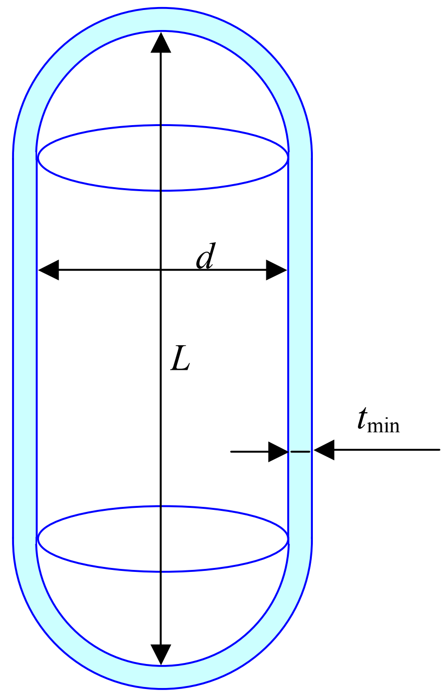
2.31 Assignment 6

Due Mon, Oct 15 at 9:30 am



A cylindrical steel tank of inside diameter d=2m and length L=5m is subjected to an internal pressure P=200 MPa. There is no external pressure. The cylindrical tank is closed at both ends by hemispherical caps.

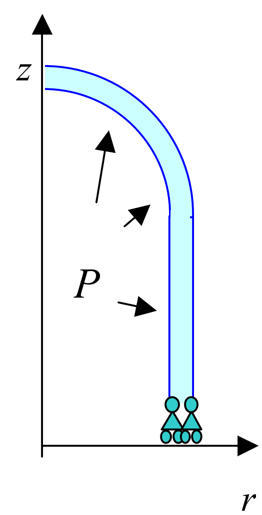
The material elastic properties are E=200 GPa, and ν=0.3. The material yield stress is 1.5 GPa. For safety reasons, we want to limit the maximum tangential stress ($\sigma_{\theta\theta}$) in the component to 1.0 GPa.

Our task is to determine the minimum (uniform) thickness of the tank, t_{min} , necessary to meet this design requirement.

1) Pen and paper work (no FE)

Idealize the tank as a thin-walled pressure vessel. Based on this idealization, obtain a first estimate for the wall thickness, t_1, which satisfies the design requirement. Obtain estimates for the three components of stress σ_{rr}, $\sigma_{\theta\theta}$, σ_{zz} at the cylindrical wall. Does thin-wall theory apply for this component? For a tank of thickness t_1, calculate the stress distributions, σ_{rr}, $\sigma_{\theta\theta}$, σ_{zz} through the tank thickness using the thick-cylinder formulation. (A quick review of the theory to obtain σ_{rr} and $\sigma_{\theta\theta}$ is attached at the end of this assignment; σ_{zz} can be considered uniform through the thickness). Does your design satisfy the requirement? Using the thick-wall solution, obtain an improved estimate for the wall thickness, t_2, which should satisfy the design requirement.

2) FE model



Create an axisymmetric FE model of the tank of thickness t_2 using quadratic full integration quadrilateral elements. You can take advantage of the symmetry of the problem and model only half of the component. Be careful when you impose the boundary conditions: the entire edge along the z axis must be constrained in r, and the entire edge along the r axis must be constrained in z.

Things to keep in mind as you set up the model:

PROPERTY : In section property you want to create a *solid* *homogeneous* section of thickness 1.
In material properties you have to input only the Mechanical props (E, ν).

MESH : Seed the assembly with a global element size of 0.1. In the mesh dialog choose standard quadratic quads, and make sure you click off the reduced integration option. (you should have CAX8 as element type).

Look at the results in the VISUALIZATION module of ABAQUS/CAE:
Plot and print the contours of s11, s22, s33 (= σ_{rr}, σ_{zz} , $\sigma_{\theta\theta}$) over the tank → attach the plots to your assignment.
Create a node path along the edge on the horizontal axis of symmetry (r-axis), and obtain and plot the profiles of σ_{rr}, σ_{zz} and $\sigma_{\theta\theta}$ as a function of r → attach the plots to your assignment. How does the FE prediction compare w/ your thick-wall theory estimate? How does it compare with a thin wall estimate? Comments?

