Methods of analysis in fluid dynamics
- Control volume and Navier-Stokes equations –

&

Dimensional Analysis

Outline:

• Introduction
  o Integral properties, material volume, control volume

• Control Volume
  o Conservation Laws (in general)
  o Mass conservation
  o Momentum conservation
  o Guidelines for control volume analysis
  o Example of control volume analysis: drag force on a body

• Navier-Stokes Equations
  o Flow between two parallel plates

• Dimensional Analysis
  o Recap of Dimensional Analysis and Similitude
  o Buckingham’s \( \pi \)-theory example
  o Characteristic scales and normalization
Introduction

Results that we normally want/need/would be satisfied with:

- "point-wise" properties of the flow – defined at every point in space, e.g. velocity, temperature, stress fields (such results are obtained by solving differential equations)

- integral properties – give general information about the flow, e.g. total flux through a pipe, drag on a wing… (can be obtained by clever bookkeeping => conservation laws)

We can define two objects which help us observe changes of volume of fluid:

- Material Volume – discussed in lecture notes
- Control Volume
- Material volume (π.V) & Material surface (π.S)

- A volume which consists of fluid particles under consideration

- Material surface moves with the local flow velocity \( \vec{u} \)

- No fluid particle can join or leave π.V.

- Lagrangian concept

- Control Volume (C.V.) & Control Surface (C.S)

- Arbitrary volume (always known, prescribed) location

- In the general case, control surface can move with any velocity

- Usually fixed boundary for analysis convenience

- Fluid particle can join or leave C.V.

- Eulerian concept when C.V. is fixed

- Easily defined in a laboratory
Conservation law for any quantity (using control volume)

- Generally (book-keeping)

RATE OF TOTAL CHANGE OF A QUANTITY IN A GIVEN VOLUME = TOTAL (INFLUX - OUTFLUX) OF THE QUANTITY IN THE VOLUME + SOURCE/sink OF THE QUANTITY (BOTH IN THE VOLUME AND ON THE BOUNDARY)

- Let's observe the change of a quantity $\mathbf{X}$ (can be scalar, vector or tensor field).

- Influx/outflux - fluid particles carry the quantity in/out of the volume.

- During time $\Delta t$, the amount of property carried out by fluid particle is:

$$\text{outflux} = \mathbf{X} \cdot \left( \frac{\Delta \mathbf{F}_{\text{in}} - \Delta \mathbf{F}_{\text{out}}}{\Delta t} \right)$$

- In the limit $\Delta t \to 0$, resultant velocity normal pointing out:

$$\Rightarrow \text{outflux} = \mathbf{X} \cdot (\mathbf{v}_{\text{in}} - \mathbf{v}_{\text{out}}) \cdot \mathbf{n} \, dS$$

- Total (net) outflux:

$$\int_{\partial S(e)} \mathbf{X} \cdot (\mathbf{v}_{\text{in}} - \mathbf{v}_{\text{out}}) \cdot \mathbf{n} \, dS$$

$$(\mathbf{v}_{\text{in}} - \mathbf{v}_{\text{out}}) \cdot \mathbf{n}$$

normal velocity out of the C.V.
- Hence, the general conservation law in integral form follows:

\[
\iiint \frac{\partial \chi}{\partial t} \, dV = - \oint \chi \cdot (\nabla \cdot \mathbf{q}) \cdot \mathbf{n} \, d\mathbf{S} + \text{source}
\]

**C.V.**

**C.S.**

**TOTAL RATE OF CHANGE**
**NET INFLUX/OUTFLUX**
**PROBLEM DEPENDENT**

We will apply this law to a couple of examples. For simplicity, we will choose \( \overline{\mathbf{v}}_{e,s} = 0 \)

- Mass conservation

\( \chi = \rho \) - density (mass per volume)

source = 0 (mass cannot be created nor destroyed)

**Integral form of mass conservation equation**

\[
\iiint \frac{\partial \rho}{\partial t} \, dV + \oint \rho \cdot (\mathbf{v} \cdot \mathbf{n}) \, d\mathbf{S} = 0
\]

**Integral form of mass conservation equation**

- We can get the differential form of mass conservation if we use Gauss' Theorem,

\[
\oint \chi \cdot (\nabla \cdot \mathbf{q}) \, d\mathbf{S} = \iiint \nabla \cdot (\chi \mathbf{q}) \, dV
\]

**Gauss (Divergence) Theorem**

\[
\Rightarrow \iiint \frac{\partial \rho}{\partial t} \, dV + \iiint \nabla \cdot (\rho \mathbf{v}) \, dV = 0
\]

Since C.V. is chosen arbitrarily,

\[
\Rightarrow \frac{2\rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0
\]

**Differential form of mass conservation**
Momentum conservation

\[ \mathbf{\tau} = \rho \mathbf{v} \]  - momentum (per volume)

source = (force in the source of momentum since \( \rho = \frac{\partial m}{\partial t} \))

\[ = \iiint F_0 \, dV + \iint \mathbf{T}_{ij} \, n_j \, dS \]

C.V. \qquad C.S.  
body forces \qquad surface forces

\[ \text{Convection Law (3D)} \Rightarrow \boxed{\int \frac{\partial}{\partial t} (\rho \mathbf{v}_i) \, dV = - \iiint \mathbf{T}_{ij} \, n_j \, dS + \iiint F_0 \, dV} \]

C.V. \quad C.S. \quad C.V. \quad C.S.

