# Modeling the Temperature of the Ocean Using a Finite Volume Scheme with Proper Orthogonal Decomposition

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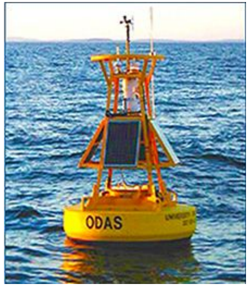
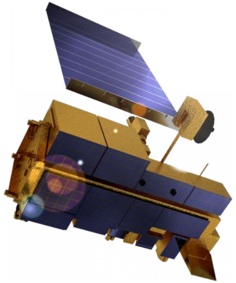


MWRA
NERACOOS Buoy



NASA Terra
MODIS

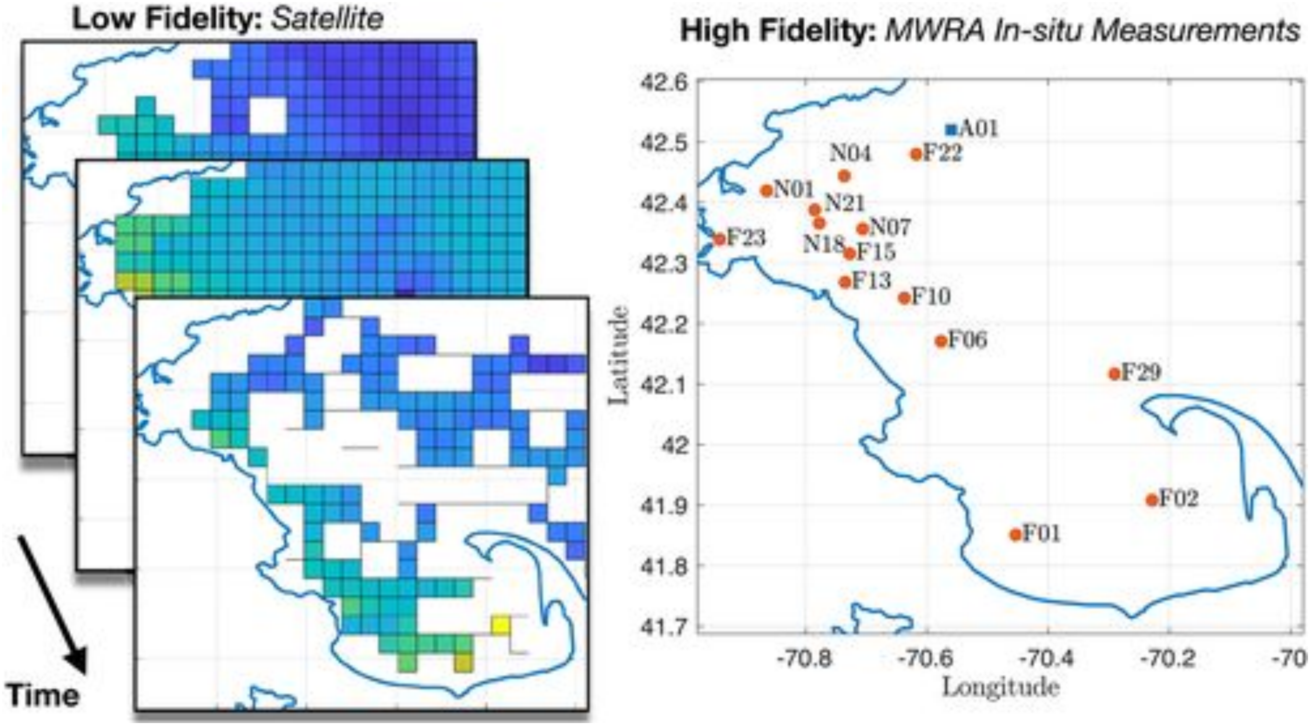
$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z} = \frac{\partial}{\partial z}(K_h\frac{\partial T}{\partial z}) + F_T$$

