

## Lecture Notes on Fluid Dynamics

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2-5Stokes.tex

## 2.5 Stokes flow past a sphere

[Refs]

Lamb: *Hydrodynamics*

Acheson : *Elementary Fluid Dynamics*, p. 223 ff

One of the fundamental results in low Reynolds-number hydrodynamics is the Stokes solution for steady flow past a small sphere. Applications range widely from the determination of electron charges to the physics of aerosols.

The continuity equation reads

$$\nabla \cdot \vec{q} = 0 \quad (2.5.1)$$

With inertia neglected, the approximate momentum equation is

$$0 = -\frac{\nabla p}{\rho} + \nu \nabla^2 \vec{q} \quad (2.5.2)$$

Physically, the pressure gradient drives the flow by overcoming viscous resistance, but does not affect the fluid inertia significantly.

Referring to Figure 2.5 for the spherical coordinate system  $(r, \theta, \phi)$ . Let the ambient velocity be upward and along the polar ( $z$ ) axis:  $(u, v, w) = (0, 0, W)$ . Axial symmetry demands

$$\frac{\partial}{\partial \phi} = 0, \quad \text{and} \quad \vec{q} = (q_r(r, \theta), q_\theta(r, \theta), 0)$$

Using a known formula for the divergence in spherical polar coordinates, Eq. (2.5.1) becomes

$$\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 q_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (q_\theta \sin \theta) = 0 \quad (2.5.3)$$

An equivalent and physically more revealing way is to write

$$\frac{\partial}{\partial r} (r^2 q_r \sin \theta) + \frac{\partial}{\partial \theta} (r q_\theta \sin \theta) = 0 \quad (2.5.4)$$

As in the case of rectangular coordinates, we define the stream function  $\psi$  to satisfy the continuity equation (2.5.4) identically

$$q_r = \frac{1}{r^2 \sin \theta} \frac{\partial \psi}{\partial \theta}, \quad q_\theta = -\frac{1}{r \sin \theta} \frac{\partial \psi}{\partial r} \quad (2.5.5)$$

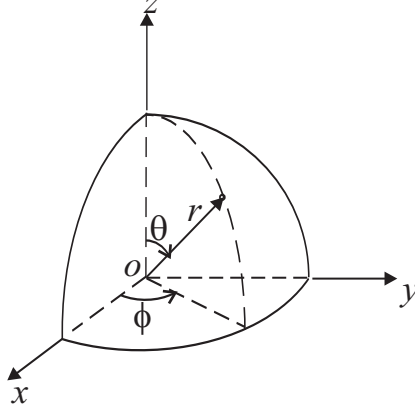


Figure 2.5.1: The spherical coordinates

At infinity, the uniform velocity  $W$  along  $z$  axis can be decomposed into radial and polar components

$$q_r = W \cos \theta = \frac{1}{r^2 \sin \theta} \frac{\partial \psi}{\partial \theta}, \quad q_\theta = -W \sin \theta = -\frac{1}{r \sin \theta} \frac{\partial \psi}{\partial r}, \quad r \sim \infty \quad (2.5.6)$$

The corresponding stream function at infinity follows by integration

$$\psi = \frac{W}{2} r^2 \sin^2 \theta, \quad r \sim \infty \quad (2.5.7)$$

Using the vector identity

$$\nabla \times (\nabla \times \vec{q}) = \nabla(\nabla \cdot \vec{q}) - \nabla^2 \vec{q} \quad (2.5.8)$$

and (2.5.1), we get

$$\nabla^2 \vec{q} = -\nabla \times (\nabla \times \vec{q}) = -\nabla \times \vec{\zeta} \quad (2.5.9)$$

Taking the curl of (2.5.2) and using (2.5.9) we get

$$\nabla \times (\nabla \times \vec{\zeta}) = 0 \quad (2.5.10)$$

After some straightforward algebra given in the Appendix, we can show that

$$\vec{q} = \nabla \times \left( \frac{\psi \vec{e}_\phi}{r \sin \theta} \right) \quad (2.5.11)$$

and

$$\vec{\zeta} = \nabla \times \vec{q} = \nabla \times \nabla \times \left( \frac{\psi \vec{e}_\phi}{r \sin \theta} \right) = -\frac{\vec{e}_\phi}{r \sin \theta} \left( \frac{\partial^2 \psi}{\partial r^2} + \frac{\sin \theta}{r^2} \frac{\partial}{\partial \theta} \left( \frac{1}{\sin \theta} \frac{\partial \psi}{\partial \theta} \right) \right) \quad (2.5.12)$$

