Application of the Continuous Galerkin Finite Element Method to 2D Compressible Flow

2.29 Class Project

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Motivation

- Goal: implement a 2D solver for PDEs with any combination of the following:
  - Viscous flux $F^v(\nabla q)$
  - Advective flux $F^{adv}(q)$
  - Source $S(q)$
  - Forcing function $f(x, y)$

➢ In particular, want to be able to solve Euler equations

Implementation: CG

Strong form of PDE:

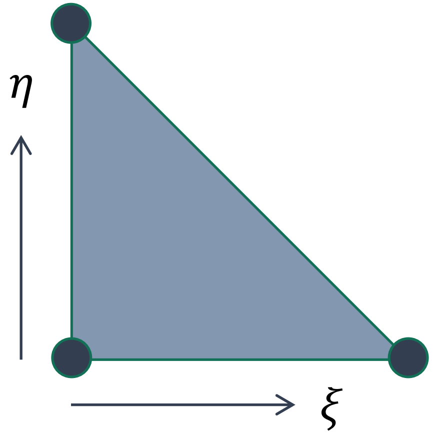
$$-\nabla \bullet (F^{v}\ (\nabla q)) + \nabla \bullet F^{adv}(q) = S(q) + f(x,y)$$

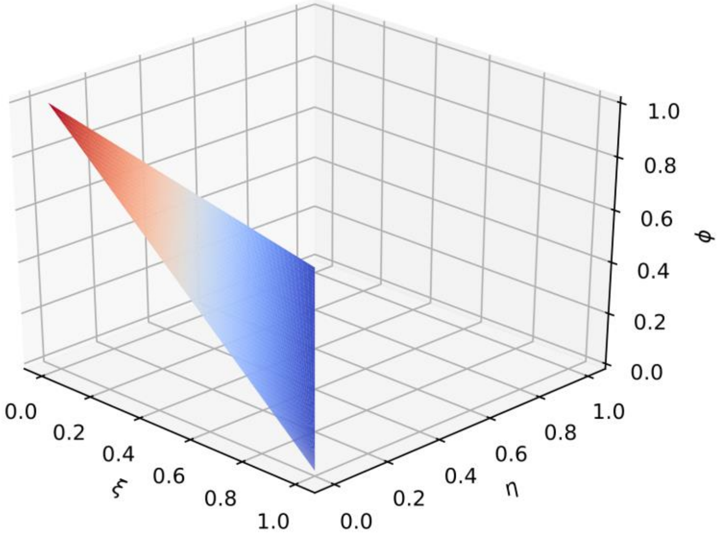
To obtain the weak form, multiply the strong form by the test function ϕ and integrate over the entire domain. Result is $\mathcal{R}(q,\phi)$

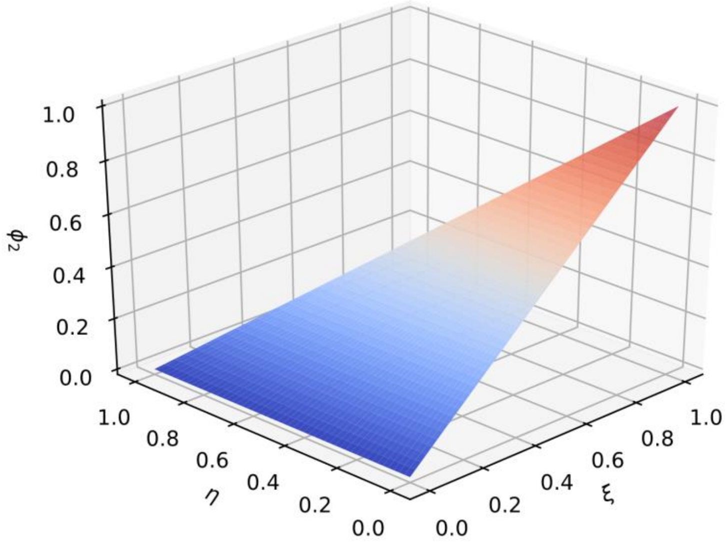
Solution can be expressed as: $q = \sum q_i \phi_i$

