Fundamental Laws of Motion for Particles, Material Volumes, and Control Volumes

Ain A. Sonin

Department of Mechanical Engineering
Massachusetts Institute of Technology
Cambridge, MA 02139, USA

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1 Basic Laws for Material Volumes

Material volumes and material particles

The behavior of a material distribution is controlled by universal physical laws. Perhaps the most ubiquitous of these are the mass conservation law, Newton's law of motion, and the first and second laws of thermodynamics. In this chapter we will review these four laws, starting with their most basic forms, and show how they can be rewritten in forms that apply to control volumes. These turn out to be very useful in engineering analysis[1]. The application of the control volume laws is illustrated in separate chapters.

The most fundamental forms of these four laws are stated in terms of a *material volume*. A material volume is a volume that contains the same particles of matter at all times[2]. A particular material volume may be defined by the closed bounding surface that envelops its material particles at a certain time. Since every point of the bounding surface moves with the local material velocity $\vec{v}$ (Fig. 1), the shape of the volume at all other times follows from the laws of dynamics.

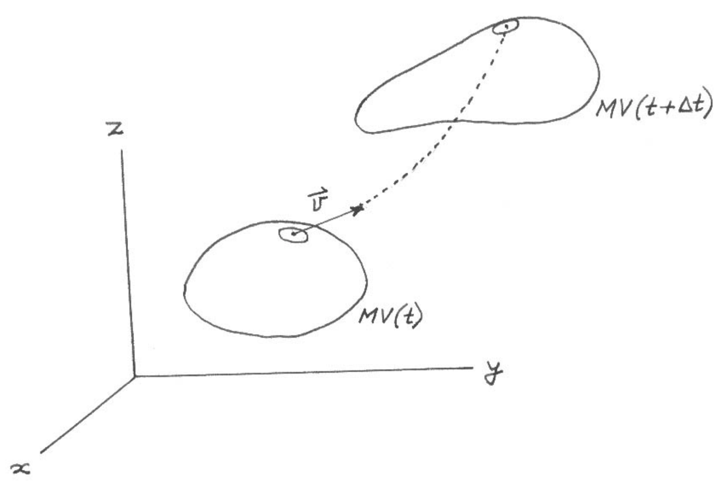

Fig. 1 A material volume moves with the material particles it encloses.

[1] For a historical note on control volume analysis in engineering, see Chapter 4 of Walter G. Vincenti's *What Engineers Know and How They know It*, John Hopkins University Press, 1990.
[2] A material volume is the same as a "closed system" in thermodynamics.

Laws for material particles

The simplest forms of the four basic laws apply to an infinitesimal *material particle* which is so small that the velocity $\vec{v}$, density ρ, thermodynamic temperature T, and other intrinsic properties are uniform throughout its volume. An observer moving with a particle ("sitting on it") would see the its properties change with time only (Fig. 2).

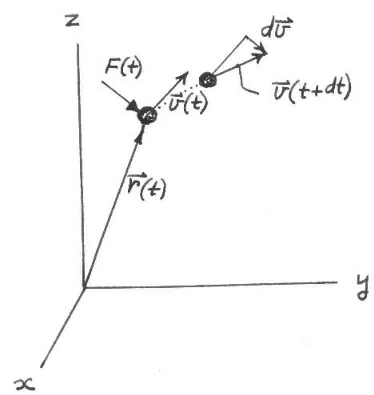

Fig. 2   Motion of a material particle between time t and time t+Δt

For a material particle with infinitesimal volume $\delta V(t)$, density $\rho(t)$, and velocity $\vec{v}$, the four laws have the following familiar forms:

<u>Mass conservation</u>

$$\frac{d}{dt}(\rho\delta V) = 0 \tag{1}$$

The mass $\delta M = \rho\delta V$ of a material particle remains invariant. The prefix δ indicates quantities that are of infinitesimal size, and the prefix d refers to changes that occur in the indicated property in time dt.