To get differential form of conservation law, use divergence theorem (here in index notation)

\[ \Rightarrow \iiint \frac{\partial}{\partial t} (\rho \mathbf{v}_i) \, dV + \iiint \frac{\partial}{\partial x_j} (\rho \mathbf{v}_j \mathbf{v}_i) \, dV = \iiint F_0 \, dV + \iiint \frac{\partial \mathbf{T}_{ij}}{\partial x_j} \, dV \]

C.V. \quad C.V. \quad C.V. \quad C.V.

- Since C.V. is arbitrary

\[ \Rightarrow \frac{\partial}{\partial t} (\rho \mathbf{v}_i) + \rho \frac{\partial \mathbf{v}_i}{\partial x_j} \frac{\partial}{\partial x_j} = F_{0i} + \frac{\partial T_{ij}}{\partial x_j} \]

\[ = \mathbf{F}_0 \mathbf{v}_i + \frac{\partial}{\partial x_j} \]

\[ \Rightarrow \mathbf{v}_j \frac{\partial}{\partial t} \mathbf{v}_i + \mathbf{v}_i \frac{\partial}{\partial x_j} + \mathbf{v}_j \frac{\partial}{\partial x_j} \mathbf{v}_i + \mathbf{v}_i \frac{\partial}{\partial x_j} \mathbf{v}_j = F_{0i} + \frac{\partial T_{ij}}{\partial x_j} \]

\[ = \mathbf{F}_0 \mathbf{v}_i + \frac{\partial T_{ij}}{\partial x_j} \]

\[ \Rightarrow \frac{\partial}{\partial t} \mathbf{v}_i = \mathbf{F}_0 \mathbf{v}_i + \frac{\partial T_{ij}}{\partial x_j} \]

\[ \text{Differential form of momentum conservation Eq.} \]

(Taylor's equation)

\[ \text{is a step away from} \]

\[ \text{Navier-Stokes Eq.)} \]
1) Select a convenient control volume for analysis
   (Note that along a streamline \( \mathbf{v} \cdot \mathbf{n} = 0 \))

2) Make a table for all the surface segments of C.S and fill out the following table:

<table>
<thead>
<tr>
<th>Surface</th>
<th>dS</th>
<th>( \mathbf{n} \cdot \mathbf{v} )</th>
<th>( \mathbf{P} )</th>
<th>( \mathbf{n} \cdot \mathbf{F} )</th>
<th>( \mathbf{n} \cdot \mathbf{v} \cdot \mathbf{n} )</th>
</tr>
</thead>
</table>

3) For the control volume, determine \( \frac{\partial \mathbf{v}}{\partial t} \), \( \frac{\partial \mathbf{P}}{\partial t} \), \( \frac{\partial \mathbf{v}^2}{\partial t} \)
   (For steady flow \( \frac{\partial \mathbf{v}}{\partial t} = \frac{\partial \mathbf{P}}{\partial t} = 0 \))

4) Apply mass conservation to obtain flux through a certain surface

5) Apply momentum conservation to obtain volume external force

   (Note: Force acting on fluid by the body
   \(-=\) Force acting on the body by the fluid)

mass conservation:
\[ \iiint_{CV} \frac{\partial \rho}{\partial t} dV + \oint_{CS} \rho \mathbf{v} \cdot d\mathbf{S} = 0 \]

momentum conservation:
\[ \iiint_{CV} \frac{\partial (\rho \mathbf{v})}{\partial t} dV + \oint_{CS} \rho \mathbf{v} \cdot (\mathbf{v} \cdot d\mathbf{n}) dS = \iiint_{CV} \mathbf{F}_a dV - \oint_{CS} \rho \mathbf{v} \cdot d\mathbf{S} + \oint_{CS} \mathbf{v} \cdot d\mathbf{S} \]
Control Volume Analysis

- Example One:

A mysterious two-dimensional closed object is being tested in a flow tunnel of height $H$. After the flow has become steady, the horizontal velocity far ahead and downstream is found to have the velocity profile as shown. The pressure and at the upstream and downstream position is uniform across the tunnel and is found to be $p_1$ and $p_2$, respectively. The fluid density is $\rho$. Viscous force on the tunnel wall is neglected.