FVCOM

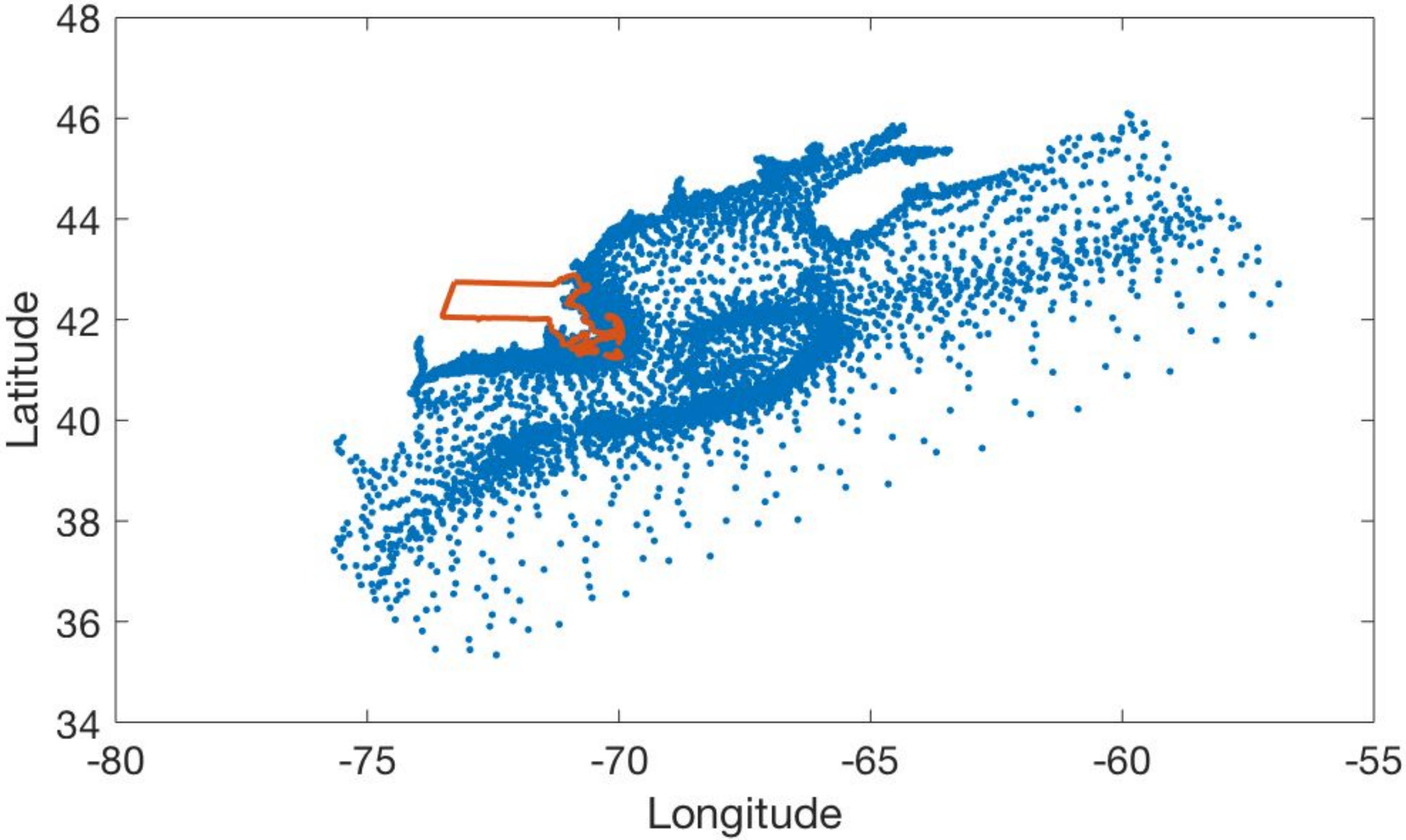
Proper Orthogonal Decomposition

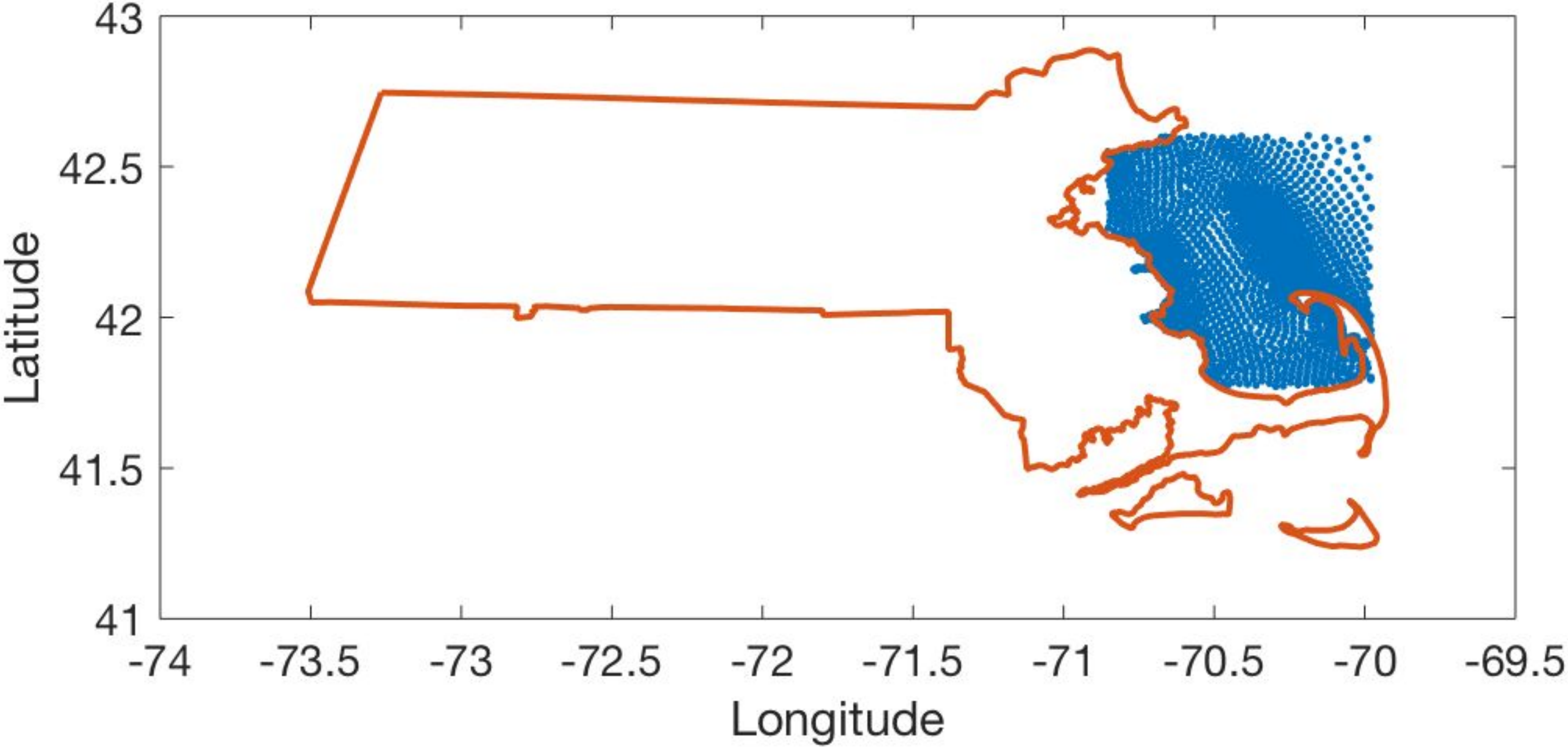
Multi-Fidelity Gaussian
Process Regression

T(x,y,z,t)




Hessam Babaee, C. Bastidas, Michael Defilippo, C. Chryssostomidis, and George Karniadakis. A multi-fidelity framework and uncertainty quantification for sea surface temperature in the massachusetts and cape cod bays. *Earth and Space Science*, 7, 01 2020


48
46
44
42
40
38
36
34
Latitude
-80
-75
-70
-65
-60
-55
Longitude


43
42.5
42
41.5
41
Latitude
-74
-73.5
-73
-72.5
-72
-71.5
-71
-70.5
-70
-69.5
Longitude

# Motivation: Why Do We Care About Uncertainty?

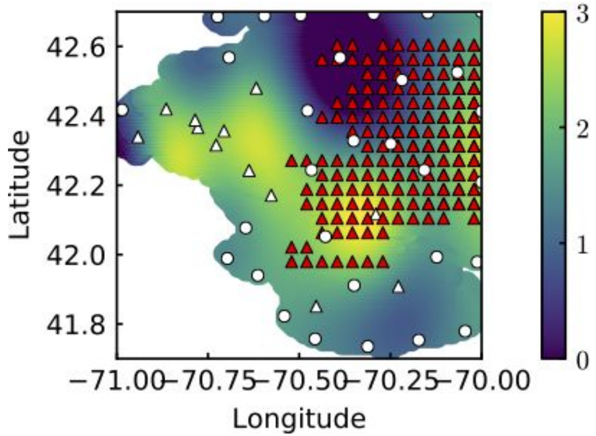
1. (Multi-Fidelity) Gaussian Process Regression

$$\bar{\mathbf{f}}(\mathbf{x}_*) = K(X_*, X)[K(X, X) + \sigma_n I]^{-1}\mathbf{y}$$

2. Optimal Sampling and Path Planning

$$a_{US}(x) = \sigma^2(x)$$

$$a_{IVR}(x) = \int [\sigma^2(x') - \sigma^2(x'; x)]dx'$$



# Simplified Problem Formulation

$$\frac{\partial\theta}{\partial t} + u\frac{\partial\theta}{\partial x} + v\frac{\partial\theta}{\partial y} + w\frac{\partial\theta}{\partial z} = \frac{\partial}{\partial z}(K_h\frac{\partial\theta}{\partial z}) + F_\theta$$

