Marine Hydrodynamics Homework #1(a) Solutions

Question 1: Tensor Notation

1.

$$
\begin{aligned}
A_i B_{kk} C_j &= A_i \left(B_{11} + B_{22} + B_{33}\right) C_j \\
&= \left(B_{11} + B_{22} + B_{33}\right) A_i C_j \\
&= \left(B_{11} + B_{22} + B_{33}\right) \begin{bmatrix} A_1 C_j \\ A_2 C_j \\ A_3 C_j \end{bmatrix} \\
&= \left(B_{11} + B_{22} + B_{33}\right) \begin{bmatrix} A_1 C_1 & A_1 C_2 & A_1 C_3 \\ A_2 C_1 & A_2 C_2 & A_2 C_3 \\ A_3 C_1 & A_3 C_2 & A_3 C_3 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\end{aligned}
$$

2.

$$
\begin{aligned}
A_i C_j \delta_{ij} B_{km} &= A_i C_i B_{km} \\
&= \left(A_1 C_1 + A_2 C_2 + A_3 C_3\right) \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix} \\
&= 43 \begin{bmatrix} 0 & 2 & -2 \\ -3 & 1 & -1 \\ 6 & -3 & -1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 86 & -86 \\ -129 & 43 & -43 \\ 258 & -129 & -43 \end{bmatrix}
\end{aligned}
$$

3.

$$
\begin{aligned}
A_i B_{jk} C_m \delta_{ij} \delta_{km} &= A_i B_{ij} C_j = \{A\}^T [B] \{C\} \\
&= A_1 B_{1j} C_j + A_2 B_{2j} C_j + A_3 B_{3j} C_j \\
&= A_1 B_{11} C_1 + A_1 B_{12} C_2 + A_1 B_{13} C_3 \\
&\quad + A_2 B_{21} C_1 + A_2 B_{22} C_2 + A_2 B_{23} C_3 \\
&\quad + A_3 B_{31} C_1 + A_3 B_{32} C_2 + A_3 B_{33} C_3 \\
&= -88
\end{aligned}
$$

4. Begin by expanding the alternating tensor, keeping only the nonzero components.

$$\varepsilon_{jkl} A_i C_k B_{li} = \left\{ \begin{array}{c} \varepsilon_{1kl} A_i C_k B_{Li} \\ \varepsilon_{2kl} A_i C_k B_{Li} \\ \varepsilon_{3kl} A_i C_k B_{Li} \end{array} \right\}$$

The ε_{jkl} tensor only has six permutations which are nonzero. Keeping those combinations yields:

$$= \left\{ \begin{array}{c} \varepsilon_{123} A_i C_2 B_{3i} + \varepsilon_{132} A_i C_3 B_{2i} \\ \varepsilon_{231} A_i C_3 B_{1i} + \varepsilon_{213} A_i C_1 B_{3i} \\ \varepsilon_{312} A_i C_1 B_{2i} + \varepsilon_{321} A_i C_2 B_{1i} \end{array} \right\}$$

Now summing over the dummy index produces the result:

$$= \left\{ \begin{array}{c} C_2 \left(A_1 B_{31} + A_2 B_{32} + A_3 B_{33}\right) - C_3 \left(A_1 B_{21} + A_2 B_{22} + A_3 B_{23}\right) \\ C_3 \left(A_1 B_{11} + A_2 B_{12} + A_3 B_{13}\right) - C_1 \left(A_1 B_{31} + A_2 B_{32} + A_3 B_{33}\right) \\ C_1 \left(A_1 B_{21} + A_2 B_{22} + A_3 B_{23}\right) - C_2 \left(A_1 B_{11} + A_2 B_{12} + A_3 B_{13}\right) \end{array} \right\}$$

$$= \left\{ \begin{array}{c} 53 \\ -15 \\ -1 \end{array} \right\}$$

Question 2: Taylor Series

The Taylor Series is defined as:

$$f(x) \approx f(0 + \Delta x) = f(0) + \Delta x \frac{df(x)}{dx}|_{x=0} + \Delta x^2 \frac{d^2 f(x)}{dx^2}|_{x=0} + H.O.T.$$

$$\begin{aligned} f(0) &= e^{2(0)} \sin(0) = 0 \\ f'(0) &= 2e^{2(0)} \sin(0) + e^{2(0)} \cos(0) = 1 \\ f''(0) &= 3e^{2(0)} \sin(0) + 4e^{2(0)} \cos(0) = 4 \end{aligned}$$

$$f(0 + 0.1) = 0 + (0.1) + 1/2(0.1)^2(4) = 0.12$$

Question 3: Supplemental Problems

1. (a)

$$\nabla \phi = \left[3x^2 y^2 z - 2x \sin\left(x^2\right)\right] \hat{\imath} + \left[2x^3 yz + \frac{2}{y}\right] \hat{\jmath} + \left[x^3 y^2 + \frac{2}{z}\right] \hat{k}$$

This is a vector

(b)

$$\nabla^2\phi = \nabla\cdot\nabla\phi = 6xy^2z - 2\sin\left(x^2\right) - 4x^2\cos\left(x^2\right) + 2x^3z - \frac{2}{y^2} - \frac{2}{z^2}$$

This is a scalar

(c)

$$\nabla\cdot\vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

This is a scalar

(d)

$$\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix}$$

This is a tensor

(e)

$$\vec{V}\cdot\nabla\vec{V} = \left\{ \begin{array}{c} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \\ u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \\ u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} \end{array} \right\}$$

This is a vector

2. C5:

$$\int_V \vec{\nabla}\cdot F dV = \int_S \vec{F}\cdot\hat{n} dA$$

Start with the volume integral first.

