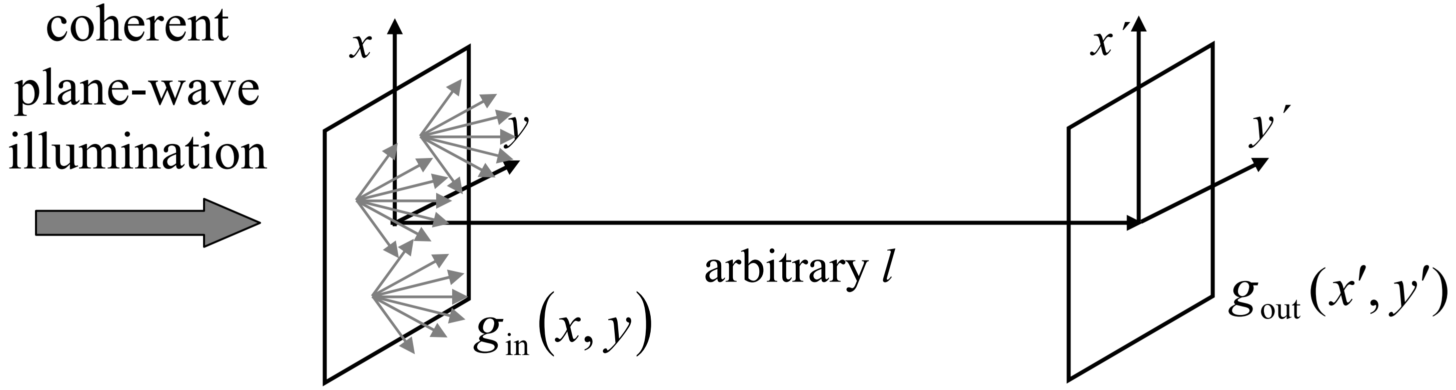
The ideal thin lens
as a Fourier transform engine

Fresnel diffraction

Reminder



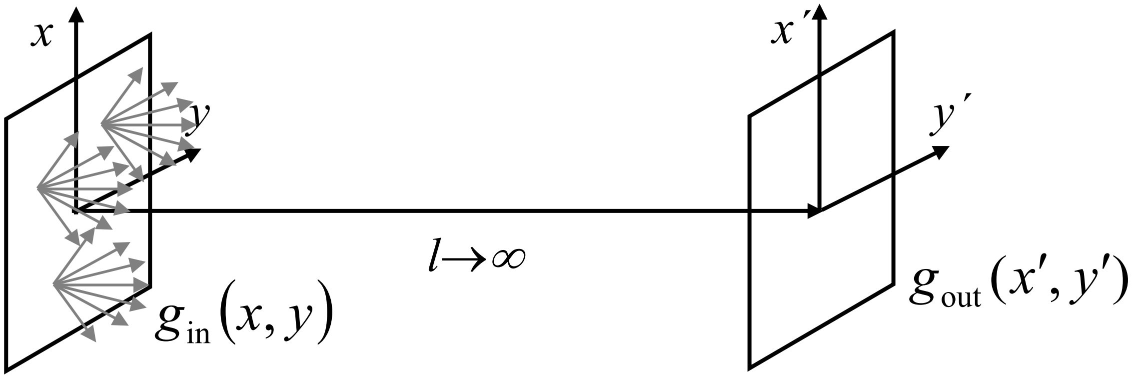
$$g_{\text{out}}(x', y'; l) = \frac{1}{i\lambda l} \exp\left\{i2\pi \frac{l}{\lambda}\right\} \iint g_{\text{in}}(x, y) \exp\left\{i\pi \frac{(x'-x)^2 + (y'-y)^2}{\lambda l}\right\} \mathrm{d}x\mathrm{d}y.$$

The diffracted field is the *convolution* of the transparency with a spherical wave