<u>Newton's law of (non-relativistic) linear motion</u>

$$(\rho\delta V)\frac{d\vec{v}}{dt} = \delta\vec{F}, \quad \text{or} \quad \frac{d}{dt}(\rho\vec{v}\,\delta V) = \delta\vec{F} \tag{2}$$

Relative to an inertial reference frame[3], the product of a particle's mass and acceleration is at every instant equal to the net force $\delta\vec{F}(t)$ exerted on it by the rest of the universe. Alternatively: The rate of change of a particle's momentum (a vector quantity) is equal at every instant to the force applied to it by the rest of the universe.

Newton's law applied to angular motion

$$\frac{d}{dt}\left(\vec{r}\times\rho\vec{v}\,\delta V\right)=\vec{r}\times\delta\vec{F} \qquad (3)$$

The rate of change of a particle's angular momentum (the quantity in brackets on the left, $\vec{r}(t)$ being the particle's position vector) is at every instant equal to the net torque exerted on the particle by the rest of the universe. This is not a new law, but one that follows from Eq. (2). Equation (3) is obtained by taking the cross product of $\vec{r}(t)$ and Eq. (2), using Eq. (1), and noting that $d\vec{r}/dt\times\vec{v}=\vec{v}\times\vec{v}=0$. Like the law it is derived from, Eq. (3) is valid only in inertial reference frames.

First law of thermodynamics

$$d(\rho e_t\delta V)=\delta W+\delta Q \qquad (4)$$

The increase of a material particle's total energy in a time interval *dt* (e_t is its total energy per unit mass) is equal to the work done in the interval dt by forces exerted by the rest of the universe on the material volume's *boundary* (that is, not counting work done by volumetric body forces), plus the heat δQ added to the particle at its boundary during this interval.

Second law of thermodynamics

$$d\left(\rho s\,\delta V\right)\geq\frac{\delta Q}{T} \qquad (5)$$

[3] An inertial reference frame is one in which the particle would move at a perceptibly constant velocity if all the accounted forces were removed.

The increase of a particle's entropy (s represents the particle's entropy per unit mass) in a time dt is greater than or equal to the heat added to the particle at its boundary during this interval divided by the absolute (thermodynamic) temperature T [4].

Laws for finite material volumes

From Eqs (1)-(5), which apply to an infinitesimal material particle, we can derive the laws for a finite material volume like the one sketched in Fig. 1. This is accomplished by applying a particular law to each of the material particles that comprise the volume under consideration, and summing. In the limit of a continuum, the sum can be viewed as an integral over the volume of material properties which are expressed as fields (that is, as functions of position $\vec{r}$ and time t), consistent with the Eulerian way of describing material flows.

The result is the following set of *rate equations*[5] for a material volume's mass, momentum, energy, and entropy:

<u>Mass conservation</u>

$$\frac{d}{dt}\int_{MV(t)}\rho(\vec{r},t)dV = 0 \quad . \tag{6}$$

The mass contained in a material volume remains invariant. $\rho(\vec{r},t)$ is the material's density field, *dV=dxdydz* represents a volume element inside the material volume, and *MV(t)* under the integral sign signifies integration over the material volume at time *t*.

<u>Motion (linear momentum)</u>

$$\frac{d}{dt}\int_{MV(t)}\rho(\vec{r},t)\vec{v}(\vec{r},t)dV = \vec{F}_{MV}(t). \tag{7}$$

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[4] According to the Second Law the temperature in Eq. (5) should be that of the "reservoir" from which the heat is supplied to the material volume. In this case the heat comes from the material that bounds the infinitesimal particle, where the temperature differs infinitesimally from the particle's own average temperature T.

[5] The usual term "conservation laws" is a bit of a misnomer, since mass is the only one of these quantities that is actually conserved in general.

This is Newton’s law of motion: The rate of change of a material volume's momentum, evaluated by integrating the local momentum per unit volume $\rho\vec{v}$ over the material volume, is at every instant equal to the vector sum $\vec{F}_{MV}(t)$ of all the forces exerted on the material volume *by the rest of the universe*. This force includes body forces acting on the material within the volume and surface forces acting at the boundary, but *not* the forces exerted between particles within the volume, which cancel each other out when the sum over all the constituent parts is taken (action is equal and opposite to reaction, according to Newton). It is understood that Eq. (7) applies only in inertial (unaccelerating) reference frames under non-relativistic conditions.