$$\vec{\nabla}\cdot\vec{F} = \frac{\partial}{\partial x}\left[z^2 - 1 - 2y^2\right] + \frac{\partial}{\partial y}\left[z^2 - 1 - 2x^2\right] + \frac{\partial}{\partial z}\left[x^2 + y^2 + z^2\right] = 2z$$

This domain is easiest to integrate in spherical cordinates where in general:

$$\int_V f(x,y,z)dV = \int_0^R \int_{\pi/2}^{\pi} \int_0^{2\pi} g(\rho,\theta\phi)\rho^2\sin(\phi)\,d\theta\,d\phi\,d\rho$$

and the spatial coordinates can be transformed by:

$$\left\{ \begin{array}{l} x = \rho\sin(\phi)cos(\theta) \\ y = \rho\sin(\phi)sin(\theta) \\ z = \rho\cos(\phi) \end{array} \right.$$

$$\begin{aligned} \int_V \vec{\nabla}\cdot F dV &= \int_0^R \int_{\pi/2}^{\pi} \int_0^{2\pi} \left[2\rho\cos(\phi)\right]\rho^2\sin(\phi)\,d\theta\,d\phi\,d\rho \\ &= -1/2\pi R^4 \end{aligned}$$

Now compare against the surface integral. Need to define the unit vector

on the sides of the hemisphere. The general way of doing this is by finding two tangent vectors, calculating the cross product, and then normalizing it such that it has a mansuetude of one. For a sphere, it is easier to define a vector starting at the origin and going to the surface of the hemisphere of radius one. This is by definition, a unit vector to the surface.

$$\hat{n} = \sin(\phi)\cos(\theta)\hat{i} + sin(\phi)\sin(\theta)\hat{j} + cos(\phi)\hat{k}$$

The unit vector on the top surface of the hemisphere is $\hat{n} = \hat{k}$

$$\begin{aligned}
\vec{F}\cdot\hat{n}_{Sides} &= \rho^2 \\
\vec{F}\cdot\hat{n}_{Sides} &= \rho^2\left[\sin(\phi)\left(\cos(\phi)^2 - 2\sin)\phi)^2\right)(\sin(\theta) - \cos(\theta)) + \cos(\phi)\right]
\end{aligned}$$

Calculate fluxes:

$$\begin{aligned}
\int_S \vec{F}\cdot\hat{n}dA &= \int_{S_{Top}} \vec{F}\cdot\hat{n}_{Top}dA + \int_{S_{Sides}} \vec{F}\cdot\hat{n}_{Sides}dA \\
&= \int_0^{2\pi}\int_0^R \left[\rho^2\right]\rho\,d\rho\,d\theta \\
&\quad + R^2\int_0^{2\pi}\int_{\pi/2}^{\pi}\left[\sin(\phi)\left(\cos(\phi)^2 - 2\sin)\phi)^2\right)(\sin(\theta) - \cos(\theta)) + \cos(\phi)\right]R^2\,d\rho\,d\theta \\
&= 1/2\pi R^4 - \pi R^4 \\
&= -1/2\pi R^4
\end{aligned}$$

3. C6:

$$\int_S \hat{n}\cdot(\nabla\times F)\,dS = \oint_C \vec{F}\cdot d\vec{x}$$

Starting with the surface integral:

$$\begin{aligned}
\vec{\nabla}\times\vec{F} &= \hat{i}\,(F_{3,y} - F_{2,z}) - \hat{j}\,(F_{3,x} - F_{1,z}) + \hat{k}\,(F_{2,x} - F_{1,y}) \\
&= -4(x+y)
\end{aligned}$$

The unit vector for the circle is $\hat{n} = \hat{k}$

$$\begin{aligned}
\int_S \hat{n}\cdot(\nabla\times F)\,dS &= \int_0^{2\pi}\int_0^R 4\rho\left[\cos(\theta) + \sin(\theta)\right]\rho d\rho d\theta \\
&= 0
\end{aligned}$$

Now evaluating the line integral

$$\begin{cases} x = R\cos(\theta) & dx = -R\sin(\theta) \\ y = R\sin(\theta) & dy = R\cos(\theta) \end{cases}$$

$$\vec{F}\cdot d\vec{x} = -\left[1 + 2R^2\sin(\theta)^2\right]R\sin(\theta) - \left[1 - 2R^2\cos(\theta)^2\right]R\cos(\theta)$$

$$
\begin{aligned}
\oint_C \vec{F} \cdot d\vec{x} &= \int_0^{2\pi} \vec{F} \cdot d\vec{x} \\
&= 0
\end{aligned}
$$

4. C10:

Use Leibniz Integral Rule which states that:

$$
\frac{dI}{dt} = \int_{a(t)}^{b(t)} \frac{\partial F}{\partial t} dx + F(b,t)\frac{db}{dt} - F(a,t)\frac{da}{dt}
$$

where

$$
\begin{cases} F(x,t) = \sin(x)\cos(2t) \\ a(t) = 4t \\ b(t) = t+5 \end{cases}
$$

Starting with the first integral:

$$
F_{,t} = -2\sin(x)\sin(2t)
$$

which produces

$$
\begin{aligned}
\int_{a(t)}^{b(t)} -2sin(x)sin(2t)dx &= -2\sin(2t)\int_{a(t)}^{b(t)} \sin(x)dx \\
&= 2\sin(2t)\left[\cos(x)\right]_{4t}^{t+5} \\
&= 2\sin(2t)\left[\cos(t+5) - \cos(4t)\right]
\end{aligned}
$$

Now combining with the boundary terms:

$$
\frac{dI(t)}{dt} = 2\sin(2t)\left[\cos(t+5) - \cos(4t)\right] + \cos(2t)\left[\sin(t+5) - 4\sin(4t)\right]
$$