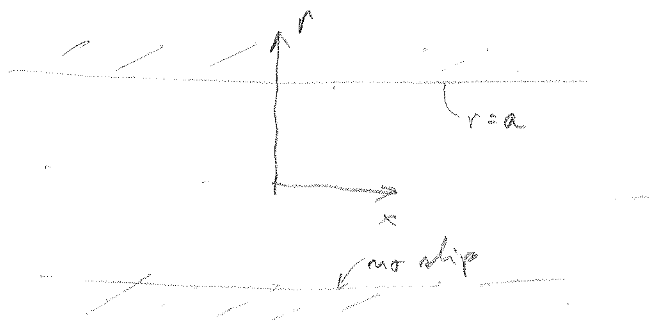
Viscous Flow

Outline:

- Poiseuille Flow
- Stokes Boundary Layer

- Poiseuille Flow - flow through a pipe



Assumptions:

$\frac{\partial}{\partial t}=0$ - steady

$\frac{\partial \vec{v}}{\partial x}=0$ - developed

$v_\theta, \frac{\partial}{\partial \theta}=0$ - axisymmetric flow

- cylindrical coordinates (x, r, θ)

- continuity eq. in cylindrical coordinates

$$\frac{1}{r}\frac{\partial r v_r}{\partial r}+\underbrace{\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}}_{=0\ (\text{axisym.})}+\underbrace{\frac{\partial v_x}{\partial x}}_{=0 \Rightarrow \text{developed}}=0 \Rightarrow r\cdot v_r \neq f(r) \Rightarrow v_r=\frac{g(x)}{r}$$

B.C. $v_r(r=a)=0 \rightarrow$ no slip

$$0=\frac{g(x)}{a} \Rightarrow g(x)=0$$

$$\Rightarrow \boxed{v_r(r)\equiv 0}$$

- Navier-Stokes $(v_r=v_\theta=0)$

$$\left.\begin{aligned} r:&\quad 0=\frac{\partial p}{\partial r} \Rightarrow p\neq f(r)\\ \theta:&\quad 0=\frac{\partial p}{\partial \theta} \Rightarrow p\neq f(\theta)\end{aligned}\right\} p=p(x) \Rightarrow \frac{\partial p}{\partial x}\rightarrow\frac{dp}{dx}$$

$$x:\ \underbrace{\frac{\partial v_x}{\partial t}}_{=0(\text{steady})}+\underbrace{v_r\frac{\partial v_x}{\partial r}}_{\substack{=0\\(v_r=0)}}+\underbrace{\frac{v_\theta}{r}\frac{\partial v_x}{\partial \theta}}_{\substack{=0\\(v_\theta=0)}}+\underbrace{v_x\frac{\partial v_x}{\partial x}}_{\substack{=0\\(\text{developed})}}=-\frac{1}{\rho}\frac{dp}{dx}+\nu\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_x}{\partial r}\right)+\underbrace{\frac{1}{r^2}\frac{\partial^2 v_x}{\partial \theta^2}}_{=0}+\underbrace{\frac{\partial^2 v_x}{\partial x^2}}_{\substack{=0\\(\text{developed})}}\right\}$$

$$\because \underbrace{\frac{\partial \vec{v}}{\partial x}, \frac{\partial \vec{v}}{\partial \theta}=0}_{\text{developed, axisymmetric}} \Rightarrow v_x=v_x(r) \Rightarrow \frac{\partial v_x}{\partial r}\rightarrow\frac{d v_x}{dr}$$

$$\Rightarrow \quad \frac{1}{r}\frac{d}{dr}\left(r\frac{dv_x}{dr}\right) = \frac{1}{\mu}\frac{dp}{dx} \quad \Big/ \cdot r \Big/ \int dr$$

$$r\frac{dv_x}{dr} = \frac{r^2}{2\mu}\frac{dp}{dx} + c_1 \quad \Big/ : r$$

$$\frac{dv_x}{dr} = \frac{r}{2\mu}\frac{dp}{dx} + \frac{c_1}{r} \quad \Big/ \int dr$$