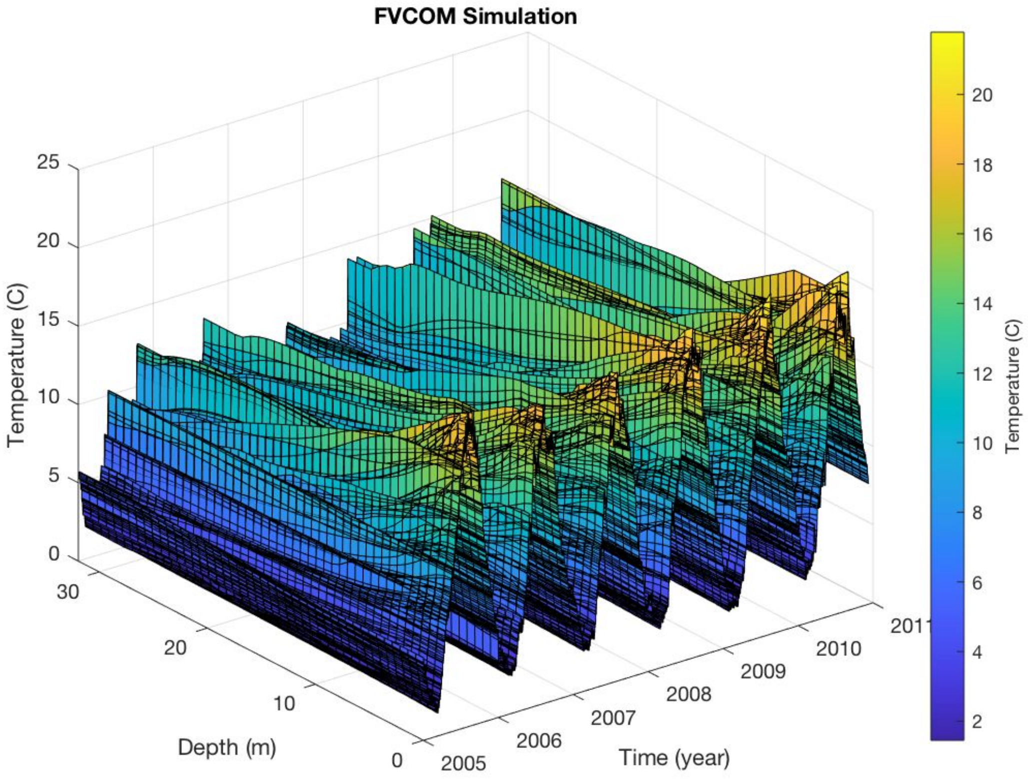
$$\downarrow$$

$$\frac{\partial T}{\partial t} + w\frac{\partial T}{\partial z} = \frac{\partial}{\partial z}(K\frac{\partial T}{\partial z})$$

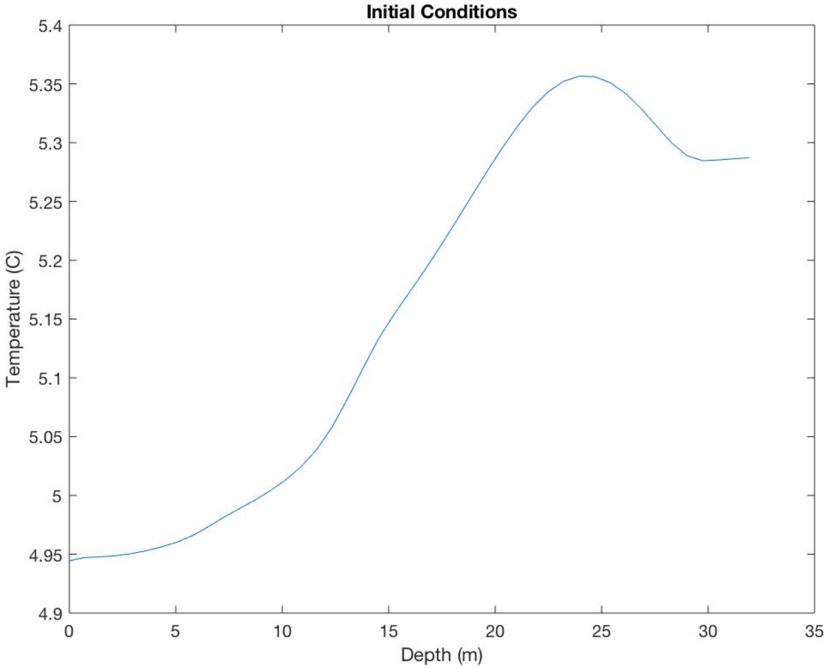
# Fractional Step Method

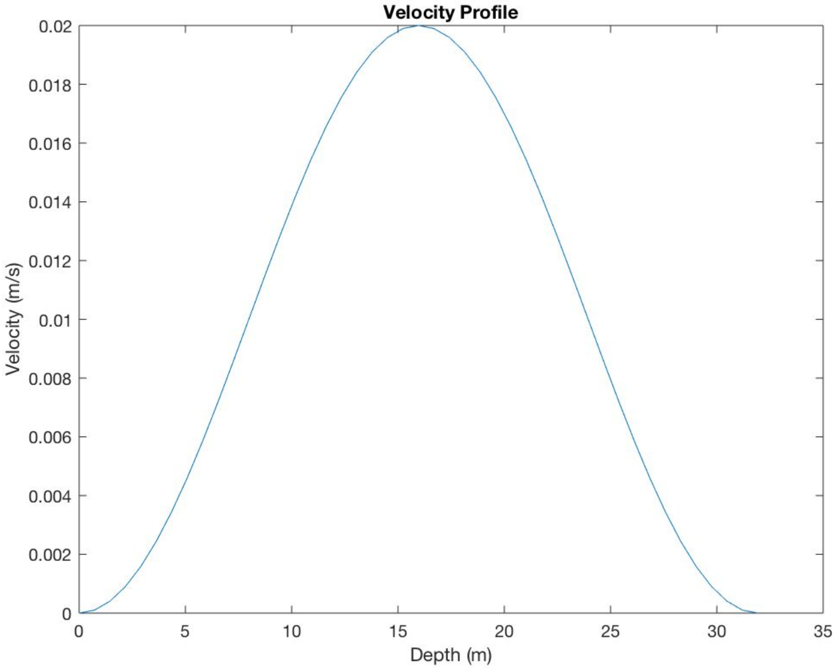
$$\frac{\partial T}{\partial t} + \text{Advection} = \text{Diffusion}$$

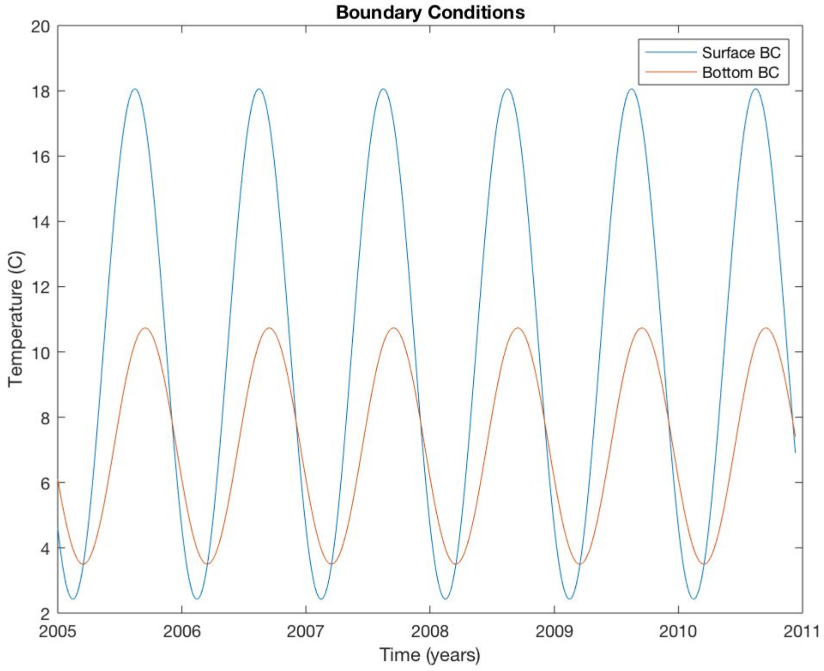
1. Finite Volume Scheme for the Advection Term $(T^n)^* = T^n - (\text{Advection}^n)\Delta t$
   a. First order: Upwind
   **b. Second order: QUICK**
   c. Runge-Kutta
2. Implicit Scheme for the Diffusion Term $T^{n+1} = (T^n)^* + (\text{Diffusion}^{n+1})\Delta t$
   **a. Second order: Finite Volume Central Difference**
   b. Higher Order Pade Scheme

