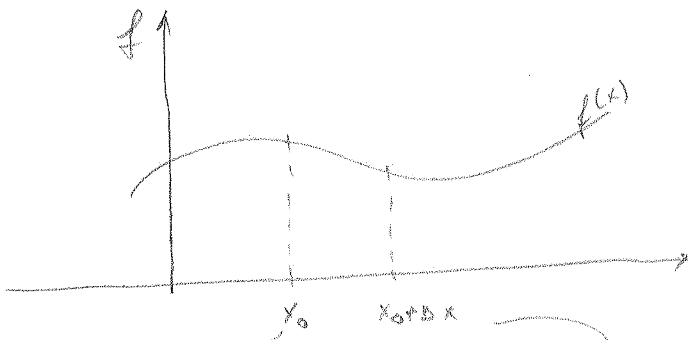
div (curl $\vec{v}$) $= \vec{\nabla}\cdot(\vec{\nabla}\times\vec{v}) \equiv \vec{0}$ (identically)

curl (curl $\vec{v}$) $= \vec{\nabla}\times(\vec{\nabla}\times\vec{v})$

— Taylor series expansion

- To calculate the value of a function in the neighborhood of point x_o

1D: $$f(x_o+\Delta x) = f(x_o) + \Delta x \left.\frac{df}{dx}\right|_{x=x_o} + \frac{(\Delta x)^2}{2}\left.\frac{d^2 f}{dx^2}\right|_{x=x_o} + O((\Delta x)^3)$$



IF (everything is known @ $x=x_o$ (i.e. value, slope, ...)) $\Rightarrow$ (what is the value of the function @ $x = x_o+\Delta x$)

2D: $$f(x_o+\Delta x, y_o+\Delta y) = f(x_o,y_o) + \left[\Delta x\frac{\partial f}{\partial x} + \Delta y\frac{\partial f}{\partial y}\right]\Bigg|_{(x_o,y_o)} + O((\Delta x)^2, \Delta x\Delta y, (\Delta y)^2)$$

or, written in vector form

$$\vec{x} = x\hat{i} + y\hat{j} \quad ; \quad \Delta\vec{x} = \Delta x\,\hat{i} + \Delta y\,\hat{j}$$

$$\Rightarrow f(\vec{x}_o+\Delta\vec{x}) = f(\vec{x}_o) + \Delta\vec{x}\cdot\nabla f\Big|_{\vec{x}=\vec{x}_o} + O((\Delta x)^2, \Delta x\Delta y, (\Delta y)^2)$$

- if f is a vector function $\vec{f}(\vec{x}) = f_1(\vec{x})\hat{i} + f_2(\vec{x})\hat{j} + f_3(\vec{x})\hat{k}$

$$\because f_i(\vec{x}_o+\Delta\vec{x}) = f_i(\vec{x}_o) + \Delta\vec{x}\cdot\nabla f_i\Big|_{\vec{x}_o} + \cdots, \quad \text{for } i=1,2,3$$

$$\therefore \vec{f}(\vec{x}_o+\Delta\vec{x}) = \vec{f}(\vec{x}_o) + \Delta\vec{x}\cdot\nabla\vec{f}\Big|_{\vec{x}_o} + \cdots$$

– Divergence (Gauss) Theorem

– Used to change volume integrals to surface integrals (or vice versa)

$$\boxed{\oiint_S \vec{v}\cdot\vec{n}\,dS = \iiint_V \nabla\cdot\vec{v}\,dV}$$

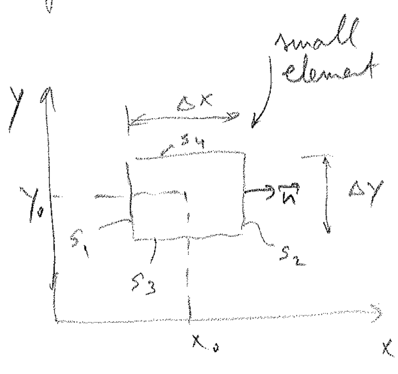
or

$$\boxed{\oiint_S v_i n_i\,dS = \iiint_V \frac{\partial v_i}{\partial x_i}\,dV}$$

} for vector fields

$$\oiint \sigma_{ij} n_j\,dS = \iiint_V \frac{\partial \sigma_{ij}}{\partial x_j}\,dV$$ – for a tensor field