(a) Calculate $U_2$;
(b) Determine the drag force $D$ on the object.
Example 1: Control Volume (integral approach)

1. Select control volume enclosed by surface segments (AB), (BC), (CD), (DE), (EF), (FA), S

2. Complete the table for all surface segments

<table>
<thead>
<tr>
<th>Surface</th>
<th>dS</th>
<th>( \overrightarrow{n} )</th>
<th>( \overrightarrow{F}_{\text{in}} )</th>
<th>( \overrightarrow{F}_{\text{out}} )</th>
<th>( \overrightarrow{T} )</th>
<th>( \overrightarrow{P} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(AB)</td>
<td>dy</td>
<td>-( \vec{y} )</td>
<td>( u_{1} \vec{y} )</td>
<td>( p_{1} \vec{y} )</td>
<td>( -T_{1} )</td>
<td>( -u_{1}^{2} \vec{y} )</td>
</tr>
<tr>
<td>(BC)</td>
<td>dx</td>
<td>-( \vec{x} )</td>
<td>( u(x) \vec{x} )</td>
<td>( \rho(x) \vec{x} )</td>
<td>( 0 )</td>
<td>( 0 )</td>
</tr>
<tr>
<td>(CD)</td>
<td>dy</td>
<td>-( \vec{y} )</td>
<td>( 2u_{1} \vec{y} )</td>
<td>( -p_{2} \vec{y} )</td>
<td>( 2u_{2} )</td>
<td>( 4u_{2}^{2} \vec{y} )</td>
</tr>
<tr>
<td>(DE)</td>
<td>dy</td>
<td>-( \vec{y} )</td>
<td>( u_{2} \vec{y} )</td>
<td>( -p_{2} \vec{y} )</td>
<td>( u_{2} )</td>
<td>( u_{2}^{2} \vec{y} )</td>
</tr>
<tr>
<td>(EF)</td>
<td>dy</td>
<td>-( \vec{y} )</td>
<td>( 2u_{2} \vec{y} )</td>
<td>( -p_{2} \vec{y} )</td>
<td>( 2u_{2} )</td>
<td>( 4u_{2}^{2} \vec{y} )</td>
</tr>
<tr>
<td>(FA)</td>
<td>dx</td>
<td>-( \vec{x} )</td>
<td>( u(x) \vec{x} )</td>
<td>( \rho(x) \vec{x} )</td>
<td>( 0 )</td>
<td>( 0 )</td>
</tr>
</tbody>
</table>

3. Steady flow \( \Rightarrow \frac{\partial \rho}{\partial t} = 0 \) and \( \frac{\partial \vec{v}}{\partial t} = 0 \)

4. Apply mass conservation

\[
\int \left( \frac{\partial \rho}{\partial t} + \nabla \cdot \vec{F} \right) dV = 0 \Rightarrow \int_{\Omega} \left( \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} (\rho u) + \frac{\partial}{\partial y} (\rho v) + \frac{\partial}{\partial z} (\rho w) \right) dV = 0
\]

\[
\Rightarrow \ \rho \left( -U_{1} \frac{h}{3} + \frac{5}{3} U_{2} h \right) = 0
\]

\[
\Rightarrow U_{2} = \frac{3}{5} U_{1}
\]

5. Apply momentum conservation

\[
\int \left( \vec{F}_{\text{in}} \cdot \vec{n} \right) \cdot dS = \int \vec{F}_{\text{out}} \cdot \vec{n} \cdot dS
\]

\( \vec{n} \cdot dS \) - vector eq.
\[- \text{ in } x \text{-direction (to get the drag)} \]

\[
\begin{align*}
\mathcal{B} \left[ -\int U_1^1 \, dy + \int 4U_2^2 \, dy + \int U_1^2 \, dy + \int 4U_2^1 \, dy \right] &= \int p_1 \, dx + \int p_2 \, dx \\
&+ \int p_0 \, dy + \int p_0 \, dy + \int (\mathcal{F}_{dy})_x \, ds
\end{align*}
\]

\[
\Rightarrow \mathcal{B} \left[ -U_1^1 \text{H} + 3U_2^1 \text{H} \right] = p_1 \text{H} - p_2 \text{H} + (\mathcal{F}_{dy})_x
\]

\[
\Rightarrow \mathcal{B} \left[ -U_1^1 + \frac{27}{25} U_1^2 \right] \text{H} = (p_1 - p_2) \text{H} + (\mathcal{F}_{dy})_x
\]

\[
\Rightarrow (\mathcal{F}_{dy})_x = (p_1 - p_2) \text{H} + \frac{2}{25} \mathcal{B} U_1^1 \text{H}
\]

\[
\Rightarrow \text{ Drag an object } \boxed{D = -(\mathcal{F}_{dy})_x = (p_1 - p_2 - \frac{2}{25} \mathcal{B} U_1^1) \text{H}}
\]
Solving Navier-Stokes equation

- Steady flow of a viscous incompressible fluid between two infinite plates (ignore gravity)
  makes problem 2D
  \[ \eta = (u, v, w) \]

- Steady flow \( \Rightarrow \frac{\partial \eta}{\partial t} = 0 \)
- Infinite plates \( \Rightarrow \frac{\partial \eta}{\partial x} = \frac{\partial \eta}{\partial y} = 0 \), \( v = 0 \) (2D)
- Incompressible flow \( \Rightarrow \nabla \cdot \eta = 0 \)

- Continuity equation
  \[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \]