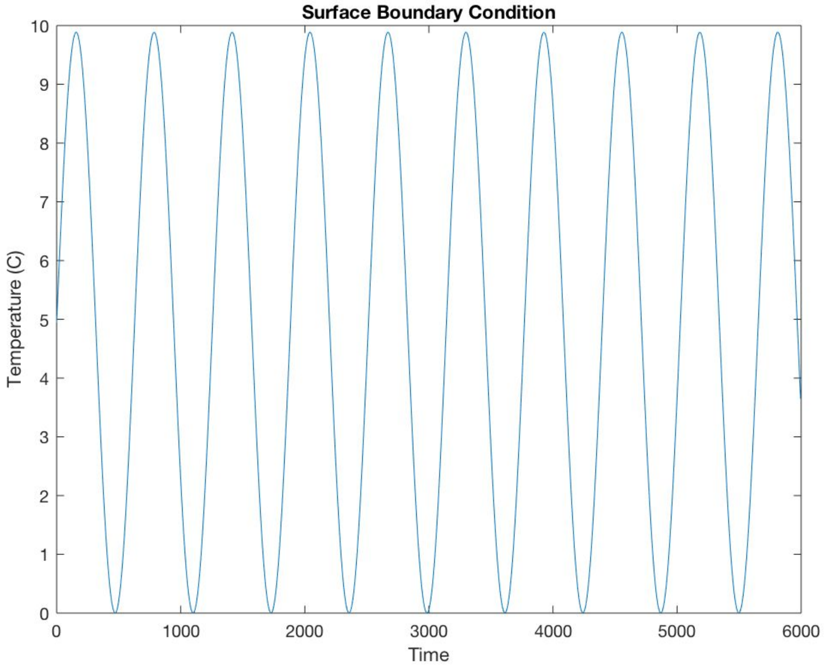



Changsheng Chen, Hedong Liu, and Robert C. Beardsley. An unstructured grid, finite-volume, three-dimensional, primitive equations ocean model: Application to coastal ocean and estuaries.Journal of Atmospheric and Oceanic Technology, 20(1):159 – 186, 2003.

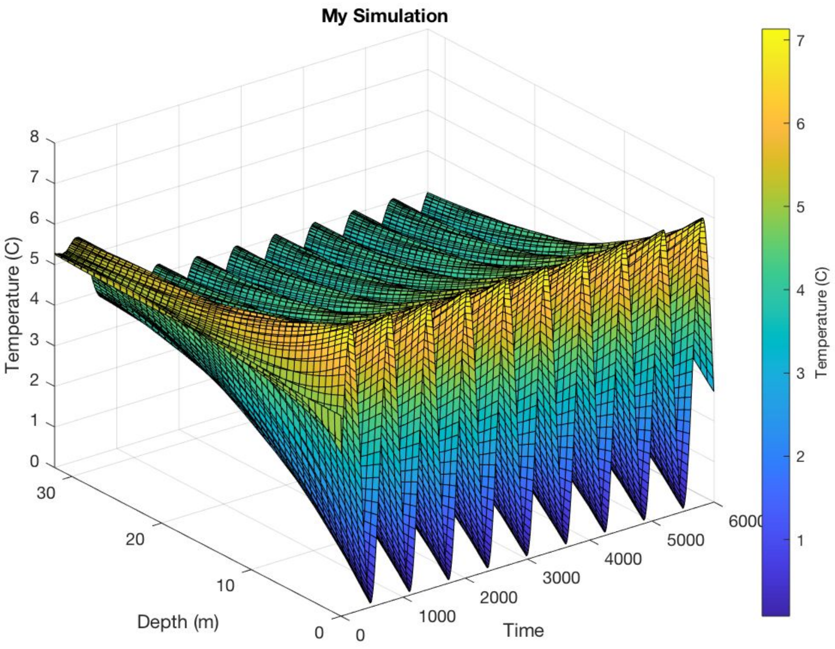

Initial Conditions
Temperature (C)
5.4
5.35
5.3
5.25
5.2
5.15
5.1
5.05
5
4.95
4.9
0
5
10
15
20
25
30
35
Depth (m)

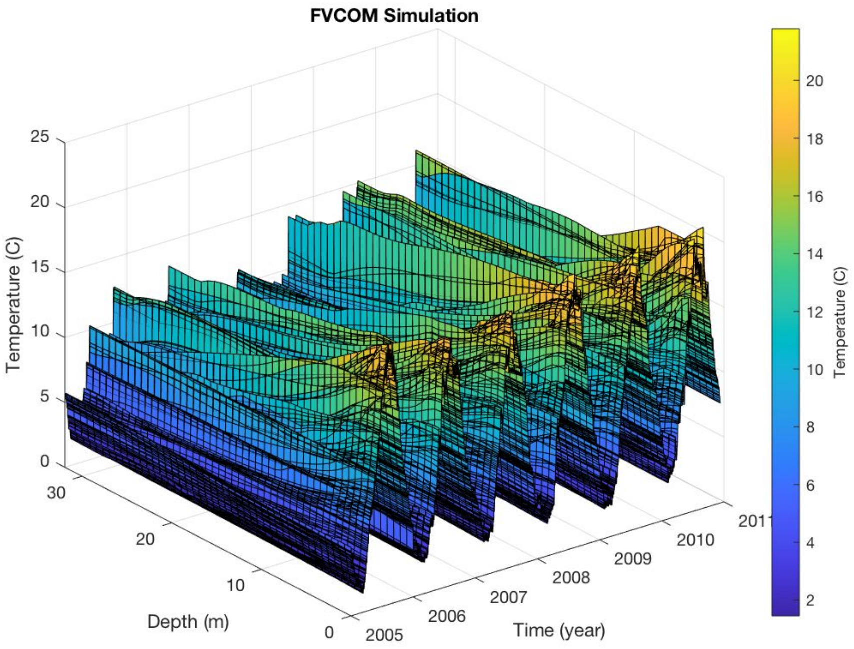

Velocity Profile
Velocity (m/s)
0.02
0.018
0.016
0.014
0.012
0.01
0.008
0.006
0.004
0.002
0
0
5
10
15
20
25
30
35
Depth (m)


Boundary Conditions
Surface BC
Bottom BC
Temperature (C)
Time (years)
20
18
16
14
12
10
8
6
4
2
2005
2006
2007
2008
2009
2010
2011

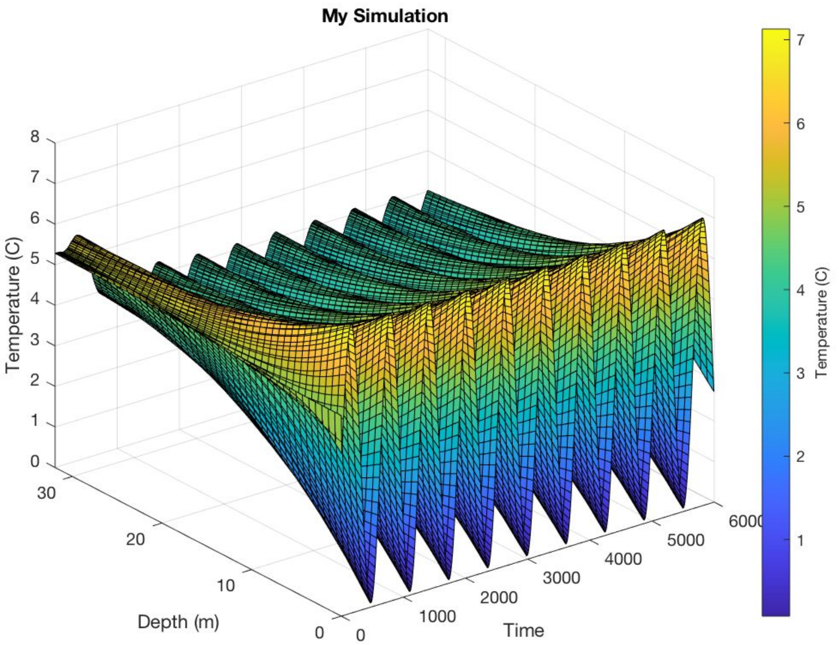

Surface Boundary Condition
Temperature (C)
Time
10
9
8
7
6
5
4
3
2
1
0
0
1000
2000
3000
4000
5000
6000