Now from (2.5.10)

$$\nabla \times \nabla \times (\nabla \times \vec{q}) = \nabla \times \nabla \times \left[ \nabla \times \left( \nabla \times \frac{\psi \vec{e}_\phi}{r \sin \theta} \right) \right] = 0$$

hence, the momentum equation (2.5.10) becomes a scalar equation for  $\psi$ .

$$\left( \frac{\partial^2}{\partial r^2} + \frac{\sin \theta}{r^2} \frac{\partial}{\partial \theta} \left( \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \right) \right)^2 \psi = 0 \quad (2.5.13)$$

The boundary conditions on the sphere are

$$q_r = 0 \quad q_\theta = 0 \quad \text{on} \quad r = a \quad (2.5.14)$$

The boundary conditions at  $\infty$  is

$$\psi \rightarrow \frac{W}{2} r^2 \sin^2 \theta \quad (2.5.15)$$

Let us try a solution of the form:

$$\psi(r, \theta) = f(r) \sin^2 \theta \quad (2.5.16)$$

then  $f$  is governed by the equi-dimensional differential equation:

$$\left[ \frac{d^2}{dr^2} - \frac{2}{r^2} \right]^2 f = 0 \quad (2.5.17)$$

whose solutions are of the form  $f(r) \propto r^n$ , It is easy to verify that  $n = -1, 1, 2, 4$  so that

$$f(r) = \frac{A}{r} + Br + Cr^2 + Dr^4$$

or

$$\psi = \sin^2 \theta \left[ \frac{A}{r} + Br + Cr^2 + Dr^4 \right]$$

To satisfy (2.5.15) we set  $D = 0, C = W/2$ . To satisfy (2.5.14) we use (2.5.5) to get

$$q_r = 0 = \frac{W}{2} + \frac{A}{a^3} + \frac{B}{a} = 0, \quad q_\theta = 0 = W - \frac{A}{a^3} + \frac{B}{a} = 0$$

Hence

$$A = \frac{1}{4} W a^3, \quad B = -\frac{3}{4} W a$$

Finally the stream function is

$$\psi = \frac{W}{2} \left[ r^2 + \frac{a^3}{2r} - \frac{3ar}{2} \right] \sin^2 \theta \quad (2.5.18)$$

Inside the parentheses, the first term corresponds to the uniform flow, and the second term to the doublet; together they represent an inviscid flow past a sphere. The third term is called the Stokeslet, representing the viscous correction.

The velocity components in the fluid are: (cf. (2.5.5)) :

$$q_r = W \cos \theta \left[ 1 + \frac{a^3}{2r^3} - \frac{3a}{2r} \right] \quad (2.5.19)$$

$$q_\theta = -W \sin \theta \left[ 1 - \frac{a^3}{4r^3} - \frac{3a}{4r} \right] \quad (2.5.20)$$

### 2.5.1 Physical Deductions

1. *Streamlines*: With respect to the the equator along  $\theta = \pi/2$ ,  $\cos \theta$  and  $q_r$  are odd while  $\sin \theta$  and  $q_\theta$  are even. Hence the streamlines (velocity vectors) are symmetric fore and aft.

2. *Vorticity*:

$$\vec{\zeta} = \zeta_\phi \vec{e}_\phi = \left( \frac{1}{r} \frac{\partial(rq_\theta)}{\partial r} - \frac{1}{r} \frac{\partial q_r}{\partial \theta} \right) \vec{e}_\phi = -\frac{3}{2} Wa \frac{\sin \theta}{r^2} \vec{e}_\phi$$

3. *Pressure* : From the  $r$ -component of momentum equation

$$\frac{\partial p}{\partial r} = \frac{\mu Wa}{r^3} \cos \theta (= -\mu \nabla \times (\nabla \times \vec{q}))$$

Integrating with respect to  $r$  from  $r$  to  $\infty$ , we get

$$p = p_\infty - \frac{3}{2} \frac{\mu Wa}{r^3} \cos \theta \quad (2.5.21)$$

4. *Stresses and strains*:

$$\frac{1}{2} e_{rr} = \frac{\partial q_r}{\partial r} = W \cos \theta \left( \frac{3a}{2r^2} - \frac{3a^3}{2r^4} \right)$$

On the sphere,  $r = a$ ,  $e_{rr} = 0$  hence  $\tau_{rr} = 0$  and

$$\sigma_{rr} = -p + \tau_{rr} = -p_\infty + \frac{3}{2} \frac{\mu W}{a} \cos \theta \quad (2.5.22)$$

On the other hand

$$e_{r\theta} = r \frac{\partial}{\partial r} \left( \frac{q_\theta}{r} \right) + \frac{1}{r} \frac{\partial q_r}{\partial \theta} = -\frac{3}{2} \frac{Wa^3}{r^4} \sin \theta$$

