Finite Difference Solutions to the Lees-Dorodnitsyn Compressible Boundary Layer Equations

Chelsea Onyeador
May 12, 2020
2.29 Final Project
Motivation

- Compressible, and specifically hypersonic, boundary layers can have complex velocity and temperature profiles
- Lees-Doroditsyn transformation results in decreased need for boundary layer scaling
- Boundary layer analysis is necessary to assess surface heating, skin friction, and external flow displacement effects
- Developing robust methods for modeling hypersonic boundary layers
TSL L-D Coordinate Transformation

\[
\xi = \int_0^x \rho_e u_e \mu_e \, dx \quad \eta = \frac{u_e}{\sqrt{2\xi}} \int_0^y \rho \, dy
\]

\[
(C f''')' + f f'' = \frac{2\xi}{u_e} \left[ (f')^2 - \frac{\rho_e}{\rho} \right] \frac{du_e}{d\xi} + 2\xi \left( f' \frac{\partial f'}{\partial \xi} - \frac{\partial f}{\partial \xi} f'' \right)
\]

\[
\frac{\partial p}{\partial \eta} = 0
\]

\[
\left( \frac{C}{p_T^e} g' \right)' + f g' = 2\xi \left[ f' \frac{\partial g}{\partial \xi} + f' g \frac{\partial h_e}{\partial \xi} - g' \frac{\partial f}{\partial \xi} + \frac{\rho_e u_e}{\rho h_e} f' \frac{du_e}{d\xi} \right] - C \frac{u_e^2}{h_e} (f'')^2
\]

\[
C = \frac{\rho \mu}{\rho_e \mu_e}, \quad g = \frac{h}{h_e}, \quad \text{and, } f' = \frac{u}{u_e}
\]
Lees-Dorodnitsyn Equations

- For 2D, Laminar, Compressible Boundary Layers
- Effectively Parabolic
- 5th Order system of coupled equations
- Introduce F, U, S, H, and Q to simplify into 5 - 1st order equations
  - $F = f = \frac{\varphi}{\sqrt{2\xi}}$ (stream function relation)
  - $U = \frac{\partial f}{\partial \eta} = \frac{u}{u_e}$ (velocity relation)
  - $S = \frac{\partial U}{\partial \eta} = \frac{\partial^2 f}{\partial \eta^2}$ (shear stress relation)
  - $H = g = \frac{h}{h_e} = \frac{T}{T_e} = \frac{\rho_e}{\rho}$ (enthalpy/temperature relation)
  - $Q = \frac{\partial g}{\partial \eta} = \frac{\partial H}{\partial \eta}$ (heat transfer relation)
Flow Parameters

FLUID MODELS

• Viscosity Sutherland’s Law
  \[ \mu = \mu_{ref} \left( \frac{T}{T_{ref}} \right)^{\frac{3}{2}} \left( \frac{T_{ref} + S_{ref}}{T + S_{ref}} \right) \]

• Calorically Perfect Gas
  \[ h = c_p T \]
  \[ \gamma = 1.4 \]

• Constant Pr
  \[ 0.71 \text{ (Van Driest), 0.75, or 1} \]

• Isothermal wall

SPECIFIED PARAMETERS

• \( \xi_{max} \)
• \( \eta_e \)
• \( T_e \to h_e \)
• \( M a_e(\xi) \to u_e \)
• \( h_w \frac{h_e}{h_e} \to h_w \)
• \( P_e = \rho_e \)
Flow Parameters

FLUID MODELS

• Viscosity Sutherland’s Law
  \[ \mu = \mu_{ref}\left(\frac{T}{T_{ref}}\right)^{\frac{3}{2}}\left(T_{ref} + S_{ref}\right) \]

• Calorically Perfect Gas
  \[ h = c_p T \]
  \[ \gamma = 1.4 \]

• Constant Pr
  \[ 0.71 \text{ (Van Driest), 0.75, or 1} \]

• Isothermal wall

SPECIFIED PARAMETERS

• \( \xi_{max} \)
• \( \eta_e \)
• \( T_e \rightarrow h_e \)
• \( M a_e(\xi) \rightarrow u_e \)
• \( \frac{h_w}{h_e} \rightarrow h_w \)
• \( P_e = \rho_e \)
Boundary Conditions

• 5 Required for a 5\textsuperscript{th} Order system:
  
  • Wall:
    ◦ \( F_{\text{wall}} = 0 \)
    ◦ \( U_{\text{wall}} = 0 \)
    ◦ \( G_{\text{wall}} = G_{\text{specified}} \)
  
