Dynamic, Data-Driven Reduced-Order Models

Tony Ryu
Reduced Order Modelling (ROM)

- Cheaper alternative to full-model solve
- Built offline, used online

Peherstorfer (2016)
Dynamic Mode Decomposition (DMD)  \(^{(1)}\)

\[
x_{k+1} = F(x_k)
\]

Discrete system

\[
x_{k+1} = A x_k
\]

Approximate, discrete, locally linear dynamical system

Kutz, et. al (2016)
Dynamic Mode Decomposition (DMD) (2)

Tall, skinny snapshot matrices

\[ X = \begin{bmatrix} x_1 & x_2 & \cdots & x_{m-1} \end{bmatrix}, \]

\[ X' = \begin{bmatrix} x_2 & x_3 & \cdots & x_m \end{bmatrix}. \]

\[ \|X' - AX\|_F \]

\[ A = X'X^\dagger \]

Kutz, et. al
Online DMD

\[ x_1, x_2, x_3, \ldots \quad x_{k-1}, x_k \]

\[ y_1, y_2, y_3, \ldots \quad y_{k-1}, y_k \]

Time

Zhang, et al. 2017
Online DMD

(1)

Rank-1 Updates

\[ P_k = (X_k X_k^T)^{-1} \]

Is \( X_k X_k^T \) invertible?

Zhang, et al. 2017
Online DMD

(1)

Rank-1 Updates

\[ P_k = (X_k X_k^T)^{-1} \]

\[ \gamma_{k+1} = \frac{1}{1 + x_{k+1}^T P_k x_{k+1}} \]

\[ P_{k+1} = (P_k^{-1} + x_{k+1} x_{k+1}^T)^{-1} = P_k - \gamma_{k+1} P_k x_{k+1} x_{k+1}^T P_k \]

\[ A_{k+1} = A_k + \gamma_{k+1} (y_{k+1} - A_k x_{k+1}) x_{k+1}^T P_k \]

Zhang, et al. 2017
Online DMD

Weighted Online DMD

\[
\tilde{X}_k = \begin{bmatrix} \sigma^{k-1} x_1 & \sigma^{k-2} x_2 & \cdots & x_k \end{bmatrix}
\]

\[
\tilde{Y}_k = \begin{bmatrix} \sigma^{k-1} y_1 & \sigma^{k-2} y_2 & \cdots & y_k \end{bmatrix}
\]

Zhang, et al. 2017
MATLAB implementation

Sudden Expansion at large kinematic viscosity

Initial DMD

Sudden Expansion at small kinematic viscosity

Online DMD

Final DMD
## Results

<table>
<thead>
<tr>
<th></th>
<th>Mode Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial DMD</td>
<td>6.5098</td>
</tr>
<tr>
<td>Online Updated DMD</td>
<td>5.6892</td>
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## Results

<table>
<thead>
<tr>
<th></th>
<th>Error (Rho = 1)</th>
<th>Error (Rho = 0.9999)</th>
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<tbody>
<tr>
<td>Initial DMD</td>
<td>6.5098</td>
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Discussion/Future Work

- Limitations of needing $\text{rank}(X_k) > n$
  Anqi Bao, et al. (2019)

- Compressed-sensing DMD
  Kutz (2016)


Anqi Bao, Eduardo Gildin, Abhinav Narasingam and Joseph S. Kwon. Data-driven model reduction for coupled flow and geomechanics based on DMD methods, 2019
Questions?
<table>
<thead>
<tr>
<th>Aspect</th>
<th>Standard</th>
<th>Batch</th>
<th>Mini-batch</th>
<th>Streaming</th>
<th>Online</th>
<th>Windowed</th>
</tr>
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<tbody>
<tr>
<td>Computational time Memory</td>
<td>$O(mn^2)$</td>
<td>$O(kn^2)$</td>
<td>$O(wn^2)$</td>
<td>$O(r^2n)$</td>
<td>$4n^2$</td>
<td>$8n^2$</td>
</tr>
<tr>
<td>Store past snapshots</td>
<td>mn</td>
<td>kn</td>
<td>wn</td>
<td>No</td>
<td>No</td>
<td>Yes</td>
</tr>
<tr>
<td>Track time variations</td>
<td>No</td>
<td>No</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>Real-time DMD matrix</td>
<td>No</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
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<tr>
<td>Exact DMD matrix</td>
<td>Yes</td>
<td>Yes</td>
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Table 1: Characteristics of the various DMD algorithms considered. Relevant parameters are state dimension $n$, total number of snapshot pairs $m \gg n$, window size $w$ such that $n < w \ll m$, low rank $r < n$, and discrete time $k > n$. Computational time denotes the required floating-point multiplies for one iteration (computing the DMD matrix).
Online DMD

\[ \hat{P}_{k+1} = \frac{1}{\rho} (\hat{P}_k - \gamma_{k+1} \hat{P}_k x_{k+1} x_{k+1}^T \hat{P}_k) \]

\[ \gamma_{k+1} = \frac{1}{1 + x_{k+1}^T \hat{P}_k x_{k+1}} \]

\[ A_{k+1} = A_k + \gamma_{k+1} (y_{k+1} - A_k x_{k+1}) x_{k+1}^T \hat{P}_k \]

Zhang, et al. 2017
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<tr>
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