Motion (angular momentum)

$$\frac{d}{dt}\int_{MV(t)} \rho\vec{r} \times \vec{v}\,dV = \vec{T}_{MV}(t) \qquad (8)$$

This equation is obtained by summing the angular momentum law for a material particle, Eq. (3), over all the particles that comprise a finite material volume. The law states that the rate of increase of a material volume’s angular momentum, expressed as the integral over the volume of the angular momentum per unit volume, is equal to the vector sum $\vec{T}_{MV}(t)$ of all torques exerted *by the rest of the universe* on the material volume. (This form of the law assumes that the torques exerted between two particles within the volume are equal and opposite, or zero.) Note again that Eq. (8) is not a new law, but a corollary of Newton’s law of motion and subject to the same restrictions.

First law of thermodynamics

$$\frac{d}{dt}\int_{MV(t)} \rho e_t\,dV = \dot{Q}_{MS}(t) + \dot{W}_{MS}(t), \qquad (9)$$

This law is obtained by summing Eq. (4) over all the particles that comprise the material volume and noting that the particle-to-particle heat transfer and work terms cancel for all particles *inside* the material volume when the sum is taken. The law states that the rate of increase of a material volume's energy (e_t is the total energy per unit mass—internal plus kinetic plus gravitational) is equal to the sum of two “source terms” which represent interactions with the rest of the universe *at the volume’s boundary*. The first source term is the net heat flow rate *into* the material volume across its bounding surface

$$\dot{Q}_{MS}(t) = -\int_{MS(t)} \vec{q} \cdot \vec{n}\, dA \ , \tag{10}$$

where

$$\vec{q} = -k\nabla T \tag{11}$$

is the conductive heat flux vector at a point on the material volume's boundary, *k* being the material's thermal conductivity, *T* its local thermodynamic temperature, $\vec{n}$ the outward-pointing unit vector at the bounding surface, *dA* an elemental area on the bounding surface, and the symbol *MS(t)* denoting integration over the closed bounding surface of the material volume at time t. The second source term is the rate at which work is done by the rest of the universe *on* the material volume *at its boundary*, which may be evaluated as

$$\dot{W}_{MS}(t) = \int_{MS(t)} \vec{\sigma} \cdot \vec{v}\, dA \tag{12}$$

were $\vec{\sigma}$ is the vector stress exerted on the boundary by the rest of the universe and $\vec{v}$ is the material's local velocity at *dA*. The quantity $\vec{\sigma} dA$ is the force exerted by the rest of the universe on the surface element dA of the control volume.

<u>Second law of thermodynamics</u>

$$\frac{d}{dt} \int_{MV(t)} \rho s\, dV \geq -\int_{MS(t)} \frac{\vec{q} \cdot \vec{n}\, dA}{T} \tag{13}$$

The rate of increase of a material volume's total entropy is greater than or equal to the sum of all the local heat inflows at the boundary when each contribution is divided by the local *thermodynamic* (absolute) temperature at the point on the material volume's surface where the transfer takes place.

This law provides a *bounding value* of the rate of entropy increase, but not the actual value, and is less useful in dynamics than the other laws. It does, however, have some important uses in dynamics. One can for example discard from the solutions that satisfy the other physical laws those that are not realizable because they violate the Second Law, and one can predict limiting cases of negligible dissipation, where the equality sign applies.

2 The Transformation to Control Volumes

The control volume

Equations (6)-(9) and (13) state universal laws that apply to all material distributions. They are, however, in a form which makes them ill suited for applications. Each equation contains a term of the form

$$\frac{d}{dt}\int_{MV(t)}\phi(\vec{r},t)dV \qquad (14)$$

in which a quantity $\phi(\vec{r},t)$ that represents something per unit volume—mass, momentum, energy, or entropy—is first integrated over a material volume and the result then differentiated with respect to time. When the material is flowing and deforming, the volume's boundary moves with it and is not known as a function of time until the problem is solved. It seems, therefore, that one must know the solution before one can apply these laws to find the solution. Clearly, we need to find a way of applying the basic laws to systems of our own choice, that is, to "control volumes."