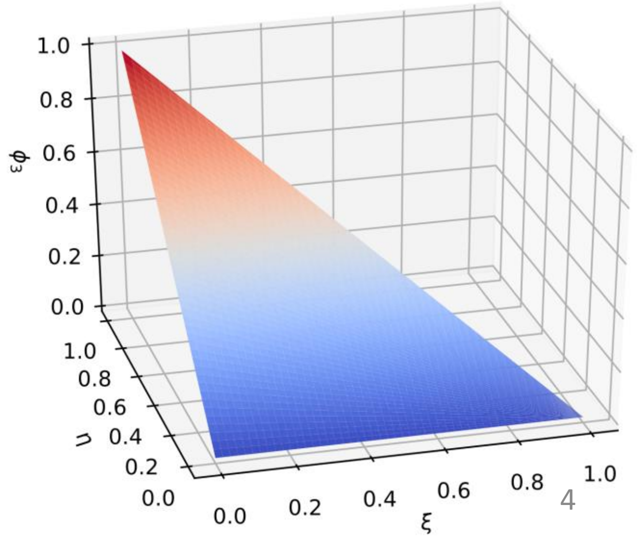
Implementation: CG

- P1 triangular elements
- Lagrange basis functions
- Gaussian quadrature used to evaluate integrals









Implementation: Nonlinear Solver

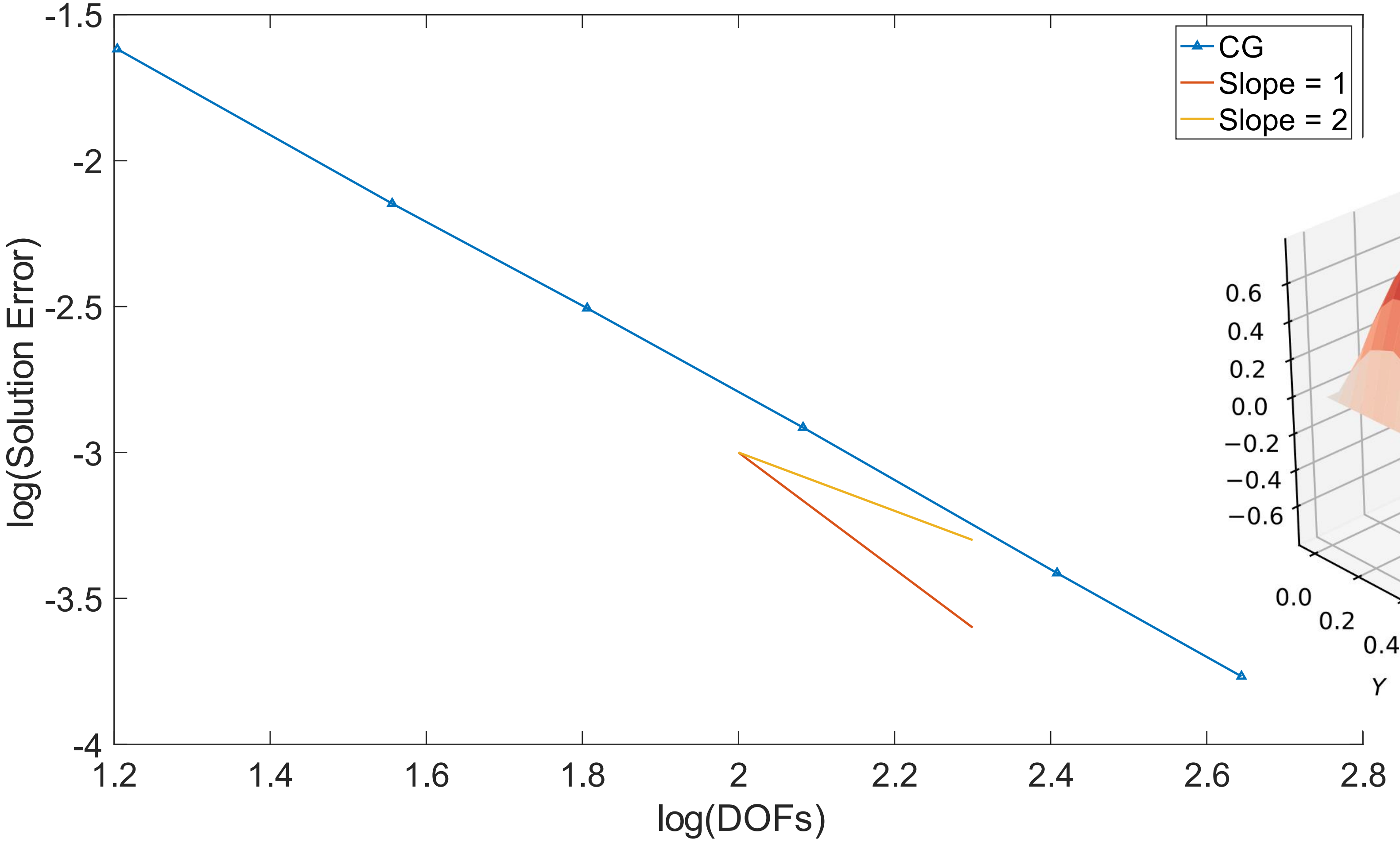
- Need to be able to solve nonlinear equation(s) for $\mathcal{R}(q, \phi) = 0$
- Use Newton's method:

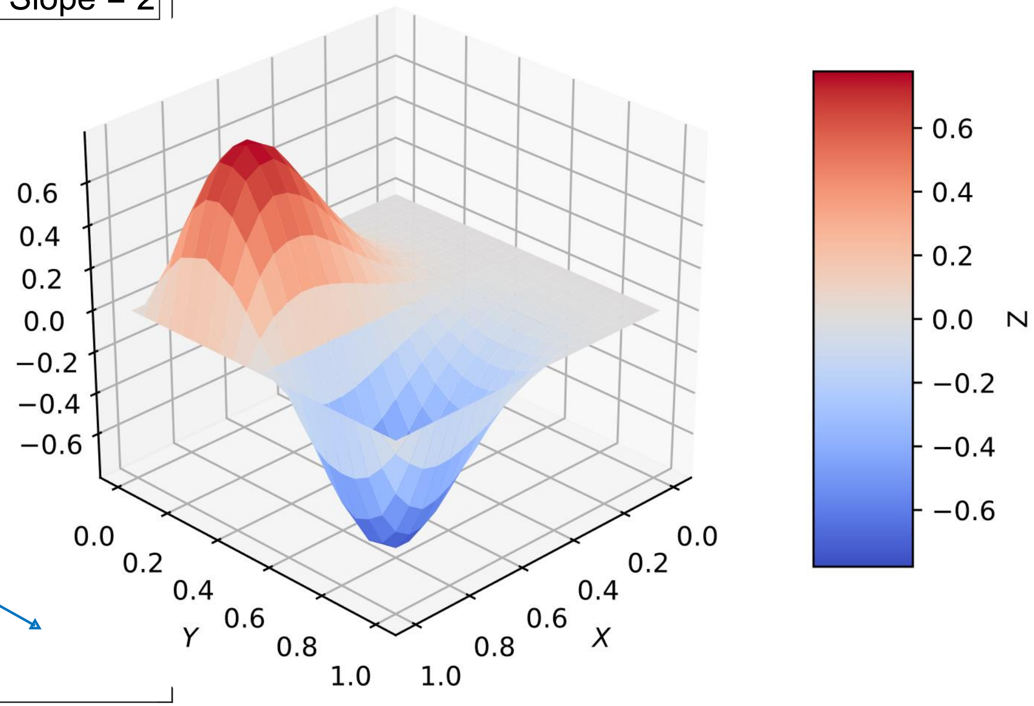
$$q^{k+1} = q^k - J^{-1}(q)\mathcal{R}(q, \phi)$$