– Proof in 2D – vector field $\vec{v}(x,y)$



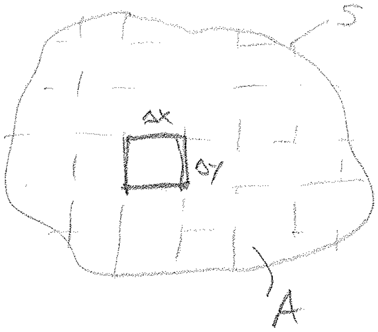
$$-\iint_{S_1} v_x(x,y)\,dy \approx -\Delta y\cdot\left[v_x(x_0,y_0) - \frac{\Delta x}{2}\frac{\partial v_x}{\partial x} + O(\Delta x^2,\Delta y^2)\right]$$

$$\iint_{S_2} v_x(x,y)\,dy \approx \Delta y\left[v_x(x_0,y_0) + \frac{\Delta x}{2}\frac{\partial v_x}{\partial x} + O(\Delta x^2,\Delta y^2)\right]$$

$$-\iint_{S_3} v_y(x,y)\,dx \approx -\Delta x\left[v_y(x_0,y_0) - \frac{\Delta y}{2}\frac{\partial v_y}{\partial y} + O(\Delta x^2,\Delta y^2)\right]$$

$$\therefore \iint_{\partial S} \vec{v}\cdot\vec{n}\,ds = -\iint_{S_1} v_x\,dy + \iint_{S_2} v_x\,dy - \iint_{S_3} v_y\,dx + \iint_{S_4} v_y\,dx = \Delta x\Delta y\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}\right)$$

- Integrating over all elements forming an area (or a volume), fluxes over neighboring elements cancel out leaving only the external surface



in the limit $\Delta x, \Delta y \to 0$

$$\boxed{\iint_S \vec{v}\cdot\vec{n}\, ds = \iiint_A \operatorname{div} \vec{v}\, dx\, dy}$$

- <u>Stokes theorem</u>

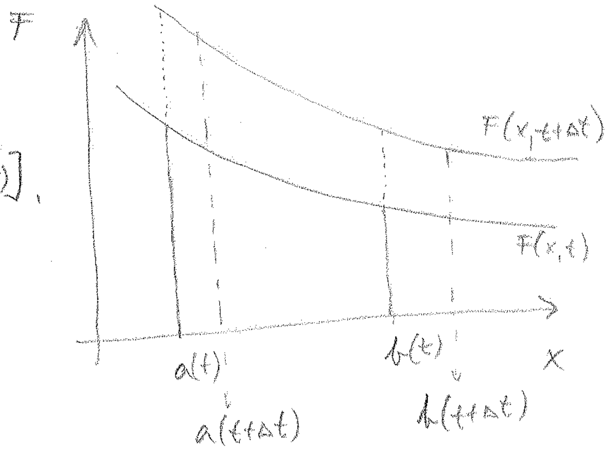
$$\oint \vec{v}\cdot d\vec{x} = \iint_S (\vec{\nabla}\times\vec{v})\cdot\vec{n}\, ds$$

- Leibniz integral rule

- For a function $I(t) = \int_{a(t)}^{b(t)} F(x,t)\,dx$ we have

$$\boxed{\frac{dI}{dt} = \int_{a(t)}^{b(t)} \frac{\partial F}{\partial t}\,dx + \frac{db}{dt}F(b,t) - \frac{da}{dt}F(a,t)}$$

- proof:



At time t, $I(t)$ is the area under $F(x,t)$ for $x \in [a(t), b(t)]$.

At time $t+\Delta t$, $I(t+\Delta t)$ is the area under $F(x, t+\Delta t)$ for $x \in [a(t+\Delta t), b(t+\Delta t)]$.

