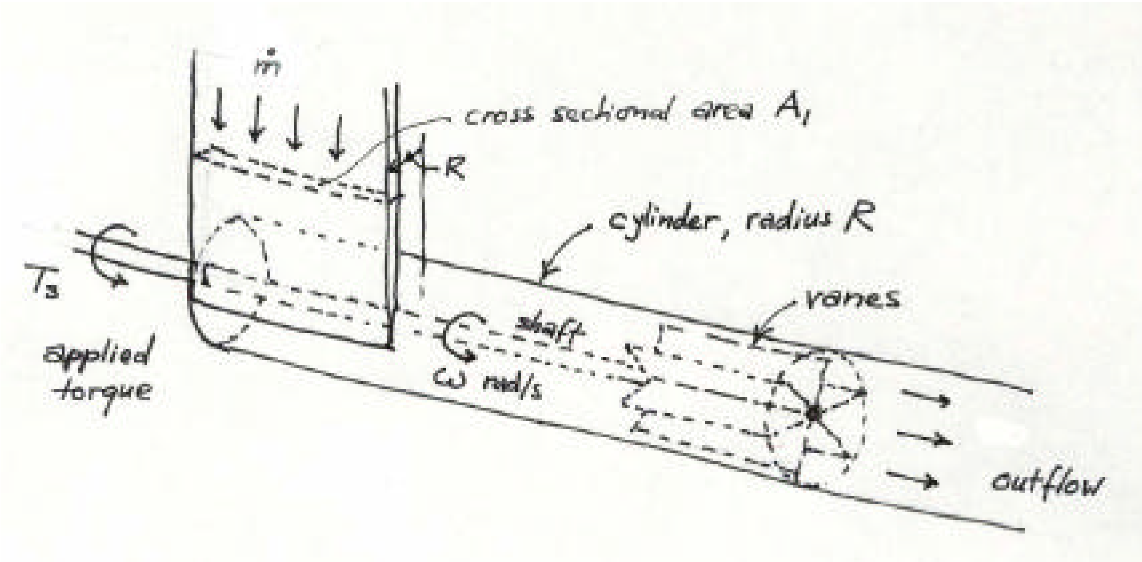
Problem 5.24 Solution



It is clear that the tangential injection causes a swirling flow tends to rotate the vanes in the direction of positive ω, and the strength of the rotation, or the torque it produces if one holds the vanes static, carries some information about the flow rate.

We apply the angular momentum theorem for steady flow,

$$(T_z)_{cv} = \int_{cs} \rho r v_\theta v_{rn} dA \ , \tag{1}$$

to a control volume that resides just inside the pipe, bounded on the left by the closed end of the pipe (where it "cuts" the shaft). The right end is just downstream of the vanes. Inflow occurs tangentially as shown. Velocity will be measured in the inertial reference frame attached to the static cylinder. The system is assumed to be in a steady state of rotation.

Assumptions:

1. The shaft is mounted on a frictionless bearing.
2. Inflow takes place tangentially at a distance R from the axis via a narrow slit-shaped opening with area A_1.
3. When the set of vanes rotates, it effectively rotates the fluid with it solid-body-like, so that $v_\theta = \omega r$ over the plane where the fluid exits.
4. The vanes are very thin, and offer no obstruction to the axial flow.
5. The axial flow velocity at the exit plane is approximately uniform (a working approximation).

Let T_s be the counterclockwise torque applied to the shaft, as shown, V_1 be the velocity at the inlet, and u the velocity at the exit.

Angular momentum theorem (equation 1):

$$T_s = \int_0^R \rho r \quad \omega r \quad u \quad 2\pi r dr - \rho R V_1 V_1 A_1 = \frac{\pi\rho\omega R^4 u}{2} - \rho R V_1^2 A_1 \qquad (2)$$

Mass conservation:

$$\rho V_1 A_1 = \dot{m} \qquad (3)$$

$$\rho u \pi R^2 = \dot{m} \qquad (4)$$

Eliminating the inflow and outflow velocities from (2) by using (3) and (4), we obtain

$$T_s = \dot{m}\frac{\omega R^2}{2} - \dot{m}^2 \frac{R}{\rho A_1} \qquad (5)$$

From this we find immediately that:

(a) If T_s=0 (shaft is free-wheeling), then

$$\dot{m} = \frac{\rho\omega R A_1}{2} \qquad (6)$$

(b) If $\omega = 0$ (shaft is held fast by applying torque), then

$$\dot{m} = \sqrt{\frac{(-T_s)\rho A_1}{R}} \qquad (7)$$

Angular momentum theorem (equation 1):

$$T_s = \int_0^R \rho r \quad \omega r \quad u \quad 2\pi r dr - \rho R V_1 V_1 A_1 = \frac{\pi\rho\omega R^4 u}{2} - \rho R V_1^2 A_1 \qquad (2)$$

Mass conservation:

$$\rho V_1 A_1 = \dot{m} \tag{3}$$

$$\rho u \pi R^2 = \dot{m} \tag{4}$$

Eliminating the inflow and outflow velocities from (2) by using (3) and (4), we obtain

$$T_s = \dot{m}\frac{\omega R^2}{2} - \dot{m}^2 \frac{R}{\rho A_1} \tag{5}$$

From this we find immediately that:

(a) If T_s=0 (shaft is free-wheeling), then

$$\dot{m} = \frac{\rho \omega R A_1}{2} \tag{6}$$

(b) If $\omega = 0$ (shaft is held fast by applying torque), then

$$\dot{m} = \sqrt{\frac{(-T_s)\rho A_1}{R}} \tag{7}$$

Equations (6) and (7) show that mass flow can indeed be measured in two ways, but only if the density of the fluid is known. (If the density is known