Hence at  $r = a$ :

$$\sigma_{r\theta} = \tau_{r\theta} = \mu e_{r\theta} = -\frac{3}{2} \frac{\mu W}{a} \sin \theta \quad (2.5.23)$$

The resultant stress on the sphere is parallel to the  $z$  axis.

$$\Sigma_z = \sigma_{rr} \cos \theta - \tau_{r\theta} \sin \theta = -p_\infty \cos \theta + \frac{3}{2} \frac{\mu W}{a}$$

The constant part exerts a net drag in  $z$  direction

$$D = \int_0^{2\pi} a d\phi \int_0^\pi d\theta \sin \theta \Sigma_z = \frac{3}{2} \frac{\mu W}{a} 4\pi a^2 = 6\pi \mu W a \quad (2.5.24)$$

This is the celebrated Stokes formula.

A drag coefficient can be defined as

$$C_D = \frac{D}{\frac{1}{2} \rho W^2 \pi a^2} = \frac{6\pi \mu W a}{\frac{1}{2} \rho W^2 \pi a^2} = \frac{24}{\frac{\rho W (2a)}{\mu}} = \frac{24}{Re_d} \quad (2.5.25)$$

5. *Fall velocity* of a particle through a fluid. Equating the drag and the buoyant weight of the particle

$$6\pi \mu W_o a = \frac{4\pi}{3} a^3 (\rho_s - \rho_f) g$$

hence

$$W_o = \frac{2}{9} g \left( \frac{a^2}{\nu} \frac{\Delta \rho}{\rho_f} \right) = 217.8 \left( \frac{a^2}{\nu} \frac{\Delta \rho}{\rho_f} \right)$$

in cgs units. For a sand grain in water,

$$\frac{\Delta \rho}{\rho_f} = \frac{2.5 - 1}{1} = 1.5, \quad \nu = 10^{-2} \text{cm}^2/\text{s}$$

$$W_o = 32,670 a^2 \text{cm/s} \quad (2.5.26)$$

To have some quantitative ideas, let us consider two sand of two sizes :

$$\begin{aligned} a = 10^{-2} \text{cm} = 10^{-4} \text{m} : \quad W_o &= 3.27 \text{cm/s}; \\ a = 10^{-3} \text{cm} = 10^{-5} \text{m} = 10 \mu\text{m}, \quad W_o &= 0.0327 \text{cm/s} = 117 \text{cm/hr} \end{aligned}$$

For a water droplet in air,

$$\frac{\Delta \rho}{\rho_f} = \frac{1}{10^{-3}} = 10^3, \quad \nu = 0.15 \text{cm}^2/\text{sec}$$

then

$$W_o = \frac{(217.8)10^3}{0.15} a^2 \quad (2.5.27)$$

in cgs units. If  $a = 10^{-3} \text{cm} = 10 \mu\text{m}$ , then  $W_o = 1.452 \text{cm/sec}$ .

## Details of derivation

Details of (2.5.11).

$$\begin{aligned}\vec{q} &= \nabla \times \left( \frac{\psi}{r \sin \theta} \vec{e}_\phi \right) = \frac{1}{r^2 \sin \theta} \begin{vmatrix} \vec{e}_r & \vec{e}_\theta & r \sin \theta \vec{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ 0 & 0 & \psi \end{vmatrix} \\ &= \vec{e}_r \left( \frac{1}{r^2 \sin \theta} \frac{\partial \psi}{\partial \theta} \right) - \vec{e}_\theta \left( \frac{1}{r \sin \theta} \frac{\partial \psi}{\partial r} \right)\end{aligned}$$

Details of (2.5.12).

$$\begin{aligned}\nabla \times \vec{q} &= \nabla \times \left( \nabla \times \frac{\psi \vec{e}_\phi}{r \sin \theta} \right) \\ &= \frac{1}{r^2 \sin \theta} \begin{vmatrix} \vec{e}_r & r \vec{e}_\theta & r \sin \theta \vec{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ \frac{1}{r^2 \sin \theta} \frac{\partial \psi}{\partial \theta} & \frac{-1}{\sin \theta} \frac{\partial \psi}{\partial r} & 0 \end{vmatrix} \\ &= \frac{\vec{e}_\phi}{r \sin \theta} \left[ \frac{\partial^2 \psi}{\partial r^2} + \frac{\sin \theta}{r^2} \frac{\partial}{\partial \theta} \left( \frac{1}{\sin \theta} \frac{\partial \psi}{\partial \theta} \right) \right]\end{aligned}$$