A *control volume* is an arbitrarily defined volume with a closed bounding surface (the control surface) that separates the universe into two parts: the part contained within the control volume, and the rest of the universe. The control surface is a mental construct, transparent to all material motion, and may be static in the chosen reference frame, or moving and expanding or contracting in any specified manner. The analyst specifies the velocity $\vec{v}_c(\vec{r},t)$ at all points of the control surface for all time.

We shall show next how the universal laws for a material volume can be rewritten in terms of an arbitrarily defined control volume. This opens the way to the application of the integral laws in engineering analysis.

Rate of change of a volume integral over a control volume

We begin by considering a time derivative like Eq. (14) for a control volume rather than a material volume. The time rate of change of the integral of some field quantity $\phi(\vec{r},t)$ over an arbitrarily defined control volume *CV(t)* is by definition

$$\frac{d}{dt}\int_{CV(t)}\phi dV = \lim_{\Delta t\to 0}\frac{\int_{CV(t+\Delta t)}\phi(\vec{r},t+\Delta t)dV - \int_{CV(t)}\phi(\vec{r},t)dV}{\Delta t}. \tag{15}$$

The first integral on the right hand side is evaluated at the advanced time over the advanced volume, and the second is evaluated at time t over the volume at time t (Fig. 3). At any point $\vec{r}$ we can write for small Δt

$$\phi(\vec{r},t+\Delta t) = \phi(\vec{r},t) + \frac{\partial\phi}{\partial t}\Delta t. \tag{16}$$

Inserting this into Eq. (15) we see immediately that

$$\frac{d}{dt}\int_{CV(t)}\phi dV = \int_{CV(t)}\frac{\partial\phi(\vec{r},t)}{\partial t}dV + \lim_{\Delta t\to 0}\frac{\int_{CV(t+\Delta t)}\phi(\vec{r},t)dV - \int_{CV(t)}\phi(\vec{r},t)dV}{\Delta t} \tag{17}$$

where the integrals on the right are evaluated based on the values of $\partial\phi/\partial t$ and ϕ at time *t*. In the limit $\Delta t\to 0$, the difference between the two volume integrals in the second term can be evaluated (see Fig. 3) by means of an integral over the material surface at time t:

$$\int_{CV(t+\Delta t)}\phi(\vec{r},t)dV - \int_{CV(t)}\phi(\vec{r},t)dV = \int_{CS(t)}\phi(\vec{r},t)\vec{v}_c\Delta t\cdot\vec{n}dA\ . \tag{18}$$

Here $\vec{v}_c(\vec{r},t)$ is the velocity of the *control surface* element *dA*, $\vec{n}$ is the outwardly-directed unit normal vector associated with *dA*, and $\vec{v}_c\cdot\vec{n}\Delta t dA$ is the control volume size increase in time Δt due to the fact that the surface element dA has moved in that time interval. The integral on the right side is taken over the entire (closed) bounding surface *CS(t)* of the control volume.

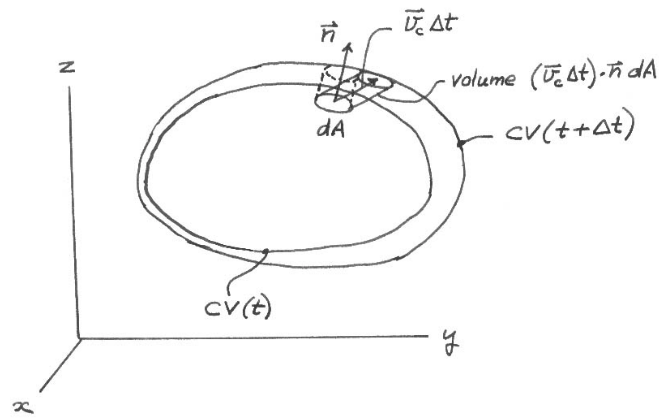

Fig. 3 Motion of a control volume between t and t+Δt for small Δt.