  \( \Rightarrow \frac{\partial w}{\partial z} = 0 \Rightarrow w = f(x, y) \)

  \( \Rightarrow \) no-slip boundary condition (since the fluid is viscous)

  \[ \eta = 3 \text{ at } z = 0, 4 \]

  \( \Rightarrow \eta = 0 \text{ at } z = 0 \Rightarrow \eta \equiv 0 \text{ everywhere} \)

  \( \Rightarrow \eta = (u, 0, 0) \)

- Momentum equation (only x component in non-trivial)

  \[ x : 3 \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) = - \frac{\partial p}{\partial x} + \nu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) \]

  \[ y : \frac{\partial v}{\partial t} = - \frac{\partial p}{\partial y} + \nu \frac{\partial^2 v}{\partial y^2} \Rightarrow \frac{\partial v}{\partial y} = 0 \Rightarrow p = f(y) \]

  \[ z : \frac{\partial w}{\partial t} = - \frac{\partial p}{\partial z} + \mu \frac{\partial^2 w}{\partial z^2} \Rightarrow \frac{\partial p}{\partial z} = 0 \Rightarrow p = f(z) \]
\[ p = p(x) \Rightarrow \frac{\partial p}{\partial x} = \frac{dp}{dx} \]

\[ u = u(z) \Rightarrow \frac{\partial u}{\partial x} = \frac{du}{dx} \quad (\text{infinite plates} \Rightarrow 2D \text{ problem}) \]

\[
\begin{align*}
\frac{d^2 u}{dz^2} &= \frac{1}{\mu} \left( \frac{d p}{dx} \right) \int dz \\
\frac{du}{dz} &= \frac{1}{\mu} \frac{dp}{dx} z + C_1 \int dz \\
\therefore u(z) &= \frac{1}{2\mu} \frac{dp}{dx} z^2 + C_1 z + C_2
\end{align*}
\]

- We boundary conditions:
- No slip: \( v = 0 \) @ \( z = 0, H \Rightarrow u = 0 @ z = 0, H \)

\[
\begin{align*}
\left. u(z) \right|_0 = C_1 = 0 & \Rightarrow C_1 = 0 \\
\left. u(z) \right|_H = \frac{1}{2\mu} \frac{dp}{dx} H^2 + C_1 H + C_2 & \Rightarrow C_2 = -\frac{1}{2\mu} \frac{dp}{dx} H
\end{align*}
\]

\[
\begin{align*}
\left. u(z) \right|_H &= \frac{1}{2\mu} \frac{dp}{dx} \left( z^2 H - 2H \right) = \frac{H^2}{2\mu} \frac{dp}{dx} \left[ \left( \frac{z^2}{H} \right) - \left( \frac{2}{H} \right) \right]
\end{align*}
\]

- e.g. for \( \frac{dp}{dx} < 0 \) (pressure drops for longer \( x \))
Dimensional Analysis
1. Dimensional Analysis (complemented by Model Testing)

1.1 Motivation

We want to find —say— the force on a body in a particular flow, which can turn out to be a very complicated problem. Some of the ways we can do this:

a) Take in situ measurements and deduce the force either directly or indirectly (eg. CV). Problems: expensive, not always possible (prototype too large, doesn’t exist etc), very specific i.e. no insight.

b) Define parameters that are of importance, use DA to reduce number of relevant parameters and find SP’s. Then do model testing: perform small-scale experiments to determine relations among parameters and use similitude to extrapolate results. Benefits: cheaper, easier to build smaller models, can obtain physical insight.

1.2 Ways to do DA — all yield same results

a) Buckingham’s Π theorem. Realize that physical events cannot depend on our choice of units of measurement and therefore it is dimensionless numbers that are of significance.

b) Non dimensionlization of governing equations. Note: It is often the case that although the governing equations are known cannot be easily solved analytically. Trick: simplifications i.e. drop terms. To justify this you need to know their relevant significance. Eg. if ratio of viscous forces to inertial forces is very small then the terms that represent the viscous forces can be neglected.

c) By inspection/ physical intuition.

1.3 Benefits of DA:

- Knowledge to design the experiment to cover the parameters efficiently
- Correlation to data to get the most useful information
- Detect errors and inconsistencies
- Establish rules of similarity for model test (scaling laws)
- Transform data from a model test (numerical or experimental) to the prototype
- Buckingham's $\Pi$ theory example

- Find the governing $\Pi$-parameters for the problem of determining the force that a wave imparts on a body.

