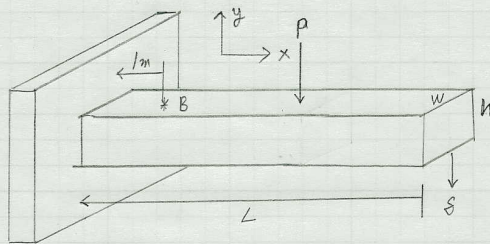




## 2.31 ASSIGNMENT #5 SOLUTION.



GIVEN:

$$P = 10 \text{ N at midspan}$$

$$L = 10 \text{ m}$$

$$w, h = 1 \text{ m}$$

$$E = 1 \text{ MPa}$$

$$\Delta = 0$$

I. Calculate  $\delta$  at the free end, and axial stress  $\sigma_{xx}(B) = \sigma_B$

a) Calculate the axial stress  $\sigma_{xx}$  at B.

From Mechanics of Materials,

$$\sigma(x) = -\frac{M(x)C}{I}$$

$$C = h/2 \text{ (For top surface POINT)}$$

$$I = wh^3/12$$

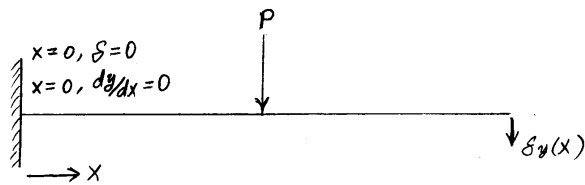
$$M(x) = P(x - L/2) \quad 0 \leq x \leq L/2$$

$$= 0 \quad L/2 < x \leq L$$

$$\text{So, } \sigma(x) = \frac{-P(x - L/2)(h/2)}{wh^3/12} = \frac{-6P(x - L/2)}{wh^2}$$

$$\sigma_B = \frac{-6P(x - L/2)}{wh^2} = \frac{-6(10 \text{ N})(1 \text{ m} - 5 \text{ m})}{(1 \text{ m})(1 \text{ m})^2} = \boxed{240 \text{ Pa (Tension)}}$$

b) Find  $\delta_y(x)$  at  $x=L$ .



$$EI \frac{d^2y}{dx^2} = M(x) = -P(x - L/2) \quad 0 \leq x \leq L/2$$

$$EI \frac{dy}{dx} = \int P(x - L/2) dx = \frac{P}{2} (x^2 - Lx) + C_1$$

Using B.C. ( $x=0, dy/dx=0$ )

$$\checkmark C_1 = 0$$

$$EI \delta_y(x) = \int \frac{P}{2} (x^2 - Lx) dx = \frac{P}{2} \left[ \frac{x^3}{3} - \frac{Lx^2}{2} \right] + C_2$$

Using B.C. ( $x=0, \delta=0$ )

$$\checkmark C_2 = 0$$

$$\therefore \delta_y(x) = \frac{P}{12EI} [2x^3 - 3Lx^2] \quad 0 \leq x \leq L/2$$

at  $x = L/2$

$$\textcircled{1} \delta_y(L/2) = \frac{P}{12EI} \left[ 2\left(\frac{L}{2}\right)^3 - 3L\left(\frac{L}{2}\right)^2 \right] = \boxed{\frac{-PL^3}{24EI}}$$

$$\textcircled{2} \frac{dy}{dx}(L/2) = \frac{P}{2EI} \left( \frac{L^2}{4} - \frac{L^2}{2} \right) = \boxed{\frac{-PL^2}{8EI}}$$

From  $L/2 < x \leq L$ ,  $\frac{dy}{dx} = \text{Const} = \frac{dy}{dx} \text{ at } L/2$

Using  $y = mx + b$

$$\delta_y(x) = \frac{-PL^2}{8EI} \left(x - \frac{L}{2}\right) - \frac{PL^3}{24EI} \quad L/2 < x \leq L$$

at  $x = L$

$$\delta_y(L) = \frac{-5PL^3}{48EI} = -0.0125m \checkmark$$

## Part II

### a) 1X10 (Linear, Full Integration)

The FE model predictions underestimate both the axial stress level and the beam deflection.