Substituting Eq. (18) into Eq. (17), we obtain for an arbitrarily chosen control volume *CV(t)*,

$$\frac{d}{dt}\int_{CV(t)} \phi dV = \int_{CV(t)} \frac{\partial \phi(\vec{r},t)}{\partial t} dV + \int_{CS(t)} \phi(\vec{r},t)\vec{v}_c \cdot \vec{n} dA \tag{19}$$

Rate of change of a volume integral over a material volume

The corresponding equation for a material volume *MV(t)* can be obtained simply by noting that a material volume is a control volume every point of which moves with the material velocity. Equation (19) thus applies to a material volume if we set the control volume velocity equal to the material velocity, $\vec{v}_c = \vec{v}$, and identify the limits of integration with the material volume. This yields for a material volume

$$\frac{d}{dt}\int_{MV(t)}\phi dV = \int_{MV(t)}\frac{\partial\phi(\vec{r},t)}{\partial t}dV + \int_{MS(t)}\phi(\vec{r},t)\vec{v}\cdot\vec{n}dA \qquad (20)$$

Reynolds' material-volume to control-volume transformation theorem

Reynolds' transformation theorem provides a recipe for transforming the fundamental laws in Eqs. (6)-(9) and (13) to control volumes. The transformation theorem is obtained by considering a control volume at time *t* and the material volume which coincides with it at that instant. The control volume *CV(t)* is chosen arbitrarily by defining its closed bounding surface *CS(t)*. The material volume is comprised of all the matter inside the control volume at time *t* (Fig. 4). The two volumes will of course diverge with time since the material volume wafts off with the particles to which it is "attached" and the control volume moves according to our specification. This is of no consequence since we are considering only a "frozen" instant when the two volumes coincide.

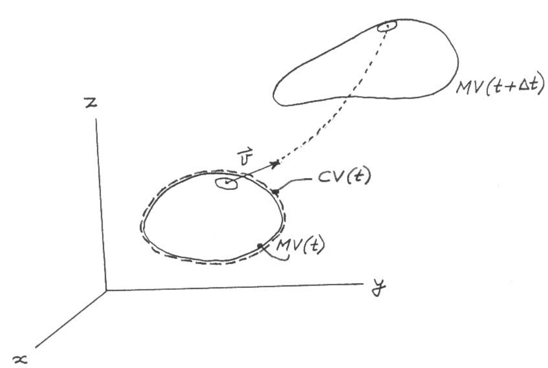

Fig. 4 The control and material volumes in the transformation theorem

We apply Eq. (19) to our *CV*, and Eq. (20) to the *MV* that coincides with it at time *t* , and note that because the volumes coincide, the integrals on the right-hand side of Eq. (20) may be evaluated over either the *CV* instead of the *MV*. This yields two alternative equations for the time derivative of an integral over a material volume, expressed in terms of a *CV* that coincides with the material at the time involved:

Form A $$\frac{d}{dt}\int_{MV(t)}\phi dV = \frac{d}{dt}\int_{CV(t)}\phi dV + \int_{CS(t)}\phi(\vec{v}-\vec{v}_c)\cdot\vec{n}dA \qquad (21)$$

Form B $$\frac{d}{dt}\int_{MV(t)}\phi dV = \int_{CV(t)}\frac{\partial\phi}{\partial t}dV + \int_{CS(t)}\phi\vec{v}\cdot\vec{n}dA \tag{22}$$

Equation (21) is obtained by subtracting Eq. (19) from Eq. (20). Equation (22) is Eq. (20) with the integrals referred to the *CV* instead of the *MV*, the two being coincident.
Recall that $\vec{v}$ is the local material velocity, $\vec{v}_c$ is the local control surface velocity at the surface element *dA*, and $\vec{n}$ is the *outward*-pointing unit normal vector associated with *dA*..

Both forms A and B are *valid for arbitrarily moving and deforming control volumes* (i.e. control volumes that may be expanding, translating, accelerating, or whatever), and for unsteady as well as steady flows. The two forms express exactly the same thing, but do the bookkeeping in different ways.

