Solving Stochastic Advection Diffusion Equation Using HDG Method

Ali Saab
Numerical Fluid Mechanics
Outline

• Stochastic Advection Diffusion Equation
• Hybridized Discontinuous Galerkin Method
  o Background
  o Discretization of The Advection Diffusion Equation
  o Global Equation
  o Examples
• Dynamically Orthogonal Field Equations
  o Definition
  o DO Field Equations for The Advection Diffusion Equation
Stochastic Advection Diffusion Equation

• Advection Diffusion Equation

\[
\frac{\partial f}{\partial t} + \nabla (uf) = \nabla .(D \nabla f)
\]

• The flow field is assumed stochastic, i.e. \( u = u(x, t; \omega) \) and \( f = f(x, t; \omega) \)

• The 1D Stochastic Advection Diffusion Equation is

\[
\frac{\partial f(x, t; \omega)}{\partial t} + \frac{\partial (u(x, t; \omega) f(x, t; \omega))}{\partial x} = \frac{\partial}{\partial x} \left( \frac{D(x, t) \partial f(x, t; \omega)}{\partial x} \right)
\]
Hybridized Discontinuous Galerkin Method

- Finite Element Methods are accurate but slow.
- HDG is competitive to CG while retaining the properties of DG.
- Each element can be solved locally given the boundary conditions.
- Solving for the boundary condition is possible by equating the fluxes on each edge to the total flux into the system.
Spatial Discretization – 1D

• Definitions
  
  • \((a, b)_{\Delta x_i} = \int_{x_i}^{x_{i+1}} ab \, dx\)
  
  • \(\lambda = \hat{f}\) is the value of \(f\) at the boundary
  
  • \(f = \sum_{i=1}^{N_e} f_i \theta_i\)

• Let \(\theta\) be a test function, then multiply the equation by the test function and integrate

\[
\int_{x_i}^{x_{i+1}} \frac{\partial f}{\partial t} \theta \, dx + \int_{x_i}^{x_{i+1}} \frac{\partial (uf)}{\partial x} \theta \, dx - \int_{x_i}^{x_{i+1}} \frac{\partial}{\partial x} \left( D \frac{\partial f}{\partial x} \right) \theta \, dx = 0
\]
Spatial Discretization - 1D

• Integrate the diffusion term by parts twice

\[
\int_{x_i}^{x_{i+1}} \frac{\partial}{\partial x} \left( D \frac{\partial f}{\partial x} \right) \theta dx = D \frac{\partial \hat{f}}{\partial x} . \tilde{n} \theta |_{x_i}^{x_{i+1}} - D \frac{\partial f}{\partial x} . \tilde{n} \theta |_{x_i}^{x_{i+1}} + \int_{x_i}^{x_{i+1}} \frac{\partial}{\partial x} \left( D \frac{\partial f}{\partial x} \right) \theta dx
\]

• The Diffusion is modeled as follows

\[
D \frac{\partial \hat{f}}{\partial x} = D \frac{\partial f}{\partial x} . \tilde{n} - \tau (f - \lambda) \tilde{n}
\]
Spatial - time Discretization – 1D

• The strong form of the Advection Diffusion Equation is

\[
\frac{(f, \theta)^{k+1}_{\Delta x_i}}{\Delta t} + \tau \theta f^{k+1} \bigg|_{x_i} + \tau \theta^{k+1} \bigg|_{x_{i+1}} - \left( \frac{\partial}{\partial x} \left( D \frac{\partial f}{\partial x}, \theta \right) \right)^{k+1}_{\Delta x_i}
\]

\[
= \tau \lambda^{k+1} \theta \bigg|_{x_i} + \tau \lambda^{k+1} \theta \bigg|_{x_{i+1}} - \left( \frac{\partial (uf)}{\partial x}, \theta \right)^{k}_{\Delta x_i} + \frac{(f, \theta)^k_{\Delta x_i}}{\Delta t}
\]

• Final Finite Element Equation

\[
f^{k+1}_i \left[ \frac{(\theta_i, \theta_j)_{\Delta x_i}}{\Delta t} + \tau \theta_i \theta_j \bigg|_{x_i} + \tau \theta_i \theta_j \bigg|_{x_{i+1}} - \left( \frac{\partial}{\partial x} \left( D \frac{\partial \theta_i}{\partial x}, \theta_j \right) \right)_{\Delta x_i} \right]
\]

\[
= \tau \lambda^{k+1} \theta_j \bigg|_{x_i} + \tau \lambda^{k+1} \theta_j \bigg|_{x_{i+1}} - \left( \frac{\partial (uf^{k})}{\partial x}, \theta_j \right) \Delta x_i + \frac{(f^k, \theta_j)_{\Delta x_i}}{\Delta t}
\]
Local Finite Element Equation Summary

\[ A_{ij}^{local} = \frac{(\theta_i, \theta_j) \Delta x_i}{\Delta t} + \tau \theta_i \theta_j \bigg|_{x_i} + \tau \theta_i \theta_j \bigg|_{x_{i+1}} - \left( \frac{\partial}{\partial x} \left( D \frac{\partial \theta_i}{\partial x} \right), \theta_j \right) \Delta x_i \]

\[ b_j^{local} = \tau \lambda^{k+1} \theta_j \bigg|_{x_i} + \tau \lambda^{k+1} \theta_j \bigg|_{x_{i+1}} - \left( \frac{\partial (uf^k)}{\partial x}, \theta_j \right) \Delta x_i + \frac{(f^k, \theta_j) \Delta x_i}{\Delta t} \]