To calculate $I(t+\Delta t)$, we first consider the integral with a & b fixed, then account for the modification due to the change of a & b.

$$I(t+\Delta t) = \int_{a(t+\Delta t)}^{b(t+\Delta t)} F(x, t+\Delta t)\,dx$$

$$= \int_{a(t)}^{b(t)} F(x, t+\Delta t)\,dx + \int_{b(t)}^{b(t+\Delta t)} F(x, t+\Delta t)\,dx - \int_{a(t)}^{a(t+\Delta t)} F(x, t+\Delta t)\,dx$$

- Apply Taylor expansion and keep only the terms up to $O(\Delta t)$

$$F(x, t+\Delta t) = F(x,t) + \Delta t \frac{\partial F}{\partial t} + O((\Delta t)^2)$$

$$a(t+\Delta t) = a(t) + \Delta t \frac{da}{dt} + O((\Delta t)^2)$$

$$b(t+\Delta t) = b(t) + \Delta t \frac{db}{dt} + O((\Delta t)^2)$$

$$\Rightarrow I(t+\Delta t) = \int_{a(t)}^{b(t)} \left(F(x,t) + \Delta t \frac{\partial F}{\partial t} \right) dx + F(b,t)\left[b(t+\Delta t) - b(t) \right] -$$

$$- F(a,t)\left[a(t+\Delta t) - a(t) \right] + O((\Delta t)^2)$$

$$\therefore \frac{dI}{dt} = \lim_{\Delta t \to 0} \frac{I(t+\Delta t) - I(t)}{\Delta t} = \int_{a(t)}^{b(t)} \frac{\partial F}{\partial t} dx + F(b,t)\frac{db}{dt} - F(a,t)\frac{da}{dt}$$

- Generalization:

<u>2D</u> $\quad I(t) = \int_{S(t)} F(\vec{x},t)\, dA \Rightarrow \frac{dI}{dt} = \int_{S(t)} \frac{\partial F}{\partial t} dA + \oint_{C(t)} F \cdot U_n \, dl$

- $C(t)$ is the boundary of the area $S(t)$
$U_n(t)$ is the normal velocity of the moving boundary $C(t)$

$\underline{3D}$: $I(t) = \int_{V(t)} F(\vec{x},t)\, dV \Rightarrow \frac{dI}{dt} = \int_{V(t)} \frac{\partial F}{\partial t} dV + \oint\!\!\oint_{S(t)} F \cdot U_n \, dA$

$S(t)$ is the boundary surface of volume $V(t)$

$U_n(t)$ is the normal velocity of the moving boundary $S(t)$

- When the integral is over a material volume, U_n is obtained from the local velocity on the boundary as $U_n = \vec{v} \cdot \vec{n}$ ($\vec{n}$ is the normal vector out of the boundary surface)

— ODE's

- In this class, you will be expected to know how to solve:

a) $\frac{d^{(n)}y}{dx^n} = \alpha$ — solved by integrating n times and applying n boundary conditions

b) Second order ODE with constant coefficients

$$\ddot{y} + a\dot{y} + by = \{0, F(t)\}$$

- For homogenous solution, assume $y_H = Ae^{\alpha t}$ and substitute

$$\alpha^2 + a\alpha + b = 0 \Rightarrow \text{solve for } \alpha$$

$$y_H = A_1 e^{\alpha_1 t} + A_2 e^{\alpha_2 t}$$

- Particular solution $y_P(t)$ depends on the form of $F(t)$; assume a similar form and substitute

- Finally: $\boxed{y(t) = y_H(t) + y_P(t)}$

c) Problems solvable by separation of variables

$$\frac{y'}{x} = 4\cos x \Rightarrow y = \int 4x\cos x\, dx$$

PDE's

In this class, PDE's will be solved by:

- Simplifying to an ODE

- Separation of variables.

i.e Given $\phi_{,x} = 2x$ $\qquad \phi_{,y} = 2y$

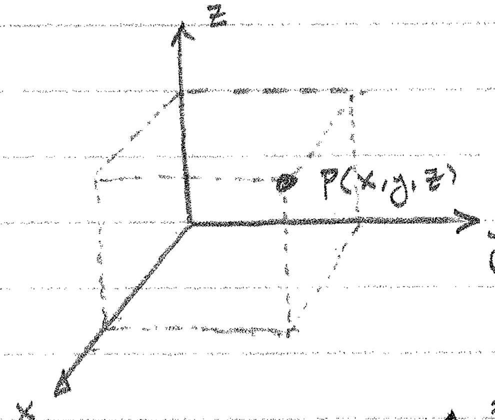
$$\int \phi_{,x}\, dx = x^2 + h(y) \qquad \int \phi_{,y} = y^2 + g(x)$$