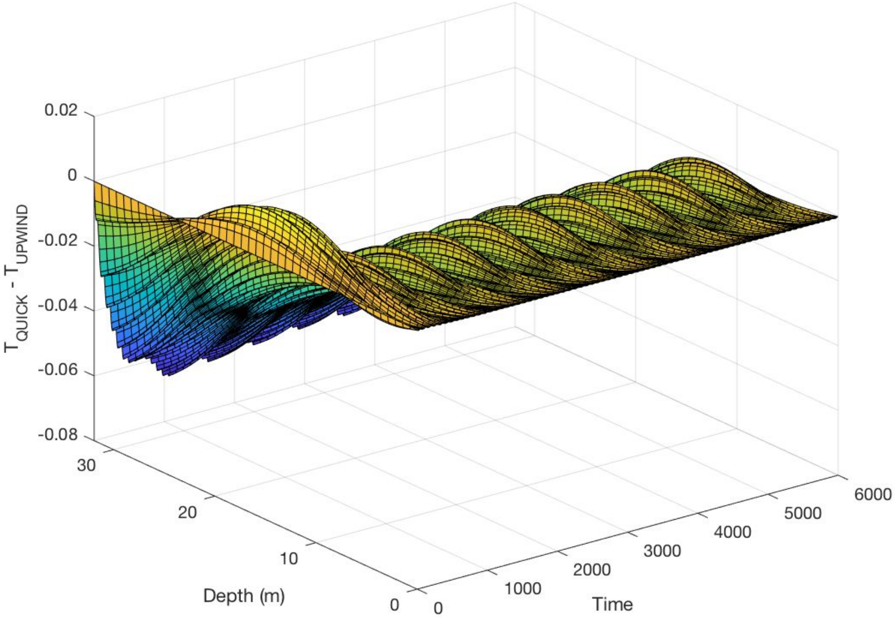
# QUICK FV Scheme



# Upwing FV Scheme




The upwind scheme leads to numerical diffusion.

# Proper Orthogonal Decomposition

$$\mathbf{T} = \begin{bmatrix} T(z_1, t_1) & \dots & T(z_1, t_m) \\ T(z_2, t_1) & \dots & T(z_2, t_m) \\ & \dots & \\ T(z_n, t_1) & \dots & T(z_n, t_m) \end{bmatrix}$$

$$\mathbf{Z} = \mathbf{T} - \bar{\mathbf{T}}$$

$$\mathbf{C} = \mathbf{Z}\mathbf{Z}^T$$

$\phi_i$ ⟶ eigenvectors of C

$\lambda_i$ ⟶ eigenvalues of C (capture the proportion of variance)

$$Z(z,t) = \sum_{i=1}^{n} q_i(t)\boldsymbol{\phi}_i(z)$$

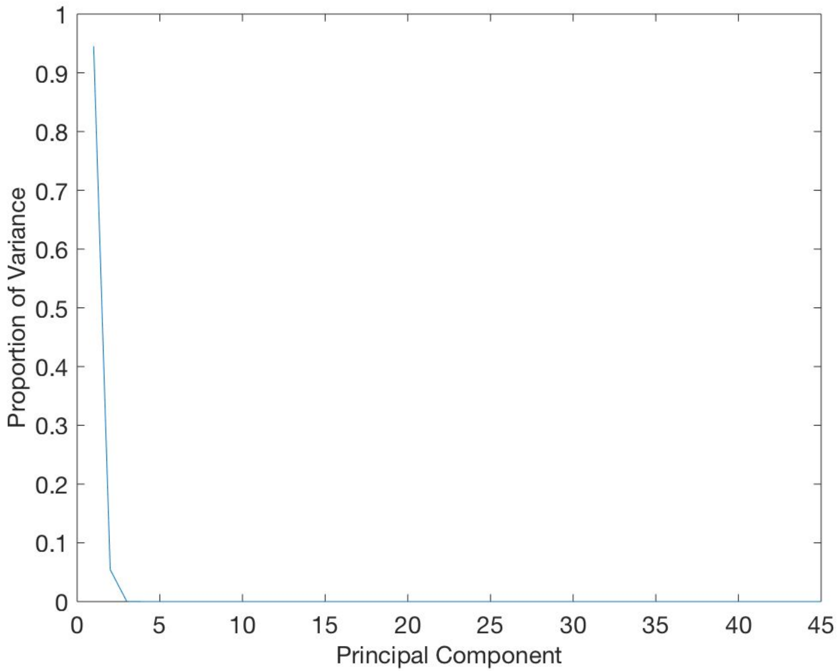
$$Z(\cdot,t) \approx \sum_{i=1}^{2} q_i(t)\boldsymbol{\phi}_i$$

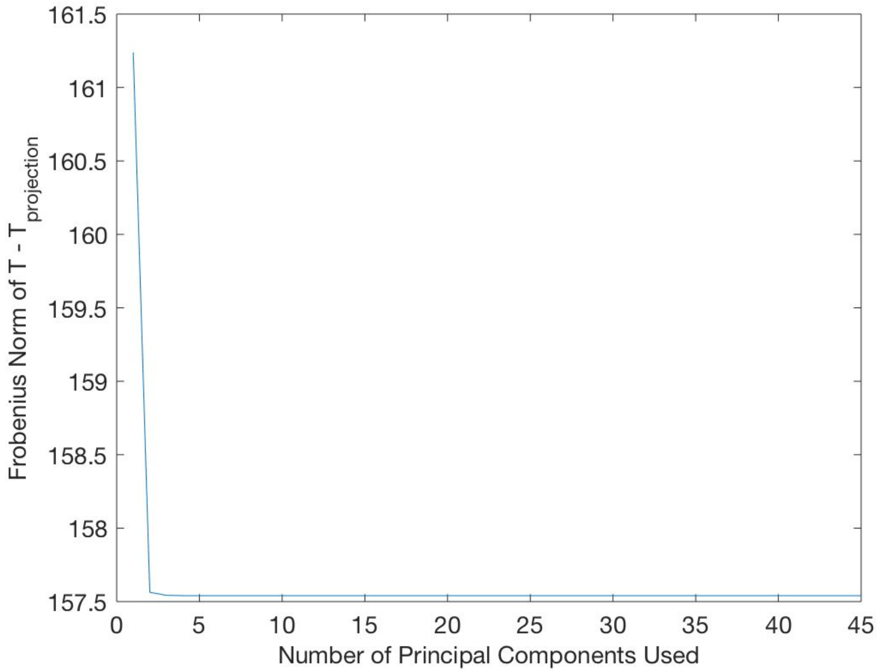
$$q_j(t) = \langle Z(\cdot,t), \boldsymbol{\phi}_j \rangle$$