Repeat the whole simulation with a more refined mesh (seed with 0.05). Compare the stress profiles between the two models and compare them to the theoretical estimate. What is your final choice for t_{min}?

2.8. Thick Cylinder

When the thickness of the cylindrical vessel is relatively large, as in the case of gun barrels, high-pressure hydraulic ram cylinders, etc., the variation in the stress from the inner surface to the outer surface becomes appreciable, and the ordinary membrane or average stress

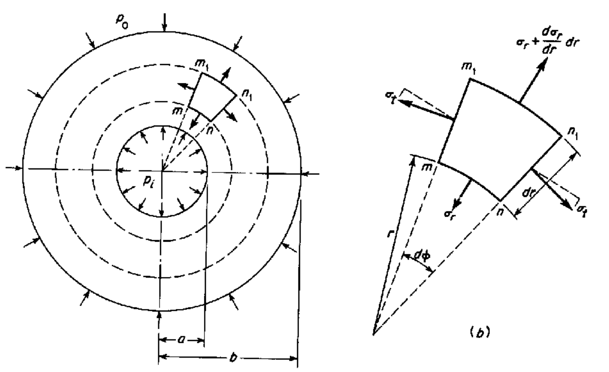


Fig. 2.16. Stresses in a Thick-Walled Cylinder

formulas are not a satisfactory indication of the significant stress. If a cylinder of constant wall thickness is subjected to an internal pressure p_i and external pressure p_0, the deformation will be symmetrical about its axis and will not change along its length, Fig. 2.16. It may be thought of as being composed of a series of concentric cylinders. If a ring is cut by two planes perpendicular to the axis at unit distance apart, it is seen that a condition of symmetry exists and hence no

shearing stresses exist on the sides of the element mnm_1n_1, Fig. 2.16*a*. This is the reason, for instance, that it is possible to construct heavy gun barrels or pressure cylinders by a multitude of concentric thin cylinders or multiple layers of bands, and no provision need be made for the transfer of shearing forces from one band to the next. Considering the element mnm_1n_1, the hoop stress acting on the sides mm_1 and nn_1 is σ_t. The radial stress normal to the side mn is σ_r, and this stress varies with the radius r in the amount of $(d\sigma_r/dr)dr$ over a distance dr. Therefore, the normal radial stress on the side m_1n_1 is

$$\sigma_r + \frac{d\sigma_r}{dr} dr \tag{2.8.1}$$

The equation of equilibrium for the element is obtained by summing up the forces in the direction of the bisector of the angle $d\phi$, noting that for small angles the sine and angle in radians are substantially equal. Then

$$\sigma_r r d\phi + \sigma_t dr d\phi - \left(\sigma_r + \frac{d\sigma_r}{dr} dr\right)(r + dr) d\phi = 0 \tag{2.8.2}$$

and if small quantities of high order are neglected,

$$\sigma_t - \sigma_r - r\frac{d\sigma_r}{dr} = 0 \tag{2.8.3}$$

The equation gives one relation between the stresses σ_t and σ_r. A second relation can be obtained from the deformation of the cylinder and from the assumption that the longitudinal strain of all fibers is equal. The deformation of the cylinder is then symmetrical with respect to the axis and consists of a radial displacement of all points in the wall of the cylinder. Hence, this displacement is constant in the circumferential direction but varies with distance along a radius. If u denotes the radial displacement of a cylindrical surface of radius r, the radial displacement of a surface of radius $r + dr$ is

$$u + \frac{du}{dr} dr \tag{2.8.4}$$

Therefore, an element mnm_1n_1 undergoes a total elongation in a radial direction of $(du/dr)dr$, or a unit elongation of

$$e_r = \left(\frac{du}{dr}\right)\frac{dr}{dr} = \frac{du}{dr} \tag{2.8.5}$$

In a circumferential direction the unit elongation of the same element is equal to the unit elongation of the corresponding radius, paragraph 2.4, or

$$e_t = \frac{u}{r} \tag{2.8.6}$$

Then from Eqs. 2.3.4 and 2.3.5 a second set of expressions for the stresses in terms of the strains becomes

$$\sigma_r = \frac{E}{1-\mu^2}\left(\frac{du}{dr} + \mu\frac{u}{r}\right) \tag{2.8.7}$$

$$\sigma_t = \frac{E}{1-\mu^2}\left(\frac{u}{r} + \mu\frac{du}{dr}\right) \tag{2.8.8}$$