The square shape of the elements in this mesh is associated with a substantial shear when the elements try to accommodate a bending deformation. The mesh responds in an artificially stiff manner (shear locking).

This is a characteristic of first-order, fully integrated elements.

### b) Mesh Density Study

1) Varying number of elements: Not surprisingly, more refined meshes give better results. The ratio between the bending component of deformation and the tensile (or compressive) component decreases if you have more elements through the beam thickness

2) Varying element shape:

The parasitic shear term, which artificially increases the element stiffness, is proportional to the square of the ratio of element length (along the bending axis) to element height (see page 99 handout 5).

Therefore it should be expected that the 1x40 mesh is the least affected by parasitic shear, while the 4x10 mesh is the most affected. Consistently, the 1x40 mesh performed best, and the 4x10 was the worst.

3) Only the beam section between the support and the load has a moment applied to it, and therefore will have some curvature. Curvature is the mode of deformation that linear elements have problems with: They perform perfectly fine if no curvature is present. It therefore makes sense to refine the mesh in areas of maximum curvature, and have a coarser mesh where the elements do not need to curve. Also, we have learned from point 2) that we want many elements along the axis rather than across the thickness.

A simple mesh that follows these ideas is one with 1x30 elements between the wall and the load and only 1x10 elements between the load and the free end.

This mesh will perform better than 4x10, 1x40 (uniform density), and 2x20 meshes.

### c) Changing element types

The Linear-Reduced integration elements yielded virtually no stress. They suffer from their own numerical problem called hourglassing.

Essentially these elements are unable to resist pure bending modes because no strain energy is generated at the single central integration point of the element.

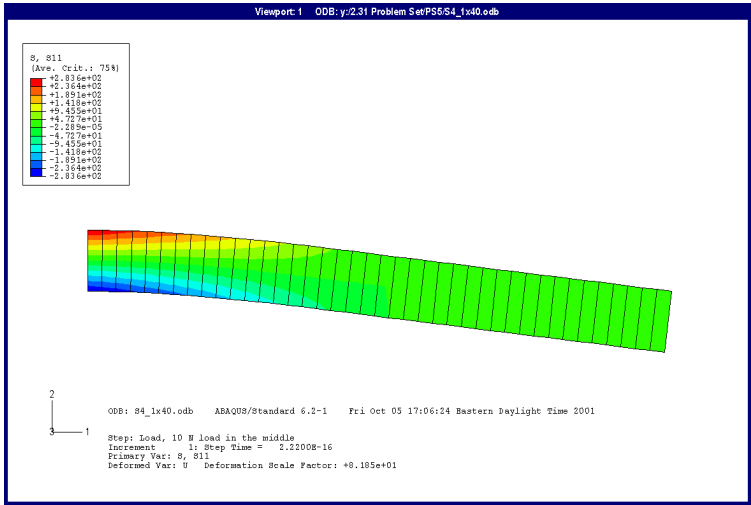
For linear elements the best response is obtained by incompatible mode elements since these elements use internal degrees of freedom to get rid of shear locking and they do not hourglass.

Quadratic elements (full or reduced integration) are able to capture the bending curvature and yield good results in bending.