- Relevant dimensional variables: $F, L, \rho, a, g, S \Rightarrow N=6$

- Units:
  - $[F] = MLT^{-2}, [L] = L, [\rho] = L, [a] = L$
  - $[g] = LT^{-2}, [S] = [ML^{-3}]$

  - All $M, L, T$ are present $\Rightarrow$
    - Choose any 3 dimensionally independent variables, e.g. $\{L, g, S\}$
    - $k = 3$

  - Number of $\Pi$-parameters: $N = 6, k = 3 \Rightarrow \Pi = N-k = 3$

- Determine $\Pi$'s

  $\Pi_1: F \cdot L^a \cdot g \cdot S^c = \Pi_1$

  $\PiLT^{-2}, L^a, L^T^{-2} \cdot \Pi C^{-3} = \Pi^0 L^0 T^0$

- Form a set of 3 equations for $M, L, T$

  $M: 1 + c = 0 \Rightarrow c = -1$

  $L: 1 + a + b - 3c = 0 \Rightarrow a = -3$

  $T: -2 - 2b = 0 \Rightarrow b = -1$

  $\Pi_1 = \frac{F}{8gL^3}$
\[ T_2: \quad \lambda L^2 + 8c = \Pi_2 \]
\[ L \cdot L^2 + T^{-2} \cdot N \cdot L^{-2}c = \Pi_0 \cdot T_0 \]

\[ \Pi: \quad C = 0 \]
\[ L: \quad 1 + a + b - 3c = 0 \quad \Rightarrow \quad a = -1 \quad \Rightarrow \quad \Pi_2 = \frac{\lambda}{L} \]
\[ T: \quad 2b = 0 \quad \Rightarrow \quad b = 0 \]

\[ T_3: \quad A \cdot L \cdot g \cdot b \cdot c = T_3 \]
\[ L \cdot L^2 + T^{-2} \cdot N \cdot L^{-2}c = \Pi_0 \cdot T_0 \]

\[ \Pi: \quad C = 0 \]
\[ L: \quad 1 + a + b - 3c = 0 \quad \Rightarrow \quad a = -1 \quad \Rightarrow \quad \Pi_3 = \frac{\lambda}{L} \]
\[ T: \quad 2b = 0 \quad \Rightarrow \quad b = 0 \]

\[ \Rightarrow \text{dependence of } F \text{ on other parameters} \]

\[ F(\Pi_1, \Pi_2, \Pi_3) = 0 \quad \text{implicit form} \]

\[ F(\Pi_2, \Pi_3) = f \quad \text{explicit form} \]

\[ F = 8gL^3 \cdot f(\Pi_2, \Pi_3) \quad \text{Force on the body} \]

(two parameters experiment needed instead of 5-parameters)
Splash volume due to a meteor impact

\[ V = f(m, S_m, S, g, U, \alpha) \]

We want to know the splash volume \( V \)

\[
\begin{array}{ccccccc}
\text{dim} & \text{d} & \text{m} & \text{m}^2 & \text{m} & \text{m} & \text{m}^-1 \\
\hline
x & d^3 & mL^2 & mL^2 & LT^2 & LT^{-1} & M
\end{array}
\]

\[ N - K = 4 \quad \Rightarrow \]

\[ N = 7 \]

\[ K = 3 \]

\[ J = N - K = 4 \quad \text{Fparameters} \]

\[ \frac{\delta V}{m} = F \left( \frac{g}{U^2} \left( \frac{m}{S} \right)^{1/3} \frac{S}{S_m} \alpha \right) \]

log \[ \frac{\delta V}{m} \]\n
from experiment (keep \( \frac{S_m}{S}, \alpha \) constant)
How do we choose appropriate scale?  
We have to know what we want to study!  

Example: say there is a velocity profile given by the following curve:

- appropriate scale
\[ v = V_v \cdot x^k, \quad V_v \approx 2 \text{ m/s} \]
\[ x = X_v \cdot x^k, \quad X_v \approx 1 \text{ m} \]

Microscale
(studying "global" properties of the flow = flux...)

Zoom in

- appropriate scale
\[ v = V_m \cdot x^k, \quad V_m \approx 7 \text{ mm/s} \]
\[ x = X_m \cdot x^k, \quad X_m \approx 1 \text{ mm} \]

Nano-scale
(e.g., bacteria swimming)

Zoom in

- appropriate scale
\[ v = V_n \cdot x^k, \quad V_n \approx 1 \text{ \mu m/s} \]
\[ x = X_n \cdot x^k, \quad X_n \approx 1 \text{ \mu m} \]
In nature, problems we can study span over very large range of scales

- Example - water/ocean

<table>
<thead>
<tr>
<th>length scale</th>
<th>velocity scale</th>
<th>interest</th>
</tr>
</thead>
<tbody>
<tr>
<td>~ 1000 km</td>
<td>~ 10 m/s</td>
<td>Synoptic scale (ocean currents)</td>
</tr>
<tr>
<td>~ 10 km</td>
<td>~ 1 m/s</td>
<td>oil spills</td>
</tr>
<tr>
<td>~ 100 m</td>
<td>~ 10 m/s</td>
<td>large ships</td>
</tr>
<tr>
<td>~ 1 m</td>
<td>~ 20 m/s</td>
<td>torpedoes</td>
</tr>
<tr>
<td>~ 1 cm</td>
<td>~ 30 m/s</td>
<td>conchotom bubble</td>
</tr>
<tr>
<td>~ 1 mm</td>
<td>~ 5 m/s</td>
<td>bugs walking on water</td>
</tr>
</tbody>
</table>