These stresses are interdependent since they are expressed in terms of one function u. By substituting the values for σ_r and σ_t from Eqs. 2.8.7 and 2.8.8 into Eq. 2.8.3, the following equation for determining u is obtained:

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} = 0 \tag{2.8.9}$$

The general solution of this equation is

$$u = C_1 r + \frac{C_2}{r} \tag{2.8.10}$$

Substituting from Eq. 2.8.10, noting $du/dr = C_1 - C_2/r^2$, into Eqs. 2.8.7 and 2.8.8 gives

$$\sigma_r = \frac{E}{1-\mu^2}\left[C_1(1+\mu) - C_2\frac{1-\mu}{r^2}\right] \tag{2.8.11}$$

$$\sigma_t = \frac{E}{1-\mu^2}\left[C_1(1+\mu) + C_2\frac{1-\mu}{r^2}\right] \tag{2.8.12}$$

The constants C_1 and C_2 can be determined from the conditions at the inner and outer surfaces of the cylinder where the pressure, i.e., the normal stresses σ_r, are known. For instance, if p_i denotes the internal pressure and p_o denotes the external pressure, the conditions at the inner and outer surfaces of the cylinder are

$$\sigma_{r_a} = -p_i \quad \text{and} \quad \sigma_{r_b} = -p_o \tag{2.8.13}$$

The negative sign on the right-hand side of these equations indicates that the stress is compressive, because the normal stress is considered positive for tension. Substituting the expressions for σ_r from Eq. 2.8.11 into Eq. 2.8.13 gives equations for determining the constants C_1 and C_2 from which

$$C_1 = \frac{1-\mu}{E}\,\frac{a^2p_i - b^2p_o}{b^2-a^2}; \qquad C_2 = \frac{1+\mu}{E}\,\frac{a^2b^2(p_i-p_o)}{b^2-a^2} \tag{2.8.14}$$

Placing the values of these into Eqs. 2.8.11 and 2.8.12 gives the general expressions for the normal stresses:

$$\sigma_r = \frac{a^2p_i - b^2p_o}{b^2-a^2} - \frac{(p_i-p_o)a^2b^2}{r^2(b^2-a^2)} \tag{2.8.15}$$

$$\sigma_t = \frac{a^2p_i - b^2p_o}{b^2-a^2} + \frac{(p_i-p_o)a^2b^2}{r^2(b^2-a^2)} \tag{2.8.16}$$

Inspection of Eqs. 2.8.15 and 2.8.16 indicates that the maximum value of σ_t occurs at the inner surface, and maximum σ_r will always be the larger of the two pressures, p_i and p_o. These equations are known as the Lamé solution, or thick-cylinder formulas. It is noted that the sum of these two stresses remains constant; hence the deformation of all elements in the axial direction is the same, and cross sections of the cylinder remain plane after deformation, thereby fulfilling the original assumption.

The maximum shearing stress at any point in the cylinder is equal to one half the algebraic difference of the maximum and minimum principal stresses at that point. Since the longitudinal (axial) stress is usually small compared to σ_r and σ_t,

$$\tau = \frac{\sigma_t - \sigma_r}{2} = \frac{(p_i-p_o)}{b^2-a^2}\,\frac{a^2b^2}{r^2} \tag{2.8.17}$$

1. *Cylinder Under Internal Pressure Only*

In this particular case which covers most of the practical vessel applications, $p_o = 0$, Eqs. 2.8.15 and 2.8.16 reduce to:

$$\sigma_r = \frac{a^2p_i}{b^2-a^2}\left(1 - \frac{b^2}{r^2}\right) \tag{2.8.18}$$

$$\sigma_t = \frac{a^2p_i}{b^2-a^2}\left(1 + \frac{b^2}{r^2}\right) \tag{2.8.19}$$