\[ A^{local} f = b^{local} \]
Global Equation

• Definition
  
  • $[[a, \vec{n}]] = a^+ \cdot \vec{n}^+ + a^- \cdot \vec{n}^-$
  
  • $< a, b >_e = \int_e ab \, de$
  
  • $< a, b >_\varepsilon = \sum_{e \in \varepsilon} < a, b >_e$

• By Equating the fluxes of the boundary to the total flux we arrive at
the global Flux Equation

$$\left\langle \left[ \left[ \left( D \frac{\partial f}{\partial x} \cdot \vec{n} - \tau (f - \lambda) \vec{n} \right) \cdot \vec{n} \right] \right], \theta_\varepsilon \right\rangle \equiv < g_N, \theta_\varepsilon >_\varepsilon$$
Global Finite Element Equation

• Assume \( f \approx f^F + \sum f^{\lambda_i} \)
• \( f^F \) is the effect of the forcing terms only
• \( f^{\lambda_i} \) is the effect of the boundary only

\[
A^{global}_{i,j} = \left\langle \left[ \left[ \left( \frac{\partial f^{\lambda_i}}{\partial x} \cdot \vec{n} - \tau (f^{\lambda_i} - \delta_{ij} \theta_{\epsilon,i}) \vec{n} \right) \cdot \vec{n} \right] \right] , \theta_{\epsilon,j} \right\rangle_{\epsilon}
\]

\[
b^{global}_{j} = \langle g_N , \theta_{\epsilon,j} \rangle_{\epsilon} - \left\langle \left[ \left[ \left( \frac{\partial f^F}{\partial x} \cdot \vec{n} - \tau (f^F) \vec{n} \right) \cdot \vec{n} \right] \right] , \theta_{\epsilon,j} \right\rangle_{\epsilon}
\]

\( A^{global} \lambda = b^{global} \)
Example 1
Example 2
Dynamically Orthogonal Field Equations

• The response of the dynamical system is assumed to have the form

\[ f(x, t; \omega) = \bar{f}(x, t) + \sum_{i=1}^{r_f} \zeta_i(t; \omega) f_i(t, x) = \bar{f} + \zeta_i f_i \]

• And the stochastic term \( u \) can be written as

\[ u(x, t; \omega) = \bar{u}(x, t) + \sum_{k=1}^{r_u} \beta_k(t; \omega) u_k(t, x) = \bar{u} + \beta_k u_k \]

• \( \zeta_i \) and \( \beta_k \) are zero mean stochastic processes
DO Condition

• The DO condition is defined as

\[
\left\langle \frac{\partial f_i(x, t)}{\partial t}, f_j(x, t) \right\rangle = 0
\]

• The above condition implies

\[
\frac{\partial}{\partial t} \left\langle f_i(x, t), f_j(x, t) \right\rangle = 0
\]

• \(\{f_i(x, t)\}_{i=1}^s\) are deterministic fields which are initially orthonormal
Do Field Equations For The Advection Diffusion Equation

\[
\frac{\partial \bar{f}}{\partial t} = \frac{\partial \bar{f}}{\partial x} \left( \frac{\partial D}{\partial x} - \bar{u} \right) + D \frac{\partial^2 f}{\partial x^2} - \frac{\partial f_i}{\partial x} u_k Cov(\beta_k, \zeta_i)
\]

\[
\frac{\partial \zeta_j}{\partial t} = - \zeta_i < \bar{u} \frac{\partial f_i}{\partial x}, f_j > - \beta_k < \frac{\partial \bar{f}}{\partial x} u_k, f_j > - \beta_k \zeta_i < u_k \frac{\partial f_i}{\partial x}, f_j >
\]

\[
Cov(\beta_k, \zeta_i) < \frac{\partial f_i}{\partial x} u_k, f_j > + \zeta_i < \frac{\partial D}{\partial x} \frac{\partial f_i}{\partial x}, f_j > + \zeta_i < D \frac{\partial^2 f_i}{\partial x^2}, f_j >
\]
Do Field Equations For The Advection Diffusion Equation

\[
\frac{\partial f_j}{\partial t} = \frac{\partial f_j}{\partial x} \left( \frac{\partial D}{\partial x} - \bar{u} \right) + D \frac{\partial^2 f_j}{\partial x^2} \\
+ < \frac{\partial f_l}{\partial x} \left( - \frac{\partial D}{\partial x} \bar{u} \right) - D \frac{\partial^2 f_l}{\partial x^2}, f_j > f_j \text{Cov}(\zeta_l \zeta_i) \text{Cov}^{-1}(\zeta_j, \zeta_i) \\
\left( < \frac{\partial f}{\partial x} u_k, f_j > f_j - \frac{\partial f}{\partial x} u_k \right) \text{Cov}(\beta_k, \zeta_i) \text{Cov}^{-1}(\zeta_j, \zeta_i) \\
+ < u_k \frac{\partial f_l}{\partial x}, f_j > f_j M_3(\beta_k, \zeta_l, \zeta_i) \text{Cov}^{-1}(\zeta_j, \zeta_i) \\
- u_k \frac{\partial f_j}{\partial x} M_3(\beta_k, \zeta_l, \zeta_i) \text{Cov}^{-1}(\zeta_j, \zeta_i)
\]
Future Work

- Solve higher dimensional problems (2D, 3D) using HDG methods
- Solve the DO Field Equations
- Compare the results with Monte Carlo Simulations