- Jacobian is determined using complex step
  - Similar to Finite Difference, but uses complex perturbation
  - Imaginary component corresponds to $f(x+h) - f(x)$ in FD
- Since physical quantities cannot be negative, must check that $q^{k+1}$is physical

Poisson's Equation

$$-\nabla^2 q = f(x, y)$$

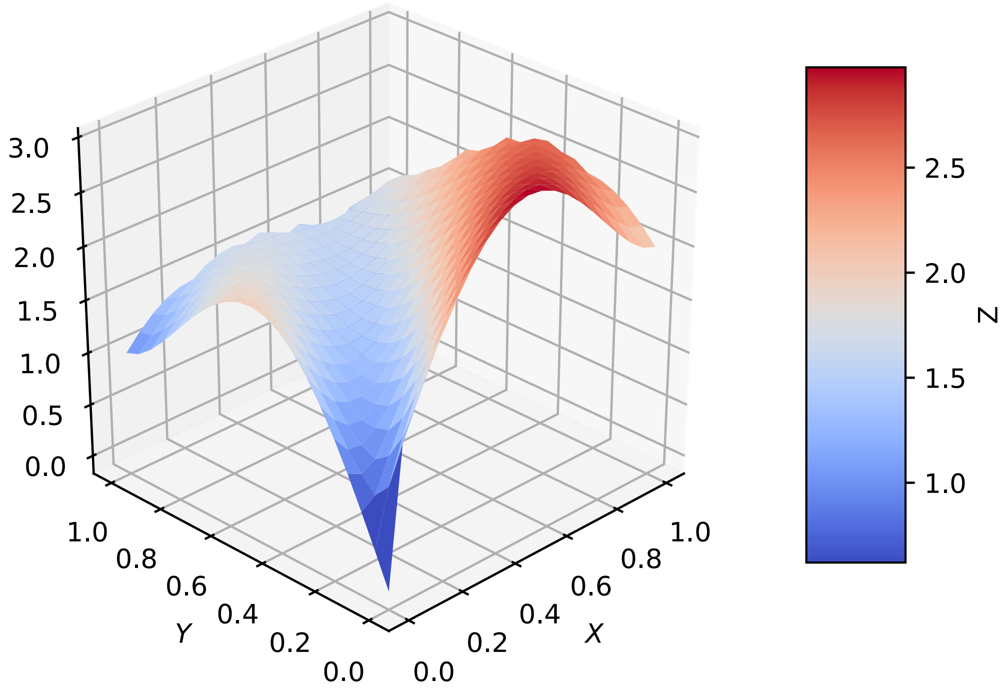


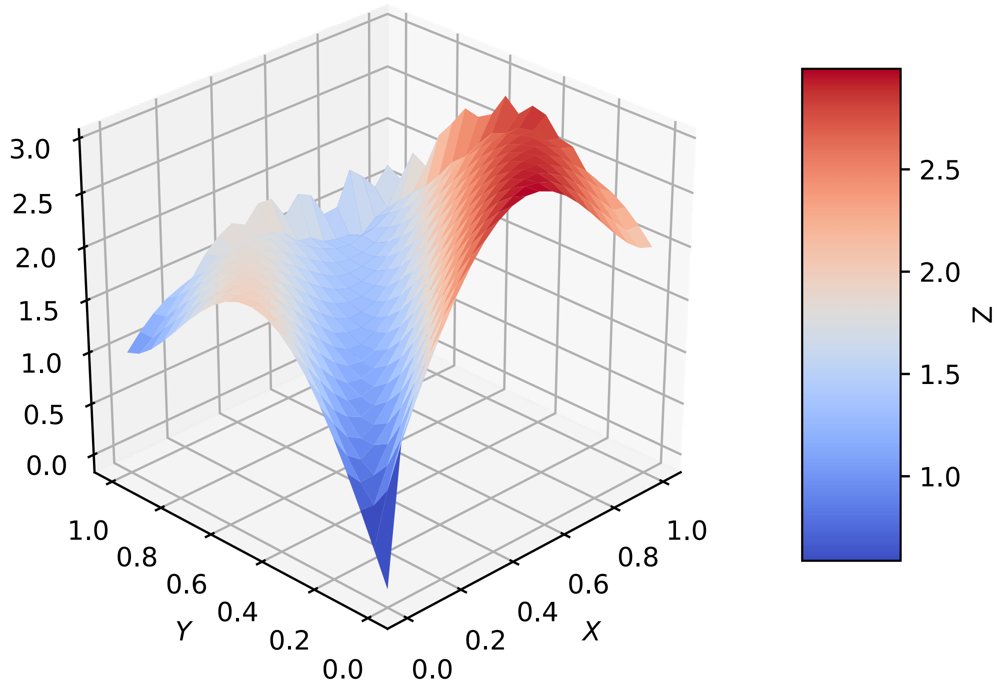


Linear Advection Diffusion

$$-\mu\nabla\bullet\nabla q+\nabla\bullet(\vec{u}\mathrm{q})=0$$

Viscosity on right is half that of the left





Begin to see spurious oscillations…
What can be done about them?

Need for Stabilization

- When Pe is high, advection dominates problem

$$Pe = \frac{advective\ transport}{diffusive\ transport} = \frac{hU}{\mu}$$

- To avoid spurious oscillations and quantities, need to stabilize PDE
  - Many ways to do this!
  - Stabilization is achieved by adding diffusion into the PDE
    - τ determines how much diffusion to add

Two Stabilization Methods

- Streamline Upwind Petrov-Galerkin (SUPG):
  - Add PDE dependent diffusion onto the weak form
  - For Euler, add the following:

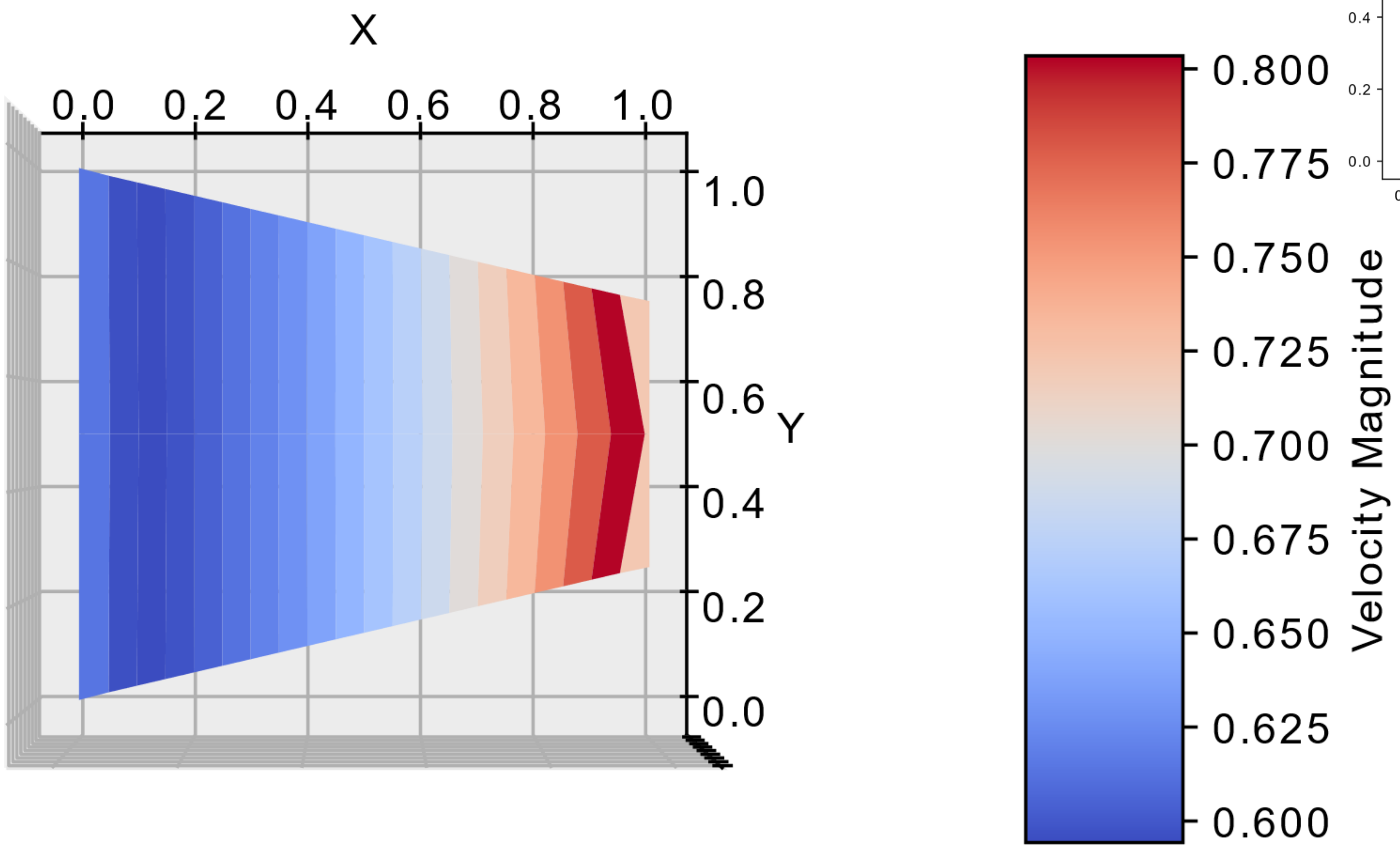
$$\int \frac{\partial \phi}{\partial x_i} . A_i \tau \underbrace{(\mathcal{L}u - f)}_{strong\ form\ residual} d\Omega$$

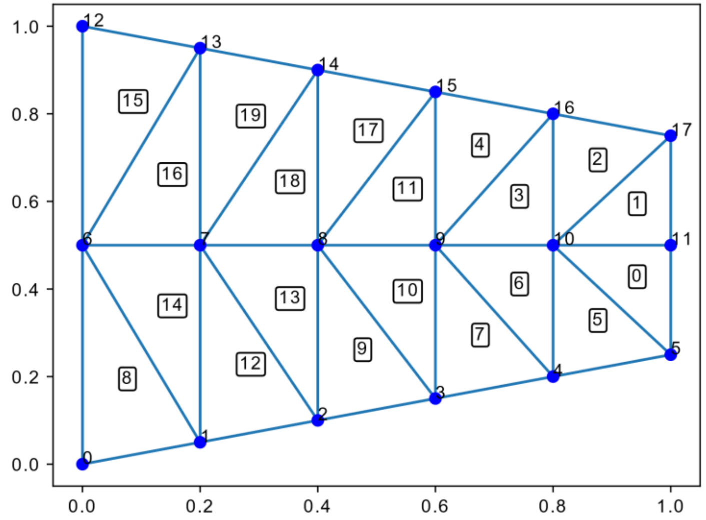
where

$$A_i = \frac{\partial F_i}{\partial u}, \text{ and } \tau \alpha \left(\frac{\|v\| + c}{vol^{\frac{1}{d}}} \right)^{-1}$$

- Simpler alternative: add in $F^v(\nabla q)$ of the form $-\mu \nabla^2 q$, where μ is a problem specific constant

Euler with simple diffusion





References

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Thank you!