⇒ in every level of these scales, there are assumptions which help us to solve the problem, i.e. we can neglect certain influences. How? ⇒

⇒ Form non-dimensional parameters (Sp's) ⇒

⇒ see how (if) they change on different scales (remember, Nature has no models or prototypes) ⇒ if some of Sp's very small, neglect the term next to them

- Example: Navier-Stokes eq., flow around a ship

\[ U \sim 10^3 \text{ m/s}, \quad L \sim 100 \text{ m}, \quad \nu \sim 10^{-6} \text{ m}^2/\text{s} \Rightarrow Re \sim 10^9 \]

\[ St \frac{\partial p}{\partial x} + \left( \frac{\partial P}{\partial x} \right)^2 = -\frac{1}{2} \nabla \cdot \nabla \rho + \frac{1}{Re} \frac{\partial^2 \rho}{\partial \theta^2} + \frac{1}{Re} \frac{\partial^2 \rho}{\partial \phi^2} \]

\[ 0(1) \quad 0(10^{-9}) \Rightarrow \text{NEGLECT} \]
Example 2.

Similitude example: Drag force on automobile prototype/model

It is proposed that the steady longitudinal drag force on an automobile be measured using a geometrically-similar 1:5 length-scale (i.e., \( L_r \equiv L_p/L_M=5 \)) model in a towing tank filled with fresh water.

(a) State the condition(s) required to ensure dynamic similitude between the prototype and the model.
(b) If the prototype speed is \( U_p=36 \text{ km/hr} \approx 10 \text{ m/s} \), at what speed \( U_M \) should the model be towed?
(c) At the speed in (b), the drag force on the model is measured to be \( F_M=10^4 \text{ N} \). What is the force \( F_p \) on the prototype automobile (in air)?
(d) Due to a soft suspension, the prototype heaves noticeably at a frequency near \( \omega_p=2 \text{ rad/s} \). At what frequency \( \omega_M \) should the model be moved up and down to achieve fluid dynamic similitude for this effect?
(e) It is pointed out that the presence of surface waves in the towing tank (an effect absent in the prototype) may distort the force measurements. In order to keep this effect due to gravity less than \( \mathcal{O}(2\%) \), how deep should the tank be for a model that 'runs' on the tank bottom?
Similitude example: Drag on a automobile

- geometrically similar: \( \lambda = \frac{L_p}{L_n} = 5 \)

a) dynamic similitude \( \Rightarrow \) ratio of forces has to be equal

\[
\begin{align*}
St &= \frac{\text{local inertia}}{\text{convective inertia}} \cdot \frac{L_p}{UT} \\
Fr &= \frac{\text{inertia}}{\text{gravity force}} \cdot \frac{U}{g} \\
Re &= \frac{\text{inertia}}{\text{viscous forces}} = \frac{UL}{\nu}
\end{align*}
\]

UNIMPORTANT (flow in steady)

UNIMPORTANT (no gravity effects)

\[\text{dynamic similitude } \Rightarrow Re_p = Re_m\]

b) \( U_p = 3.6 \text{ km/h} \approx 10 \text{ m/s} \)
\( V_p = U_p \times 10^{-5} m^3/s \)
\( V_n = V_m \times 10^{-5} m^3/s \)

\[
Re_p = Re_m \\
\left( \frac{U_p \cdot L_p}{U_p} \right) = \left( \frac{U_n \cdot L_n}{U_n} \right) \Rightarrow U_n = U_p \cdot \left( \frac{L_p}{L_n} \right) \cdot \left( \frac{V_m}{U_p} \right)
\]

\( U_n = 10 \cdot 5 \cdot 10^{-1} = 5 \text{ m/s} \)
c) Similitude \( \Rightarrow \) Drag force \( \mathbf{T} \) parameter has to be the same for prototype and the model

\[
\mathbf{T} = \frac{F}{\frac{1}{2} \rho U^2 L^2} = C_D \quad \text{drag coefficient}
\]

\[
C_D = \frac{F_n}{\frac{1}{2} \rho_n U_n L_n^2}
\]

\[
F_p = \frac{1}{2} \rho_p U_p^2 L_p^2 \cdot C_D = F_n \cdot \left( \frac{\rho_p}{\rho_n} \right) \cdot \left( \frac{U_p}{U_n} \right)^2 \cdot \left( \frac{L_p}{L_n} \right)^2
\]

\[
\rho_p = 1 \text{ kg/m}^3
\]

\[
\rho_n = 1000 \text{ kg/m}^3
\]

\[
F_p = 10^4 \cdot 10^{-3} \cdot 2^2 \cdot 5^2 = 10^3 \text{N}
\]

d) Dynamic similitude \( \Rightarrow \) ratio of forces equal

Transient effects \( \Rightarrow \) St comes into play

(Along with Re)

\[
\Rightarrow \quad \text{St}_p = \text{St}_n
\]

\[
\text{St} = \frac{L}{U T} = \frac{L W}{2 \pi U}
\]