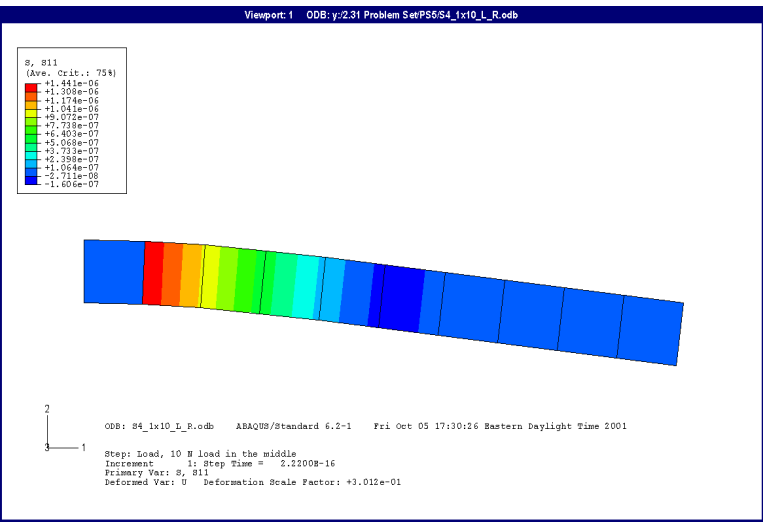
**Table of Deflection and Stress**

No.	Mesh / Method	$\sigma_{11}/\sigma_{theory}$	$\delta y/\delta_{theory}$
1	1X10 (Linear, Full Integration)	0.667	0.672
2	2X20 (Linear, Full Integration)	0.888	0.896
3	4x40 (Linear, Full Integration)	0.970	0.978
4	4X10 (Linear, Full Integration)	0.668	0.672
5	1X40 (Linear, Full Integration)	0.970	0.977
6	1X(30+10) (Linear, FI, Improved Mesh)	0.986	0.993
7	1X10 (Linear, Reduced Integration)	0.000	265.600
8	1X10 (Linear, Incompatible Modes)	1.000	1.004
9	1X10 (Quadratic, Full Integration)	1.000	1.008
10	1X10 (Quadratic, Reduced Integration)	1.000	1.008

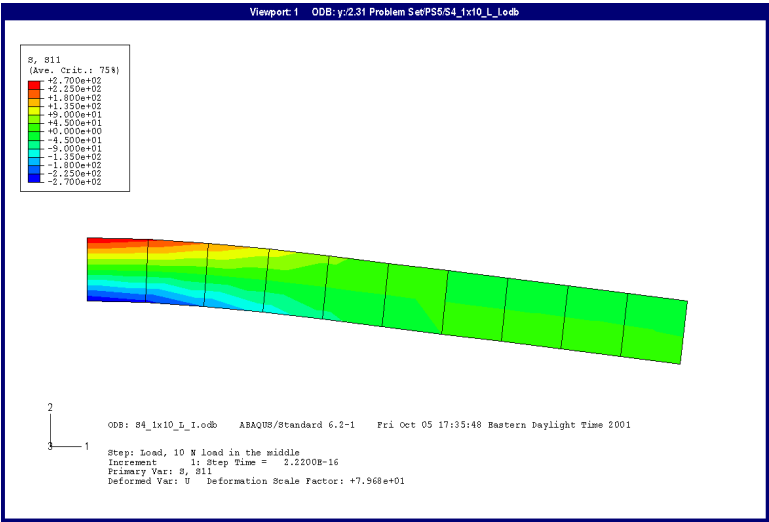
Result for 1X10 Mesh (Linear, No Reduced Integration)



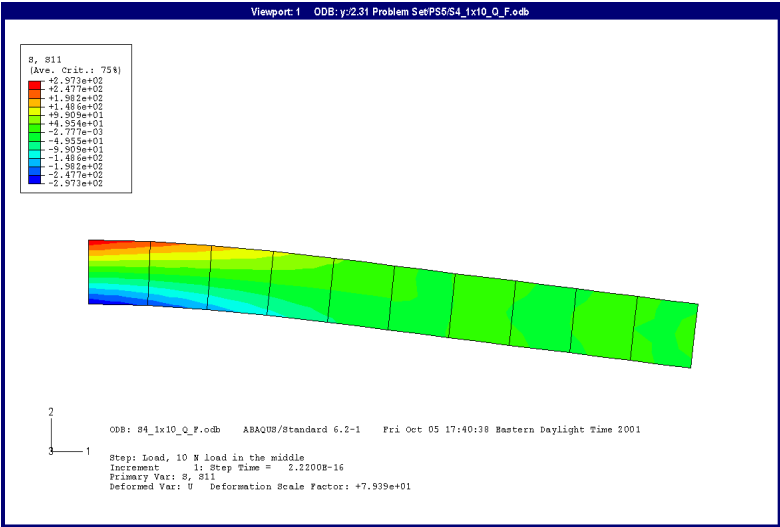
Result for 1X10 Mesh (Linear, Reduced Integration)