These equations show that both stresses are maximum at the inner surface where r has the minimum value; σ_r is always a compressive stress, and smaller than σ_t; and σ_t a tensile stress which is maximum at the inner surface of the cylinder equal to

$$\sigma_{t_{\max.}} = \frac{p_i(a^2+b^2)}{b^2-a^2} \tag{2.8.20}$$

From Eq. 2.8.20 it is seen that $\sigma_{t_{\max.}}$ is always numerically greater than the internal pressure, but approaches this value as b increases. The minimum value of σ_t is at the outer surface of the cylinder and is always less than that at the inner surface by the value of the internal pressure p_i. Figure 2.17 illustrates this variation through the wall of a thick cylinder of ratio $K = \dfrac{\text{outside radius}}{\text{inside radius}} = 2.0$. In designing for very high pressure, these observations point out the necessity of using comparably high yield point materials, or using design-construction features that will create an initial residual compressive stress on the inner surface to help counterbalance the high applied stress at this location, such as hoops shrunk on the barrels of guns, or cylinders strained beyond the yield point by hydraulic pressure so that upon release of the pressure the metal at the bore remains in a state of residual compression and the outer layers in moderate tension.

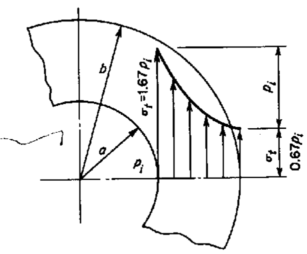


Fig. 2.17. Variation in Tangential Stress Through the Wall of a Thick Cylinder, $K = b/a = 2$

A comparison between the maximum stress obtained by the thick-cylinder formula, Eq. 2.8.20, and that obtained by the thin cylinder or average stress formula, Eq. 2.2.4, is shown in Table 2.1 for various values of K, and indicates that for small wall thicknesses there is little difference; for instance, with a wall thickness of 20 per cent of the inside radius, the maximum stress is only 10 per cent higher than the average stress.

The shearing stress is a maximum on the inner surface and from Eq. 2.8.17 for $r = a$ gives

$$\tau = \frac{p_i b^2}{b^2-a^2} \tag{2.8.21}$$

This equation for the shearing stress is particularly significant from a design viewpoint as a criteria of failure, since it correlates very well with actual rupture tests of thick cylinders, paragraph 5.15.

2. *Cylinder Under External Pressure Only*

When the internal pressure is zero and the cylinder is subject to only the action of the external pressure p_o, Eqs. 2.8.15 and 2.8.16 reduce

$$\sigma_r = -\frac{p_o b^2}{b^2-a^2}\left(1-\frac{a^2}{r^2}\right) \tag{2.8.22}$$

$$\sigma_t = -\frac{p_o b^2}{b^2-a^2}\left(1+\frac{a^2}{r^2}\right) \tag{2.8.23}$$

These equations show that both σ_r and σ_t are compressive stresses with σ_t always being numerically greater than σ_r just as it was in the case of internal pressure only. The maximum tangential compressive stress σ_t occurs on the inside surface of value

TABLE 2.1. RATIO OF MAXIMUM TO AVERAGE STRESS ON A CYLINDER FOR VARIOUS VALUES OF K (RATIO OF OUTSIDE TO INSIDE RADIUS)

	$K = b/a = 1 + h/a$					
	1.1	1.2	1.4	1.6	1.8	2.0
$\sigma_{\max}/\sigma_{\text{avg.}}$	1.05	1.10	1.23	1.37	1.51	1.67

$$\sigma_{t_a} = -\frac{2p_o b^2}{b^2-a^2} \tag{2.8.24}$$

whereas the maximum radial stress σ_r is at the outer surface and equal to p_o; i.e., the maximum values of σ_t and σ_r do not occur at the same point in the cylinder in this case. When the external pressure is reversed in direction, as could result from a vacuum surrounding the cylinder or a series of outwardly directed uniformly applied loads, p_o is replaced by $-p_o$. When the ratio b/a becomes very large, it is noted that the maximum stress approaches twice the value of the external pressure which agrees with that found in paragraph 6.3 for a small hole in a large plate subject to uniformly distributed radial forces.