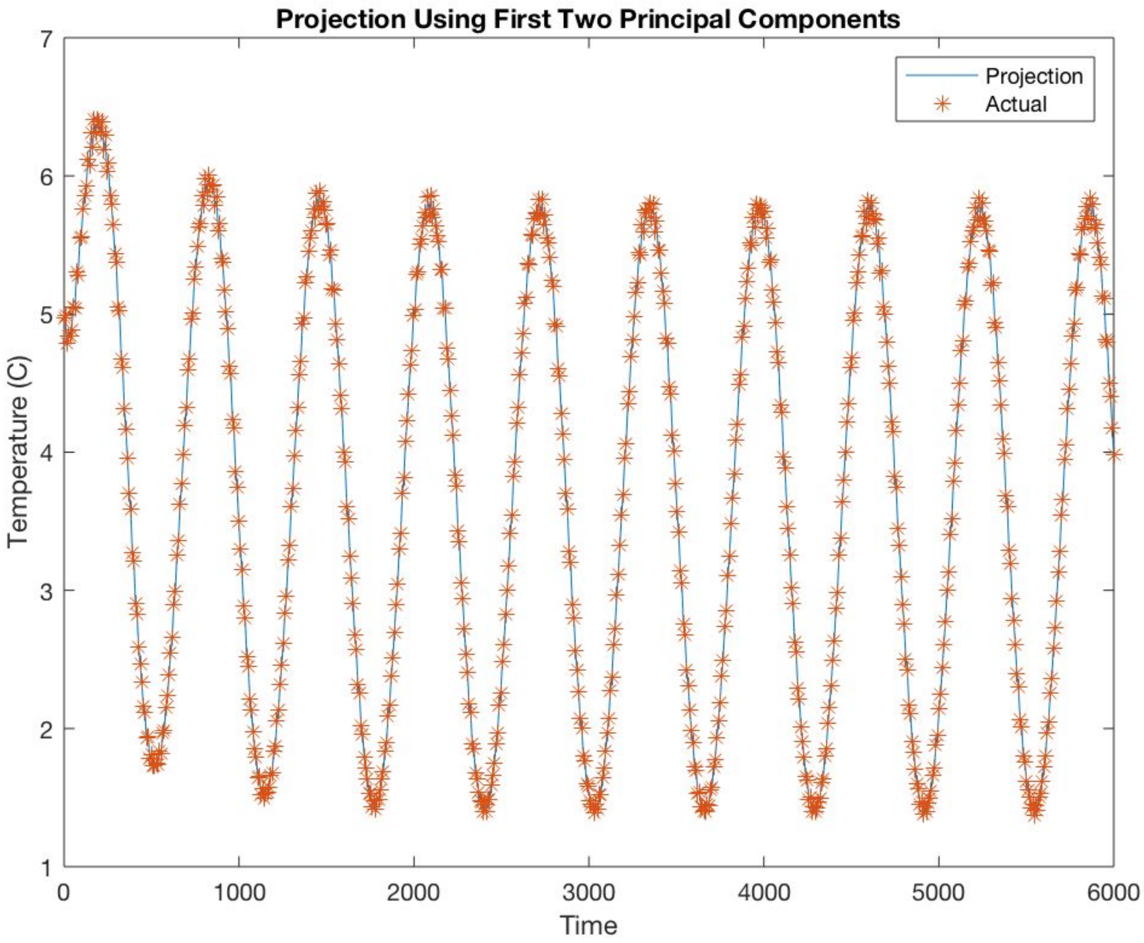
$$\mathbf{T}_{proj}(\cdot,t) = \sum_{i=1}^{2} q_i(t)\boldsymbol{\phi}_i + \bar{\mathbf{T}}(t)$$

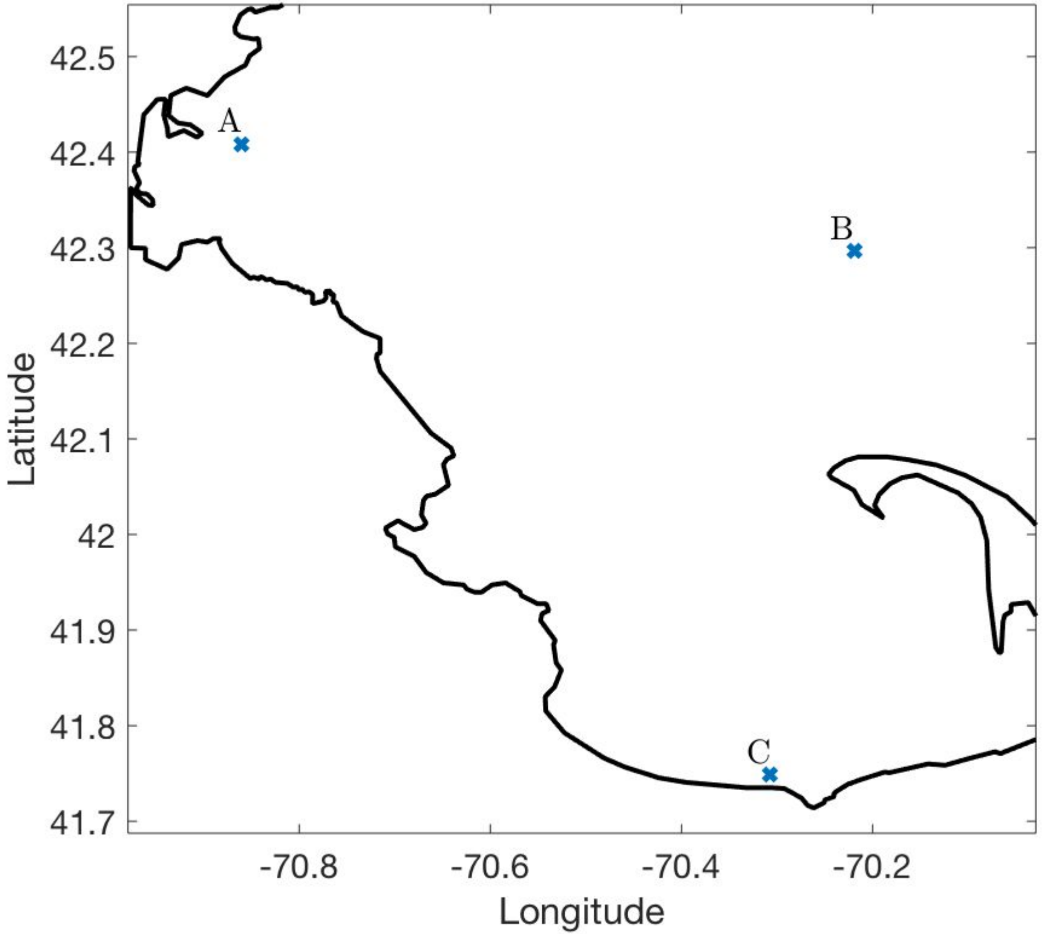
Data is reduced from $N_z \times N_t$ to $k \times (N_z + N_t)$ where k is the number of principal components that you retain

Philip Holmes, John L. Lumley, and Gal Berkooz. Proper orthogonal decomposition, page 86–128. Cambridge Monographs on Mechanics. Cambridge University Press, 1996


Proportion of Variance
Principal Component
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0 5 10 15 20 25 30 35 40 45