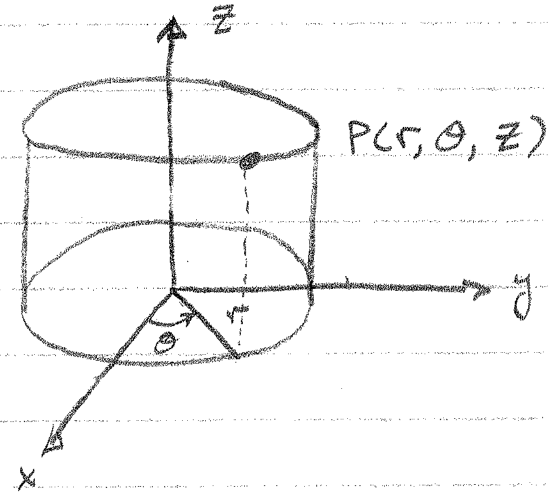
$$\phi = x^2 + h(y) = y^2 + g(x)$$

$$\phi = x^2 + y^2 + C$$

- If it is complicated enough, we will show you how to solve or give you solution.

Cylindrical Coordinates:



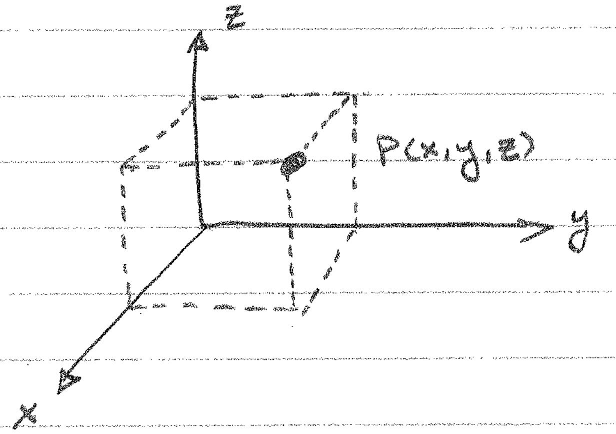


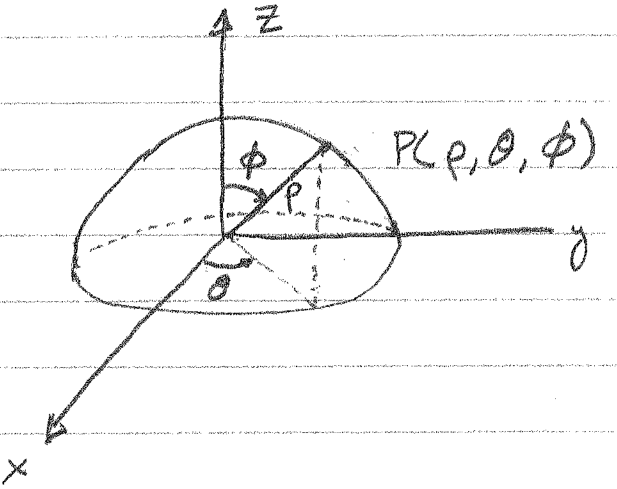
$x = r\cos(\theta)$ $\qquad$ $y = r\sin(\theta)$ $\qquad$ $z = z$

$r = \sqrt{x^2+y^2}$ $\qquad$ $\tan(\theta) = y/x$ $\qquad$ $z = z$

$$\iiint_G f(x,y,z)\,dV = \iiint_{\substack{\text{Appropriate}\\ \text{Limits}}} f(r\cos(\theta), r\sin(\theta), z)\,dz\,r\,dr\,d\theta$$

Spherical Coordinates:





$x = \rho \sin(\phi)\cos(\theta) \qquad y = \rho \sin(\phi)\sin(\theta) \qquad z = \rho\cos(\phi)$

$\rho = \sqrt{x^2+y^2+z^2} \qquad \tan(\theta) = \dfrac{y}{x} \qquad \cos(\phi) = \dfrac{z}{\sqrt{x^2+y^2+z^2}}$

$$\iiint_G f(x,y,z)\, d\forall = \iiint f(\rho\sin(\phi)\cos(\theta), \rho\sin(\phi)\sin(\theta), \rho\cos(\phi))\,\rho^2\sin(\phi)\, d\rho\, d\phi\, d\theta$$

Appropriate
Limits