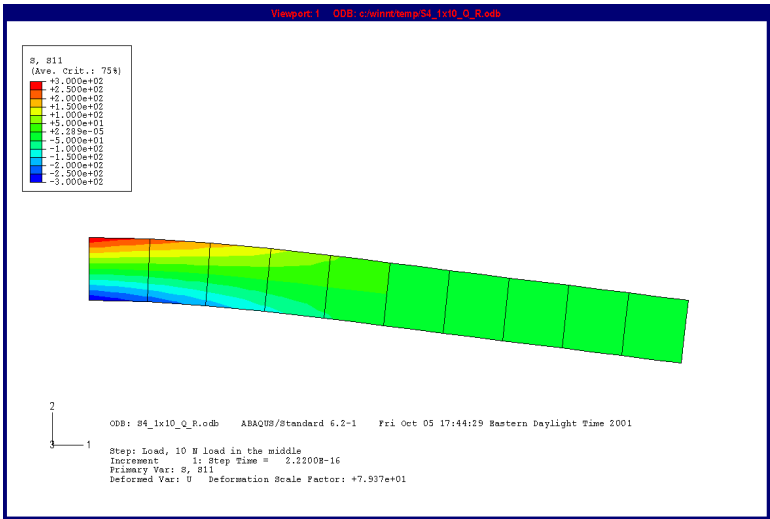
Result for 1X10 Mesh (Linear, Incompatible Modes)



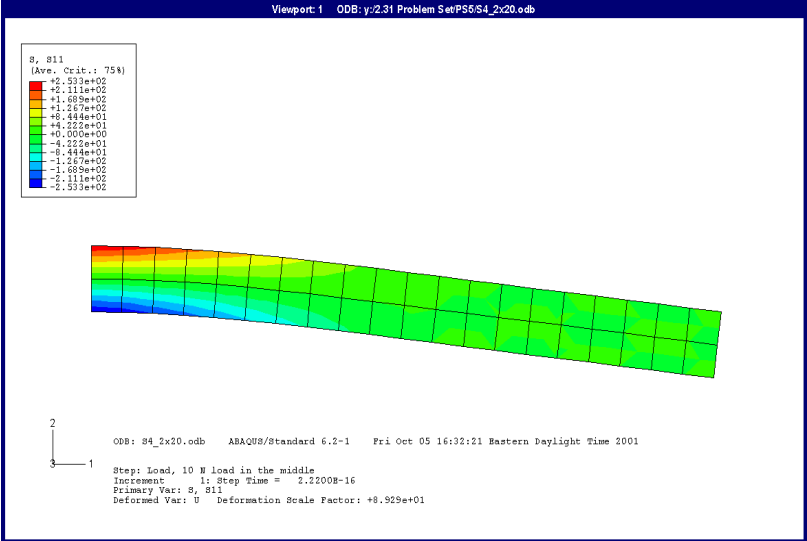
Result for 1X10 Mesh (Quadratic, Full Integration)