\[
\frac{L_p \cdot W_p}{2 \pi U_p} = \frac{L_n \cdot W_n}{2 \pi U_n} \Rightarrow \text{St}_n = \text{St}_p \cdot \left( \frac{L_p}{L_n} \right) \cdot \left( \frac{U_n}{U_p} \right)
\]

\[
\text{St}_n = 2 \cdot 5 \cdot \frac{1}{2} = 5 \text{ rad/s}
\]
c) Surface waves ⇒ gravity effects
⇒ Froude number comes into play

\[
(Fr)^2 = \frac{\text{inertia}}{\text{gravity forces}} = \frac{U_m^2}{gH_{1/2}} < 0.25
\]

\[
H_{1/2} > \frac{4U_m^2}{g} = \frac{4 \cdot 25}{10} = 10 \text{ m}
\]
<table>
<thead>
<tr>
<th>Parameter</th>
<th>Expression</th>
<th>Flow situations where parameter is important</th>
</tr>
</thead>
<tbody>
<tr>
<td>Euler number</td>
<td>$\frac{\Delta p}{\rho V^2}$</td>
<td>Flows in which pressure drop is significant: most flow situations</td>
</tr>
<tr>
<td>Reynolds number</td>
<td>$\frac{\rho lV}{\mu}$</td>
<td>Flows that are influenced by viscous effects: internal flows, boundary layer flows</td>
</tr>
<tr>
<td>Froude number</td>
<td>$\frac{V}{\sqrt{g}}$</td>
<td>Flows that are influenced by gravity: primarily free surface flows</td>
</tr>
<tr>
<td>Mach number</td>
<td>$\frac{V}{c}$</td>
<td>Compressibility is important in these flows, usually if $V &gt; 0.3 , c$</td>
</tr>
<tr>
<td>Weber number</td>
<td>$\frac{\rho V^2 l}{\sigma}$</td>
<td>Surface tension influences the flow; flow with an interface may be such a flow</td>
</tr>
<tr>
<td>Strouhal number</td>
<td>$\frac{f l}{V}$</td>
<td>Flow with an unsteady component that repeats itself periodically</td>
</tr>
</tbody>
</table>
### Table 5.1 Dimensions of Fluid-Mechanics Properties

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Symbol</th>
<th>Dimensions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length</td>
<td>$L$</td>
<td>$L^n$</td>
</tr>
<tr>
<td>Area</td>
<td>$A$</td>
<td>$L^n$</td>
</tr>
<tr>
<td>Volume</td>
<td>$V$</td>
<td>$L^n$</td>
</tr>
<tr>
<td>Velocity</td>
<td>$V$</td>
<td>$L/T$</td>
</tr>
<tr>
<td>Acceleration</td>
<td>$a$</td>
<td>$L/T^2$</td>
</tr>
<tr>
<td>Speed of sound</td>
<td>$c$</td>
<td>$L/T$</td>
</tr>
<tr>
<td>Volume flow</td>
<td>$Q$</td>
<td>$L^3/T$</td>
</tr>
<tr>
<td>Mass flow</td>
<td>$m$</td>
<td>$M/T^2$</td>
</tr>
<tr>
<td>Pressure, stress</td>
<td>$\rho$, $\sigma$, $\tau$</td>
<td>$ML^{-1}T^{-2}$, $ML^{-2}T^{-1}$, $ML^{-2}T^{-1}$</td>
</tr>
<tr>
<td>Strain rate</td>
<td>$e$</td>
<td>$T^{-1}$</td>
</tr>
<tr>
<td>Angle</td>
<td>$\theta$</td>
<td>None</td>
</tr>
<tr>
<td>Angular velocity</td>
<td>$\omega$, $\Omega$</td>
<td>$T^{-1}$</td>
</tr>
<tr>
<td>Viscosity</td>
<td>$\mu$</td>
<td>$ML^{-1}T^{-1}$, $FL^{-2}$</td>
</tr>
<tr>
<td>Kinematic viscosity</td>
<td>$\nu$</td>
<td>$L^2/T$</td>
</tr>
<tr>
<td>Surface tension</td>
<td>$\gamma$</td>
<td>$M/T$</td>
</tr>
<tr>
<td>Force</td>
<td>$F$</td>
<td>$ML^{-2}T^{-2}$, $FL^2$</td>
</tr>
<tr>
<td>Moment, torque</td>
<td>$M$</td>
<td>$ML^{-3}$, $FL$</td>
</tr>
<tr>
<td>Power</td>
<td>$P$</td>
<td>$ML^{-2}T^{-3}$, $FLT^{-1}$</td>
</tr>
<tr>
<td>Work, energy</td>
<td>$W$, $E$</td>
<td>$ML^{2}T^{-2}$, $FL$</td>
</tr>
<tr>
<td>Density</td>
<td>$\rho$</td>
<td>$ML^{-3}$, $FT^{-2}L^{-4}$</td>
</tr>
<tr>
<td>Temperature</td>
<td>$T$</td>
<td>$\Theta$</td>
</tr>
<tr>
<td>Specific heat</td>
<td>$c_p$, $c_v$</td>
<td>$L^2/T^2\Theta^{-1}$, $L^2/T^2\Theta^{-1}$</td>
</tr>
<tr>
<td>Specific weight</td>
<td>$\gamma$</td>
<td>$ML^{-2}T^{-2}$, $FL^{-3}$</td>
</tr>
<tr>
<td>Thermal conductivity</td>
<td>$k$</td>
<td>$ML^{-2}T^{-1}\Theta^{-1}$, $FT^{-1}\Theta^{-1}$</td>
</tr>
<tr>
<td>Expansion coefficient</td>
<td>$\beta$</td>
<td>$T^{-1}$</td>
</tr>
</tbody>
</table>
### 5.2 Dimensionless Groups in Fluid Mechanics