$$v_x(r) = \frac{r^2}{4\mu}\frac{dp}{dx} + c_1 \cdot \ln r + c_2$$

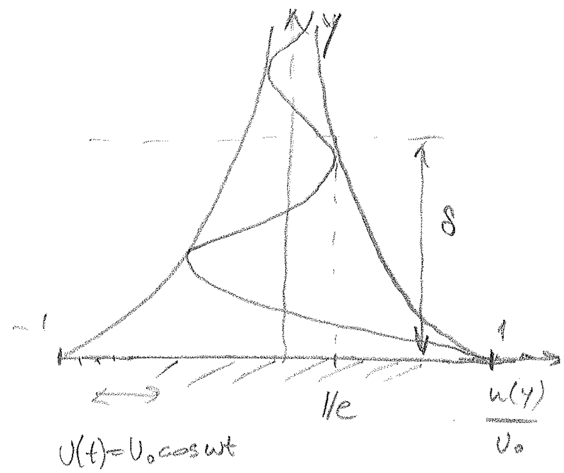
- use B.C.
  - for $r = 0$, v_x is finite $\Rightarrow c_1 = 0$
  - $r = a$, $v_x = 0$

$$0 = \frac{a^2}{4\mu}\frac{dp}{dx} + c_2 \quad \Rightarrow \quad c_2 = -\frac{a^2}{4\mu}\frac{dp}{dx}$$

$$\Rightarrow \boxed{v_x(r) = \frac{a^2}{4\mu}\frac{dp}{dx}\left[\left(\frac{r}{a}\right)^2 - 1\right]}$$

- Stokes Boundary Layer – oscillating infinite plate

- velocity profile



$$\frac{u(y,t)}{U_0} = e^{-y/\delta} \cdot \cos\left(-\frac{y}{\delta} + \omega t\right)$$

$$\delta = \delta_{1/e} = \sqrt{\frac{2\nu}{\omega}}$$

- skin friction

$$\tau_w = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0}$$

$$\frac{\partial u}{\partial y} = U_0\left(-\frac{1}{\delta}e^{-y/\delta}\cos\left(-\frac{y}{\delta}+\omega t\right) + \frac{1}{\delta}e^{-y/\delta}\sin\left(-\frac{y}{\delta}+\omega t\right)\right)$$

$$= \frac{U_0}{\delta}e^{-y/\delta}\left(\sin\left(-\frac{y}{\delta}+\omega t\right) - \cos\left(-\frac{y}{\delta}+\omega t\right)\right)$$

$$\left.\frac{\partial u}{\partial y}\right|_{y=0} = \frac{U_0}{\delta}\left(\sin(\omega t) - \cos(\omega t)\right)$$

$$\Rightarrow \tau_w = \mu\frac{U_0}{\delta}\left(\sin\omega t - \cos\omega t\right) = \mu U_0\sqrt{\frac{\omega}{2\nu}}\left(\sin\omega t - \cos\omega t\right)$$

$$\sin\omega t - \cos\omega t = \sqrt{2}\left(\cos\frac{\pi}{4}\sin\omega t - \sin\frac{\pi}{4}\cos\omega t\right) = \sqrt{2}\sin\left(\omega t - \frac{\pi}{4}\right)$$

$$\Rightarrow \boxed{\tau_w = \mu U_0\sqrt{\frac{\omega}{\nu}}\sin\left(\omega t - \frac{\pi}{4}\right)} = \mu U_0\sqrt{\frac{\omega}{\nu}}\cos\left(\omega t - \frac{3\pi}{4}\right)$$

$$|\tau_w|_{max} \text{ @ } \sin\left(\omega t - \frac{\pi}{4}\right) = \pm 1 \Rightarrow \omega t - \frac{\pi}{4} = (2n-1)\cdot\frac{\pi}{2}, \quad n = 1, 2, 3\ldots$$

$$\Rightarrow \boxed{\omega t = n\pi - \frac{\pi}{4}, \quad n = 1, 2, 3\ldots}$$