Remember that ϕ represents something per unit volume. Both forms express the material-volume time derivative on the left as a sum of two terms that refer to the control volume that coincides with the material volume at the instant *t*. In form A, the first term on the right is the rate of change of the amount of ϕ inside the control volume at time t (the volume integral is evaluated first, then the time derivative), and the second term is the net rate of outflow of ϕ across the control volume's boundary.

In form B, the first term on the right is the volume integral of the *partial* time derivative of ϕ over the control volume at time t (the *CS* is held fixed at its position at time t while the integration is performed). The second term accounts for the fact that the material volume's boundary (on the left) does not in fact maintain the shape it has at time t, but envelops more volume (and more of the quantity ϕ) when it expands, every point moving with the local material velocity $\vec{v}$. *The control surface velocity does not enter at all in form B.*

We shall see that Form A is usually more convenient in unsteady applications than Form B. This is particularly true in cases where $\partial\phi/\partial t$ is singular at some surface inside the control volume (as it is at a moving flame front inside a solid-propellant rocket, for example, if ϕ is the material density distribution in the rocket), in which case it is difficult to evaluate the volume integral in Form B. The volume integral in From A, on the other hand, can be calculated straightforwardly and then differentiated with respect to time.

3 Basic laws for control volumes.

The basic physical laws expressed by Eqs (6)-(9) and (13) in material-volume terms are transferred to a control volume as follows. We transform the left sides by setting ϕ equal to either ρ, $\rho\vec{v}$, $\vec{r}\times\rho\vec{v}$, e_t, or *s*, in form A or form B of Reynolds' transformation theorem [Eqs. (21) and (22)]. The right hand sides are transformed by noting that since the *MS* and *CS* coincide at the instant being considered (see Fig. 4), the force, torque, and heat flow terms on the right hand side of Eqs (7)-(9) and (13) are the same for the *CV* as for the *MV*. (Note, however, that the rate at which work is being done on the *CS* is *not* equal to the rate at which work is being done on the *MS* because these surfaces move at different velocities.)

Two alternative forms are obtained for each equation, depending on whether Form A [Eq. (21)] or form B [Eq. (22)] of the transformation theorem is used. The alternative forms are expressions of the same physical law, stated in somewhat different terms. Both apply to *any* control volume at every instant in time no matter how the control surface is moving and deforming, provided the reference frame is one where the basic equations apply.

We remind the reader (see Fig. 3) that in what follows,

$$v_n = \vec{v}\cdot\vec{n} = v\cos\theta \tag{23}$$

is the outward normal component of the material's *absolute* velocity at the control surface, θ being the angle between $\vec{v}$ and the outward-pointing normal unit vector $\vec{n}$, and

$$v_{rn} = (\vec{v}-\vec{v}_c)\cdot\vec{n} = v_n - v_{cn} \tag{24}$$

is the outward normal component of the material's velocity *relative to the control surface*, v_{cn} being the outward normal component of the control surface's velocity.

<u>Mass conservation</u>

Setting $\phi=\rho$ in Eqs. (21) and (22), we transform Eq. (6) into two alternative forms for a CV:

Form A $$\frac{d}{dt}\int_{CV(t)} \rho dV + \int_{CS(t)} \rho v_{rn} dA = 0 \quad (25A)$$

Form B $$\int_{CV(t)} \frac{\partial \rho}{\partial t} dV + \int_{CS(t)} \rho v_n dA = 0 \quad (25B)$$

Equation (25A) states the mass conservation principle as follows: The rate of increase of the mass contained in the *CV*, plus the *net* mass flow rate out through the (generally *moving*) *CS*, equals zero at every instant.

Equation (25B) states the same principle in different but equally correct terms: The rate of increase of the mass contained in the *fixed* volume defined by the control surface at time t, plus the net mass outflow rate through the *fixed* bounding surface of that volume, equals zero at all times.