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Definition</th>
<th>Qualitative ratio of effects</th>
<th>Importance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reynolds number</td>
<td>$Re = \frac{\rho U L}{\mu}$</td>
<td>Inertia / Viscosity</td>
<td>Always</td>
</tr>
<tr>
<td>Mach number</td>
<td>$Ma = \frac{U}{a}$</td>
<td>Flow speed / Sound speed</td>
<td>Compressible flow</td>
</tr>
<tr>
<td>Froude number</td>
<td>$Fr = \frac{U^2}{gL}$</td>
<td>Inertia / Gravity</td>
<td>Free-surface flow</td>
</tr>
<tr>
<td>Weber number</td>
<td>$We = \frac{\rho U L}{Y}$</td>
<td>Inertia / Surface tension</td>
<td>Free-surface flow</td>
</tr>
<tr>
<td>Rossby number</td>
<td>$Ro = \frac{U}{\Omega r L}$</td>
<td>Flow velocity / Coriolis effect</td>
<td>Geophysical flows</td>
</tr>
<tr>
<td>Cavitation number</td>
<td>$Ca = \frac{P - P_a}{\rho U^2}$</td>
<td>Pressure / Inertia</td>
<td>Cavitation</td>
</tr>
<tr>
<td>(Euler number)</td>
<td>$Pe = \frac{\mu U}{k}$</td>
<td>Dissipation / Conduction</td>
<td>Heat convection</td>
</tr>
<tr>
<td>Prandtl number</td>
<td>$Pr = \frac{\mu c_p}{k}$</td>
<td>Conductivity</td>
<td></td>
</tr>
<tr>
<td>Eckert number</td>
<td>$Ec = \frac{U^3}{c_p L_0}$</td>
<td>Kinetic energy / Entropy</td>
<td>Dissipation</td>
</tr>
<tr>
<td>Specific-heat ratio</td>
<td>$k = \frac{L_0 c_v}{c_p}$</td>
<td>Enthalpy / Internal energy</td>
<td>Compressible flow</td>
</tr>
<tr>
<td>Strouhal number</td>
<td>$St = \frac{A_o L}{U}$</td>
<td>Oscillation / Mean speed</td>
<td>Oscillating flow</td>
</tr>
<tr>
<td>Roughness ratio</td>
<td>$e = \frac{e}{L}$</td>
<td>Wall roughness / Body length</td>
<td>Turbulent, rough walls</td>
</tr>
<tr>
<td>Grashof number</td>
<td>$Gr = \frac{\beta \Delta T p L^3}{\mu^2}$</td>
<td>Buoyancy / Viscosity</td>
<td>Natural convection</td>
</tr>
<tr>
<td>Rayleigh number</td>
<td>$Ra = \frac{\beta \Delta T p L^3}{\mu k}$</td>
<td>Buoyancy / Viscosity</td>
<td>Natural convection</td>
</tr>
<tr>
<td>Temperature ratio</td>
<td>$\frac{T_w}{T_b}$</td>
<td>Wall temperature / Stream temperature</td>
<td>Heat transfer</td>
</tr>
<tr>
<td>Pressure coefficient</td>
<td>$C_p = \frac{P - P_a}{\frac{1}{2} \rho U^2}$</td>
<td>Static pressure / Dynamic force</td>
<td>Aerodynamics, hydrodynamics</td>
</tr>
<tr>
<td>Lift coefficient</td>
<td>$C_L = \frac{L}{\frac{1}{2} \rho A U^2}$</td>
<td>Lift force / Dynamic force</td>
<td>Aerodynamics, hydrodynamics</td>
</tr>
<tr>
<td>Drag coefficient</td>
<td>$C_D = \frac{D}{\frac{1}{2} \rho A U^2}$</td>
<td>Drag force / Dynamic force</td>
<td>Aerodynamics, hydrodynamics</td>
</tr>
<tr>
<td>Friction factor</td>
<td>$f = \frac{h}{\frac{1}{2} \rho U^2 A}$</td>
<td>Friction head loss / Velocity head</td>
<td>Pipe flow</td>
</tr>
<tr>
<td>Skin friction coefficient</td>
<td>$C_f = \frac{\tau_{wall}}{\rho V^2/2}$</td>
<td>Wall shear stress / Dynamic pressure</td>
<td>Boundary layer flow</td>
</tr>
</tbody>
</table>