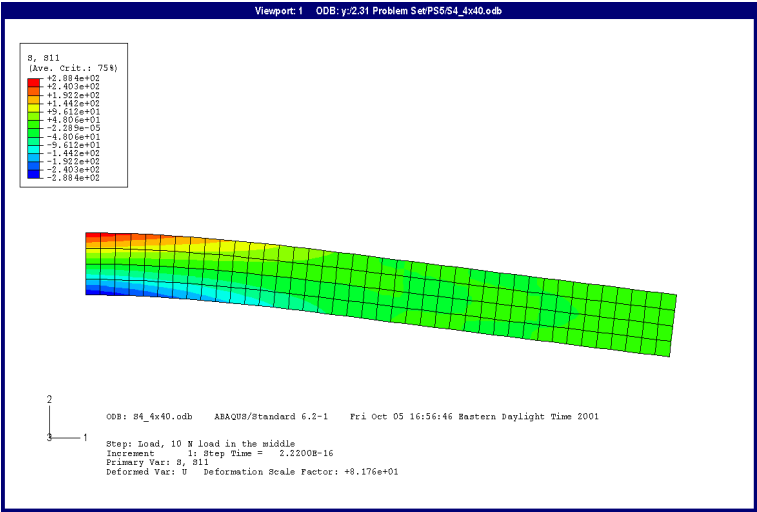
Result for 1X10 Mesh (Quadratic, Reduced Integration)



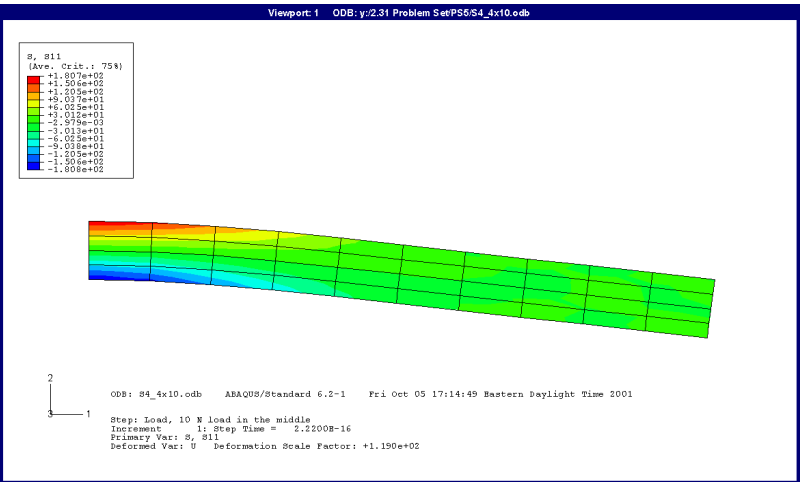
Result for 2X20 Mesh (Linear, No Reduced Integration)



Result for 4X40 Mesh (Linear, No Reduced Integration)

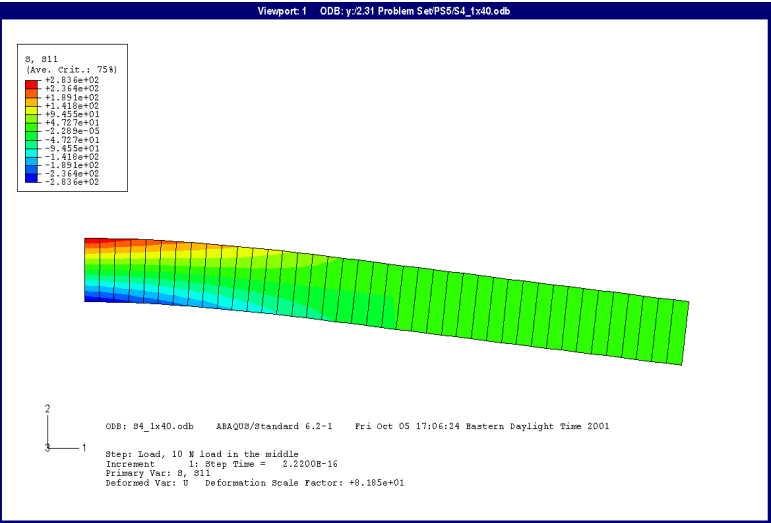


Result for 4X10 Mesh (Linear, No Reduced Integration)





Result for 1X40 Mesh (Linear, No Reduced Integration)



Result for 1X40 Mesh (Optimized)