<u>Linear momentum</u>

Putting $\phi = \rho\vec{v}$ in Eqs. (21) and (22) and substituting into (7), we obtain the following alternative forms for the equation of motion applied to a *CV*:

Form A $$\frac{d}{dt}\int_{CV(t)} \rho\vec{v} dV + \int_{CS(t)} \rho\vec{v} v_{rn} dA = \vec{F}_{CV}(t) \quad (26A)$$

Form B $$\int_{CV(t)} \frac{\partial(\rho\vec{v})}{\partial t} dV + \int_{CS(t)} \rho\vec{v} v_n dA = \vec{F}_{CV}(t) \quad (26B)$$

Here, $\vec{F}_{CV}(t)$ is the vector sum of all the forces exerted at time t by the rest of the universe on the control volume, including volumetric forces and stresses exerted on the control volume's boundaries. For a continuous distribution of surface and body forces,

$$\vec{F}_{CV}(t) = \int_{CS} \vec{\sigma} dA + \int_{CV} \rho\vec{G} dV \quad (27)$$

Equation (26A) states that the rate at which the linear momentum contained in the *CV* increases with time, plus the net flow rate of linear momentum *out* through the control surface, is equal at every instant to the force exerted by the rest of the universe on the material within the control surface.

Equation (26B) states it in different terms: The rate of increase of the momentum contained in a *fixed* volume identical with the control surface at time t, plus the net mass outflow rate through the *fixed* bounding surface of that volume, is equal at all times to the force exerted by the rest of the universe on the material in the control volume.

Angular momentum

Setting $\phi = \rho\vec{r} \times \vec{v}$ in either (21) or (22) and substituting into (8) yields the angular momentum theorem for a *CV* in two alternative forms:

Form A $$\frac{d}{dt}\int_{CV(t)}(\rho\vec{r} \times \vec{v})dV + \int_{CS(t)}(\rho\vec{r} \times \vec{v})v_{rn}dA = \vec{T}_{CV}(t) \qquad (28A)$$

Form B $$\int_{CV(t)}\frac{\partial}{\partial t}(\rho\vec{r} \times \vec{v})dV + \int_{CS(t)}(\rho\vec{r} \times \vec{v})v_{n}dA = \vec{T}_{CV}(t) \qquad (28B)$$

Here $\vec{r}$ is the position vector from an arbitrary origin, $\vec{T}_{CV}(t)$ is the sum of all the torques (relative to the chosen origin) that the rest of the universe exerts on the control volume, including those resulting from both surface forces (pressure and shear) and volumetric body forces (e.g. gravity). An inertial reference frame is presumed. For a continuous distribution of surface and body forces,

$$\vec{T}_{CV}(t) = \int_{CS}\vec{r} \times \vec{\sigma}dA + \int_{CV}\rho\vec{r} \times \vec{G}dV\,. \qquad (29)$$

where $\vec{\sigma}$ is the vector stress exerted on the boundary element *dA* by the rest of the universe, and $\vec{G}$ is the body force exerted by the rest of the universe on unit mass of material within the volume.

Equation (28A) states the following: The rate at which the angular momentum inside the control volume increases with time, plus the net rate at which angular momentum flows out of the control surface, is equal to the net torque exerted by the rest of the universe on the matter in the control volume (on the boundary as well as the mass within). The reader will be able to interpret (28B) based on the comments been made above with reference to (25B) and (26B).

First law of thermodynamics (energy equation)

Setting $\phi = e_t$ in Eqs. (21) and (22) and substituting into (9), we obtain two forms of the first law for a *CV:*

$$\text{Form A} \qquad \frac{d}{dt}\int_{CV(t)} \rho e_t dV + \int_{CS(t)} \rho e_t v_{rn} dA = -\int_{CS(t)} \vec{q}\cdot\vec{n} dA + \int_{CS(t)} \vec{\sigma}\cdot\vec{v} dA \qquad (30A)$$

$$\text{Form B} \qquad \int_{CV(t)} \frac{\partial(\rho e_t)}{\partial t} dV + \int_{CS(t)} \rho e_t v_n dA = -\int_{CS(t)} \vec{q}\cdot\vec{n} dA + \int_{CS(t)} \vec{\sigma}\cdot\vec{v} dA \qquad (30B)$$

Equation (30A) states that the rate at which the total energy contained in the *CV* increases with time, plus the net rate at which total energy flows out of the *CS*, is equal to the sum of two terms on the right. The first term is the rate at which heat is conducted *into* the *CV* via the control surface. The second is the rate at which the rest of the universe does work on the *material volume* whose bounding surface coincides with the *CS* at the instant in question. The work done at the *control surface*,

$$\dot{W}_{CS}(t) = \int_{CS(t)} \vec{\sigma}\cdot\vec{v}_c dA \ , \qquad (31)$$

depends on the control surface velocity distribution, which is chosen at will by the analyst and obviously has no place in a law that pretends to universal obeisance.