  • Edge:
    ◦ \( U_{\text{edge}} = 1 \)
    ◦ \( G_{\text{edge}} = 1 \)

• Initial condition @ \( \xi = 0 \)
  ◦ 1D Flat plate similarity solution (calculated with Newton-Raphson method)
Residual Form

\[ F' - U = 0 \]
\[ U' - S = 0 \]

\[ (CS)' + FS + \frac{2\xi}{ue} [H - (U)^2] \frac{du_e}{d\xi} + 2\xi \left( \frac{\partial F}{\partial \xi} S - U \frac{\partial U}{\partial \xi} \right) = 0 \]

\[ \left( \frac{C}{Pr} Q \right)' + FQ - 2\xi \left[ U \frac{\partial H}{\partial \xi} + \frac{U H \partial h_e}{h_e \partial \xi} - Q \frac{\partial F}{\partial \xi} + \frac{Hu_e}{h_e} U \frac{du_e}{d\xi} \right] + C \frac{u_e^2}{h_e} (S)^2 = 0 \]

\[ H' - Q = 0 \]

\( F, U, S, H, Q \) are unknowns
Finite Difference Stencil

BOX SCHEME
- Space-march in $\xi$ direction
- 2nd Order
- Centered Difference in both directions
- Similar to Crank Nicholson

3-POINT BACKWARD DIFFERENCE
- Space-march in $\xi$ direction
- 2nd Order
- Centered Difference in $\eta$
- 3-pt Backward Difference in $\xi$
Box Stencil

\[ X = X^j_{i+\frac{1}{2}} = \frac{1}{4} (X^j_i + X^j_{i+1} + X^j_{i+1} + X^j_{i+1}) \]

\[ X' = X'^j_{i+\frac{1}{2}} = \frac{\left( X^j_{i+\frac{1}{2}} - X^j_{i+\frac{1}{2}} \right)}{\Delta \eta} = \frac{\left( \frac{1}{2} (X^j_{i+1} + X^j_{i+1}) - \frac{1}{2} (X^j_i + X^j_i) \right)}{\Delta \eta} \]

\[ \frac{\partial X}{\partial \xi} = \frac{\partial X}{\partial \xi} \bigg|_{i+\frac{1}{2}} = \frac{\left( X^j_{i+1} - X^j_i \right)}{\Delta \xi} = \frac{\left( \frac{1}{2} (X^j_{i+1} + X^j_{i+1}) - \frac{1}{2} (X^j_i + X^j_i) \right)}{\Delta \xi} \]
3pt - Backward Stencil

\[ X = X_i^{j+\frac{1}{2}} = \frac{1}{2} (X_i^j + X_i^{j+1}) \]

\[ X' = X_i'^{j+\frac{1}{2}} = \frac{(X_i^{j+1} - X_i^j)}{\Delta \eta} \]

\[ \frac{\partial X}{\partial \xi} = \frac{\partial X|^{j+\frac{1}{2}}}{\partial \xi_i} = \frac{\left( \frac{3}{2} X_i^{j+\frac{1}{2}} - 2 X_i^{j+\frac{1}{2}} + \frac{1}{2} X_i^{j+\frac{1}{2}} \right)}{2 \Delta \xi} = \frac{\left( \frac{3}{2} (X_i^{j+1} + X_i^j) - 2 (X_i^{j+1} + X_i^j) + \frac{1}{2} (X_i^{j+1} + X_i^j) \right)}{4 \Delta \xi} \]
3pt - Backward Stencil

\[ X = X_i^{j+1/2} = \frac{1}{2} (X_i^j + X_i^{j+1}) \]

Finite difference approximations:
\[ X' = X_i^{j+1/2} = \frac{(X_i^{j+1} - X_i^j)}{\Delta \eta} \]

\[
\frac{\partial X}{\partial \xi} = \left. \frac{\partial X}{\partial \xi} \right|_i^{j+1/2} = \left. \left( \frac{3}{2} X_i^{j+1/2} - 2X_i^{j+1} + \frac{1}{2} X_i^{j+1} \right) \right|_{2\Delta \xi} = \left( \frac{3}{2} (X_i^{j+1} + X_i^j) - 2(X_{i-1}^{j+1} + X_{i-1}^j) + \frac{1}{2} (X_{i-1}^{j+1} + X_{i-1}^j) \right) \right|_{4\Delta \xi}
\]