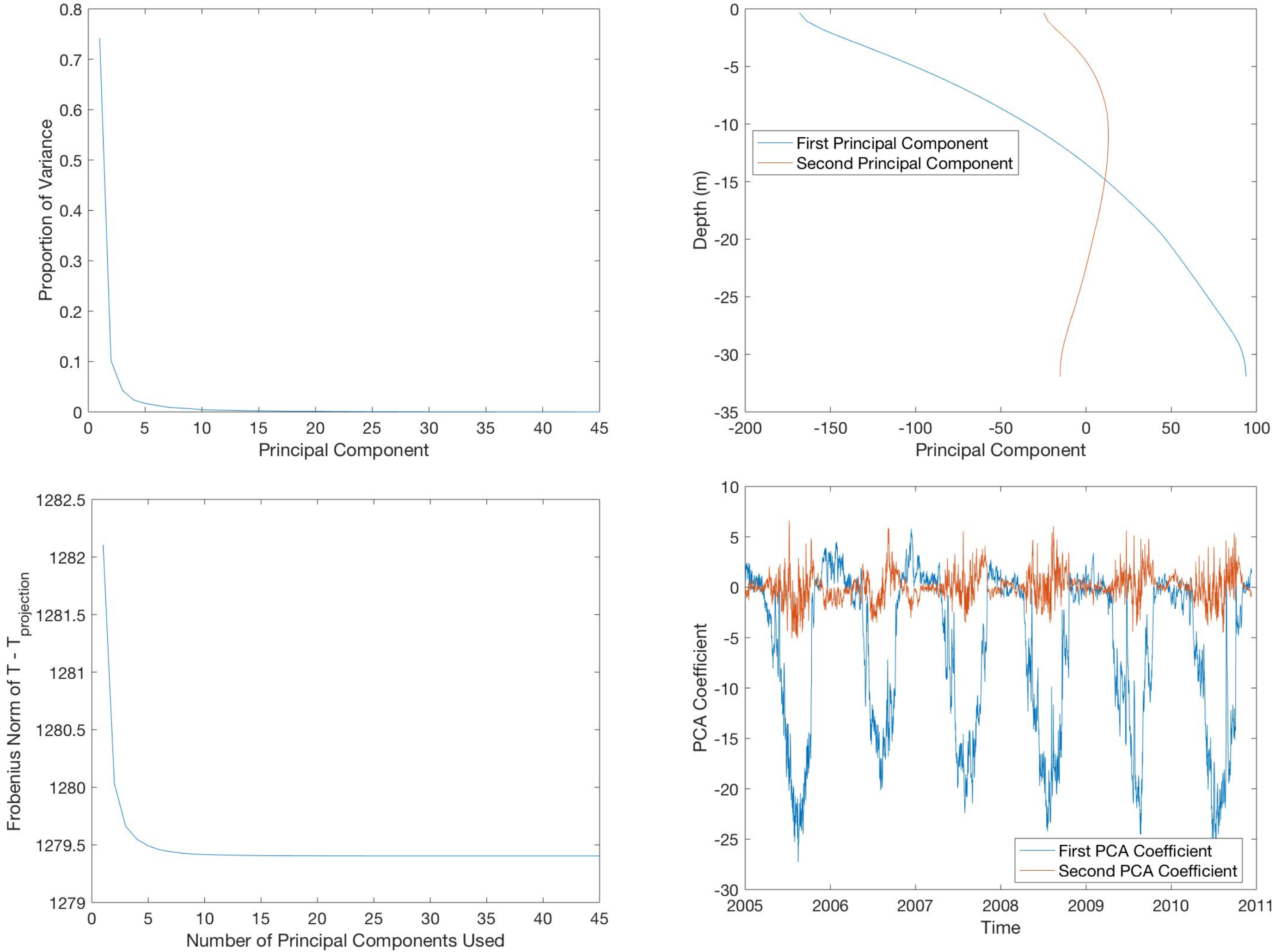

Frobenius Norm of T - T projection
Number of Principal Components Used
161.5
161
160.5
160
159.5
159
158.5
158
157.5
0 5 10 15 20 25 30 35 40 45

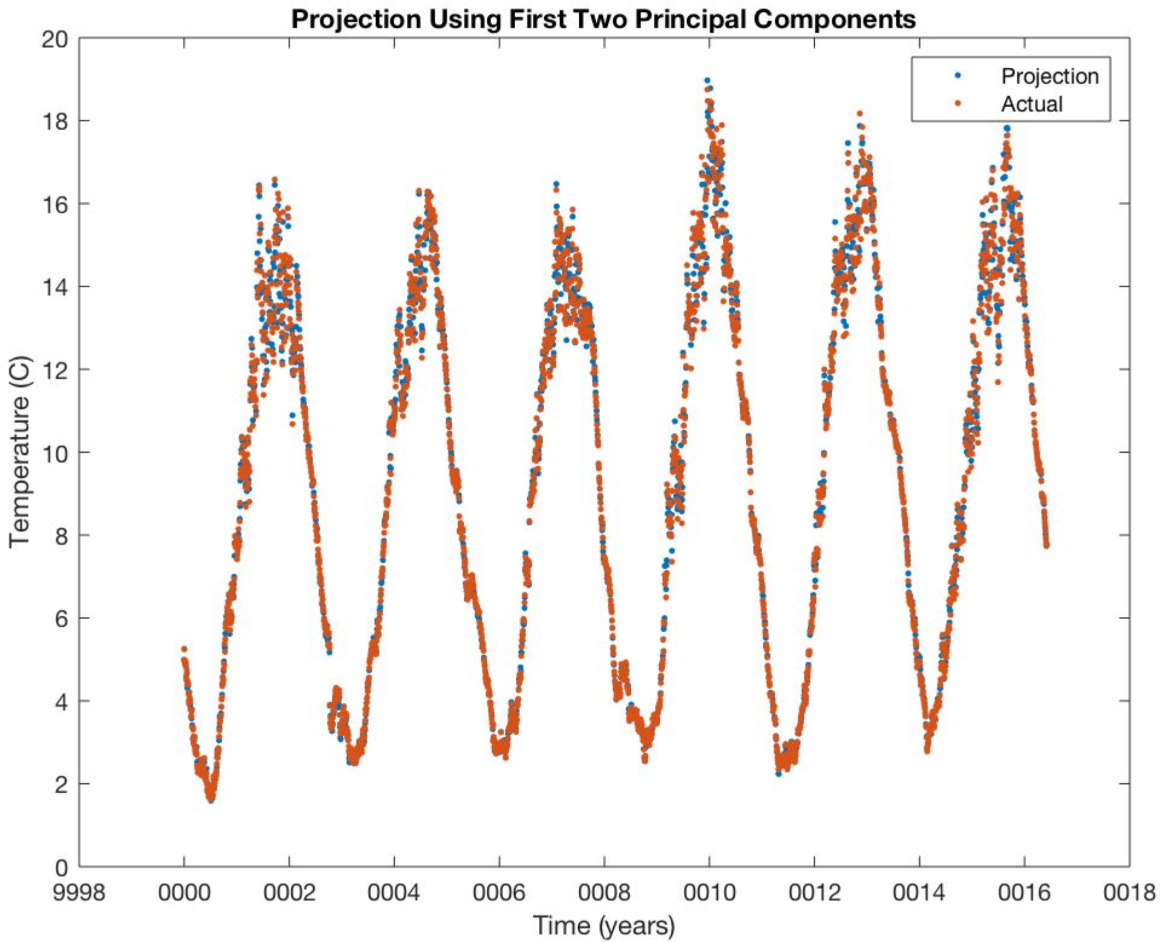

Projection Using First Two Principal Components
Projection
Actual
Temperature (C)
Time
7
6
5
4
3
2
1
0
1000
2000
3000
4000
5000
6000