Second law of thermodynamics

$$\text{Form A} \qquad \frac{d}{dt}\int_{CV(t)} \rho s dV + \int_{CS(t)} \rho s v_{rn} dA \geq -\int_{CS(t)} \frac{\vec{q}\cdot\vec{n}}{T} dA \qquad (32A)$$

$$\text{Form B} \qquad \int_{CV(t)} \frac{\partial(\rho s)}{\partial t} dV + \int_{CS(t)} \rho s v_n dA \geq -\int_{CS(t)} \frac{\vec{q}\cdot\vec{n}}{T} dA \qquad (32B)$$

Equation (32A) states that the rate of increase of the entropy contained in the *CV* (*s* is the entropy per unit mass), plus the net rate of entropy convection out of the control surface, never exceeds the integral over the control surface of the normal heat influx divided by the local absolute temperature.

4 Procedure for Control Volume Analysis

The application of any one of the integral laws involves consideration of the following nine steps:

Step 1

Identify the reference frame in which the problem is viewed and velocity and other properties are measured. If Newton's law is involved in the problem, the reference frame must be an inertial (non-accelerating) frame.

Step 2

Identify your chosen control surface by specifying its location at some instant (e.g. $t=0$) and at all times thereafter. The surface *must* be closed. It may be multiply connected. It may move in the chosen reference frame and expand and distort as it does so. All this is your choice. If the *CS* runs parallel to a fluid-solid interface, take care to specify whether your control surface is just on the fluid side, or just on the solid side. It must be on one side or the other, so that quantities like ρ, $\vec{v}$, e_t, etc. have well defined values.

Step 3

Write down the integral law that you wish to apply.

Step 4

Identify the local values of the properties $\rho, \vec{v}, \vec{v}_c, e_t, \vec{\sigma}, \vec{q}$, and s at every element dA of the control surface (that is, determine their *distributions* over the control surface), and calculate the surface integrals. Select the control volume so that the bounding surface passes as much as possible through regions where you know the properties, or can easily deduce them. Wherever you don't know some quantities, introduce them as unknowns, expecting to determine them as you proceed.

Step 5

Identify the local values of ρ, $\vec{v}$, $\vec{\sigma}, e_t, s$ and $\vec{G}$ at every volume element *dV* inside the control volume (determine their distributions inside the *CV*, and evaluate the volume integrals in your integral equation.

Step 6

Calculate the time derivative of the volume integral that appears on left-hand side of your integral equation.

Step 7

From steps (4), (5) and (6), substitute into your integral equation.

Step 8

If you wish to solve a practical problem using the control volume theorems, you must write down enough equations to ensure that their number equals the number of unknowns in the equations. The four integral laws that we have described are totally general and rigorous, but the laws themselves will in most cases not provide enough equations to solve for the unknowns. You will need to draw on other physical laws (e.g. gravitational theory to characterize the external body force field) and constitutive equations (e.g. the thermodynamic equations of state). Above all you will need to *make simplifying approximations* wherever they are appropriate. Uniform flow conditions over any given cross-sections (quasi-one-dimensional flow) is a typical approximation, for integral relations by themselves provide no information about distributions of properties. If you believe the flow may be approximated as inviscid, you invoke Bernoulli's equation. If based on the equation of state you think density varies little, you write ρ=constant. (Note: Bernoulli's equation is derived from Newton's equation of motion, just like the linear momentum theorem. By adding Bernoulli, are we not simply writing down the same equation twice? We are not. The linear momentum equation expresses a condition that *applies generally* . Bernoulli's equation expresses the *additional constraint* that the flow is inviscid.) Whenever approximations are made, justification should be provided à posteriori.

Step 9

Solve for the unknowns.