Analytical Evaluations:
\[
\xi = \xi_i \quad \frac{\partial h_e}{\partial \xi} = \left. \frac{\partial h_e}{\partial \xi} \right|_{i} = \frac{\partial h_e}{\partial \xi}(\xi_i)
\]
\[
u_e = \nu_e_i = \nu_e(\xi_i) \quad \frac{\partial \nu_e}{\partial \xi} = \left. \frac{\partial \nu_e}{\partial \xi} \right|_{i} = \frac{\partial \nu_e}{\partial \xi}(\xi_i)
\]
Newton-Raphson Solver

• Necessary for Non-linear system
• 10 – Diagonal Sparse Matrix
• Analytically calculated jacobian
• Utilized MATLAB built in matrix solver
• Quadratic convergence
• Convergence criteria based on magnitude of max residual
Newton vs. MATLAB FSolve

- Compared the Newton-Raphson Method against MATLAB’s nonlinear solver
- Fsolve is a quasi-Newton method
- Assumed both solution methods had similar accuracies

<table>
<thead>
<tr>
<th></th>
<th>Newton-Raphson</th>
<th>MATLAB FSolve</th>
</tr>
</thead>
<tbody>
<tr>
<td>Runtime</td>
<td>6-7 sec</td>
<td>116 sec</td>
</tr>
<tr>
<td># of Iterations</td>
<td>3-4</td>
<td>2</td>
</tr>
<tr>
<td>Implementation Complexity</td>
<td>~650 lines</td>
<td>~125 lines</td>
</tr>
<tr>
<td># of Function Evaluations</td>
<td>525</td>
<td>243</td>
</tr>
</tbody>
</table>

(Solved for 30 x 30 system)
Selected Results
Error Analysis

- L2 Error Analysis of $\delta^*(\xi)$, displacement thickness

$$
\delta^* = \int_0^{\eta e} \left( 1 - \frac{\rho}{\rho_e} U \right) \frac{\partial y}{\partial \eta} d\eta = \int_0^{\eta e} \left( 1 - \frac{\rho}{\rho_e} U \right) \frac{H \sqrt{2 \xi}}{u e \rho_e} d\eta
$$

- Compared to solution with fine resolution
- Roughly 2nd order convergence, as expected
- Comparable performance of Box Scheme and 3-pt scheme
Conclusions

• Newton-Raphson (in this case) is significantly preferable to FSolve
• Stability is not a significant concern for the backwards difference scheme
• Oscillations are not present in the box scheme
• There is no clear advantage to using the box scheme instead of the 3-pt backwards difference
• Further sampling of the solution space may be necessary to determine overall behavior of solution methods
Thank you

Questions?
Appendix
Analytical Evaluations:

\[ \xi = \xi_{i+\frac{1}{2}} \]
\[ u_e = u_{e_{i+\frac{1}{2}}} = u_e(\xi_{i+\frac{1}{2}}) \]
\[ h_e = h_{e_{i+\frac{1}{2}}} = h_e(\xi_{i+\frac{1}{2}}) \]

\[ \frac{\partial h_e}{\partial \xi} = \frac{\partial h_e}{\partial \xi}_{i+\frac{1}{2}} = \frac{\partial h_e}{\partial \xi}(\xi_{i+\frac{1}{2}}) \]
\[ \frac{\partial u_e}{\partial \xi} = \frac{\partial u_e}{\partial \xi}_{i+\frac{1}{2}} = \frac{\partial u_e}{\partial \xi}(\xi_{i+\frac{1}{2}}) \]
Chapman Rubesin factor

1. $C = C(g)$

2. $C_{i+\frac{1}{2}}^{j+\frac{1}{2}} = C\left(g_{i+\frac{1}{2}}^{j+\frac{1}{2}}\right)$

3. $C_{i+\frac{1}{2}}^{j+\frac{1}{2}} = \frac{1}{4}[C(g_i^j) + C(g_{i+1}^j) + C(g_i^{j+1}) + C(g_{i+1}^{j+1})]$
Similarity Equations

2 - Coupled 2nd and 3rd Order Diff. Eqns

\[
(Cf'')' + ff'' = 0
\]

\[
\left(\frac{C}{Pr}g'\right)' + fg' + C\frac{u_e^2}{\varepsilon} (f'')^2 = 0
\]

\[
f = F
\]

\[
g = H
\]

5 - Coupled 1st Order ODEs (Drela Notation)

\[
F' = U
\]

\[
U' = S
\]

\[
(CS)' + FS = 0
\]

\[
H' = Q
\]

\[
\left(\frac{C}{Pr}Q\right)' + FQ + C\frac{u_e^2}{\varepsilon} (S)^2 = 0
\]

Where \( \phi' = \frac{\partial \phi}{\partial \eta} \)