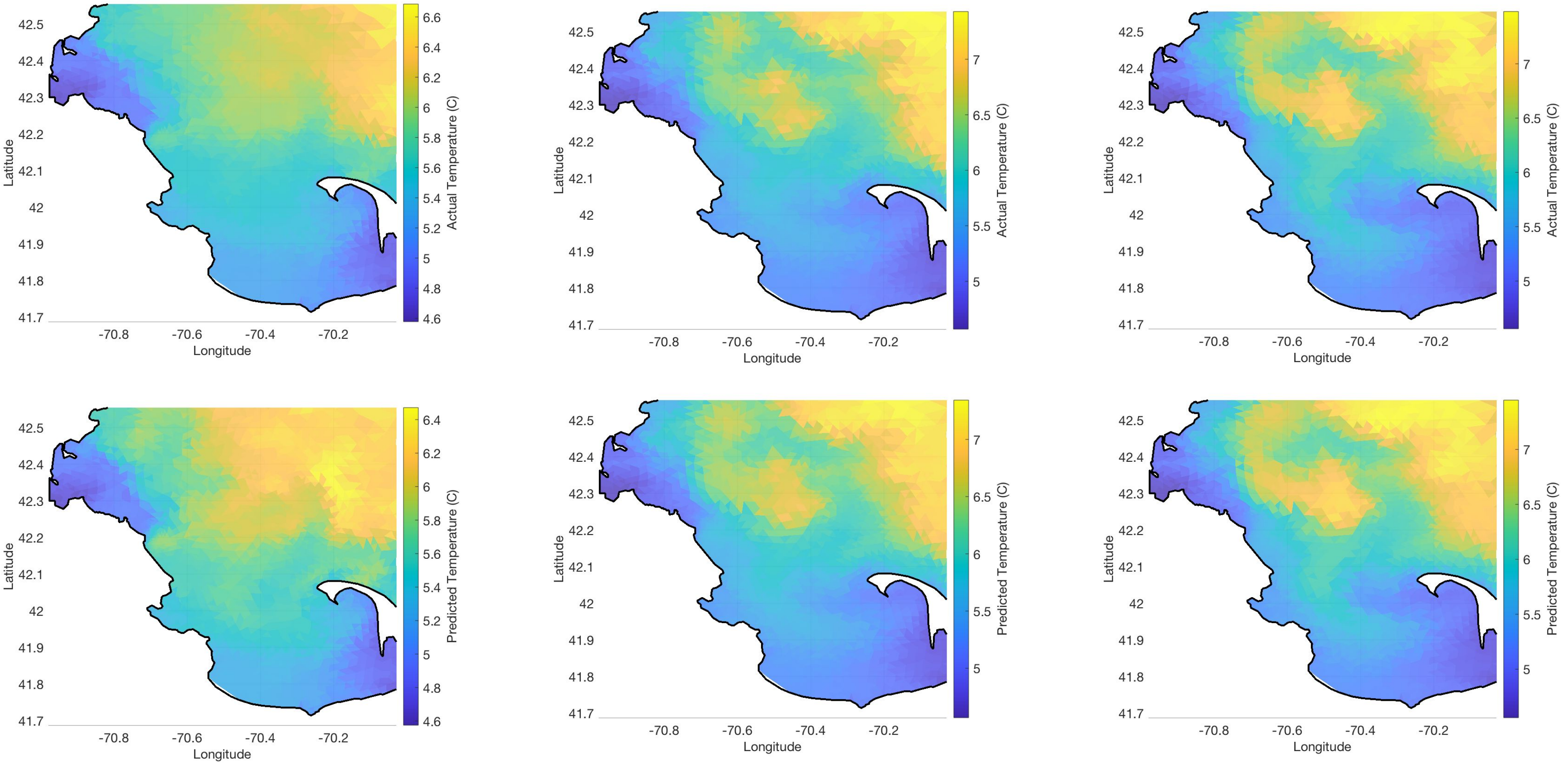


A
B
C
42.5
42.4
42.3
42.2
42.1
42
41.9
41.8
41.7
Latitude
-70.8
-70.6
-70.4
-70.2
Longitude

# Point A


Proportion of Variance
Principal Component
First Principal Component
Second Principal Component
Depth (m)
Principal Component
Frobenius Norm of T - $T_{projection}$
Number of Principal Components Used
PCA Coefficient
First PCA Coefficient
Second PCA Coefficient
Time


Projection Using First Two Principal Components
Projection
Actual
Temperature (C)
20
18
16
14
12
10
8
6
4
2
0
9998
0000
0002
0004
0006
0008
0010
0012
0014
0016
0018
Time (years)


Latitude
Longitude
Actual Temperature (C)
Predicted Temperature (C)
42.5
42.4
42.3
42.2
42.1
42
41.9
41.8
41.7
-70.8
-70.6
-70.4
-70.2
6.6
6.4
6.2
6
5.8
5.6
5.4
5.2
5
4.8
4.6
7
6.5
5.5

# Error Analysis

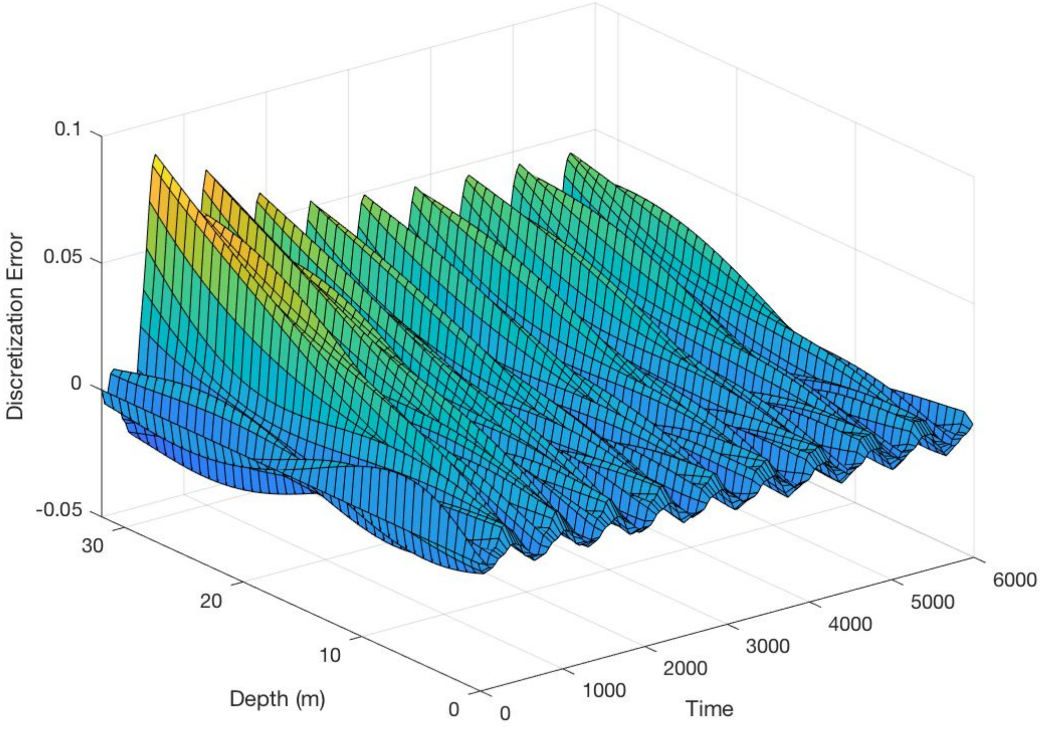
1. Finite Volume Scheme
    a. Discretization error

$$\varepsilon_{\Delta x} \approx \frac{T_{\Delta x} - T_{2\Delta x}}{2^p - 1}$$

2. Proper Orthogonal Decomposition
    a. Truncation error results from only using the first two modes
    b. Proportion of variance is captured by the corresponding eigenvalues

# Discretization Error

$$\varepsilon_{\Delta x} \approx \frac{T_{\Delta x} - T_{2\Delta x}}{2^p - 1}$$



# Future Work

1. Repeat my analysis for other numerical schemes
2. Build a neural network to predict the coefficients of the principal components as a function of sea surface temperature
    a. Quantify the uncertainty of this new projection
3. Build a 4-D (x, y, z, t) multi-fidelity model using data from the buoys, the satellites, and the numerical simulation

# References

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