3. *Deformation of a Thick Cylinder*

The radial displacement of any point in the wall of the cylinder can be found from Eq. 2.8.10 by substituting the values of the constants C_1 and C_2 from Eq. 2.8.14, which gives

$$u = \frac{1-\mu}{E}\frac{a^2p_i - b^2p_o}{b^2-a^2}r + \frac{1+\mu}{E}\frac{a^2b^2(p_i-p_o)}{(b^2-a^2)r} \tag{2.8.25}$$

In the case of a cylinder subjected to internal pressure p_i only, the radial displacement at the inner surface $r = a$ from Eq. 2.8.25 is:

$$u_a = \frac{p_i a}{E}\left(\frac{a^2+b^2}{b^2-a^2}+\mu\right) \tag{2.8.26}$$

and at the outer surface is:

$$u_b = \frac{2p_i a^2 b}{E(b^2-a^2)} \tag{2.8.27}$$

When the cylinder is subjected to external pressure p_o only, the displacement of the inner surface, $r = a$, is:

$$u = -\frac{2p_o ab^2}{E(b^2-a^2)} \tag{2.8.28}$$

and at the outer surface is:

$$u = -\frac{bp_o}{E}\left(\frac{a^2+b^2}{b^2-a^2}-\mu\right) \tag{2.8.29}$$

The minus sign indicates that the displacement is toward the axis of the cylinder.

2.9. Shrink-Fit Stresses in Builtup Cylinders

Cylindrical vessels can be reinforced by shrinking on an outer cylindrical liner so that a contact pressure is produced between the two. This is usually done by making the inside radius of the outer cylinder smaller than the outside radius of the inner one and assembling the two after first heating the outer one. (The reverse procedure of cooling the inner cylinder with Dry Ice or liquid gases has also been used.) A contact pressure is developed after cooling dependent upon the initial interference of the two cylinders. Its magnitude and the stresses it produces are calculable by the equations of paragraph 2.8.3. As an example, prior to assembly the outside radius b of the inner cylinder in Fig. 2.18 was larger than the inside radius of the outer cylinder by an amount δ, creating a pressure p between the cylinders after assembly. Its value can be determined from the condition that the increase in the inner radius of the outer cylinder plus the decrease in the outer radius of the inner cylinder must equal δ. Thus, from Eqs. 2.8.26 and 2.8.29,

$$\frac{bp}{E}\left(\frac{b^2+c^2}{c^2-b^2}+\mu\right) + \frac{bp}{E}\left(\frac{a^2+b^2}{b^2-a^2}-\mu\right) = \delta \tag{2.9.1}$$

or

$$p = \frac{E\delta}{b}\,\frac{(b^2-a^2)(c^2-b^2)}{2b^2(c^2-a^2)} \tag{2.9.2}$$

If such a builtup cylinder is now subjected to internal pressure, the

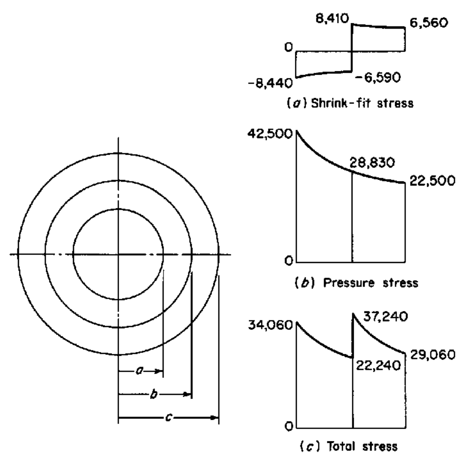


Fig. 2.18. Shrink-Fit Tangential Stresses in a Cylinder

stresses produced by this pressure are the same as those in a solid-wall cylinder of thickness equal to the sum of those of the individual cylinders $c-a$. These stresses are superposed on the shrink-fit stresses discussed previously. The latter are compressive at the inner surface of the cylinder which reduces the maximum tangential tensile stress due to