Q: how can we “undo” the convolution optically**?**

Fraunhofer diffraction

Reminder



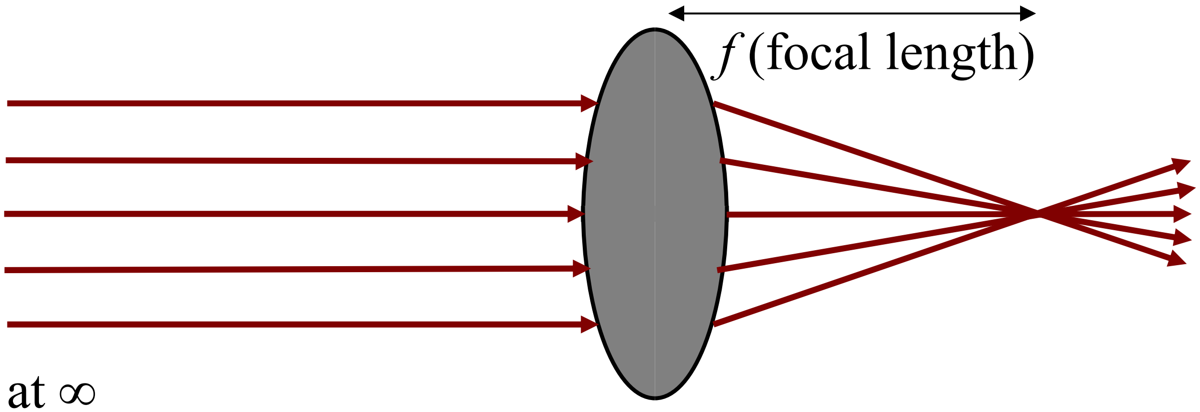
$$g_{\text{out}}(x',y';l) \propto \int g_{\text{in}}(x,y)\exp\left\{-i2\pi\left[x\left(\frac{x'}{\lambda l}\right)+y\left(\frac{y'}{\lambda l}\right)\right]\right\}\mathrm{d}x\mathrm{d}y$$

The **“far-field”** (i.e. the diffraction pattern at a large longitudinal distance *l* equals the **Fourier transform** of the original transparency calculated at spatial frequencies

$$f_x = \frac{x'}{\lambda l} \qquad f_y = \frac{y'}{\lambda l}$$

Q: is there another optical element who can perform a Fourier transformation without having to go too far (to ∞) **?**

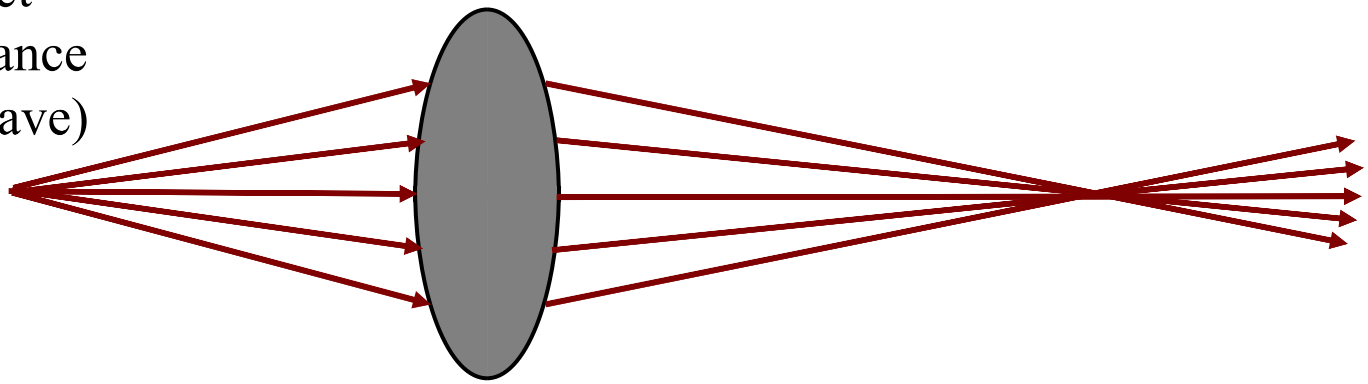
The thin lens (geometrical optics)



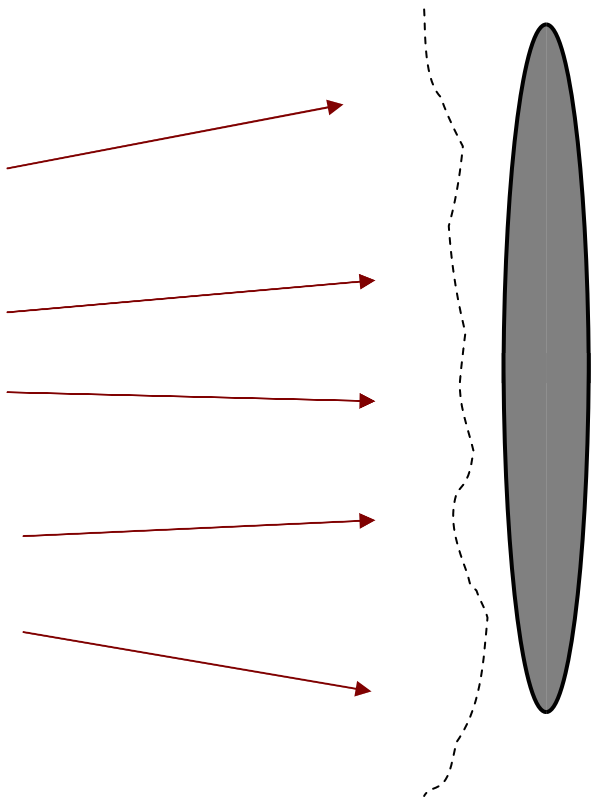
object at ∞
(plane wave)

Ray bending is proportional
to the distance from the axis

point object
at finite distance
(spherical wave)



The thin lens (wave optics)



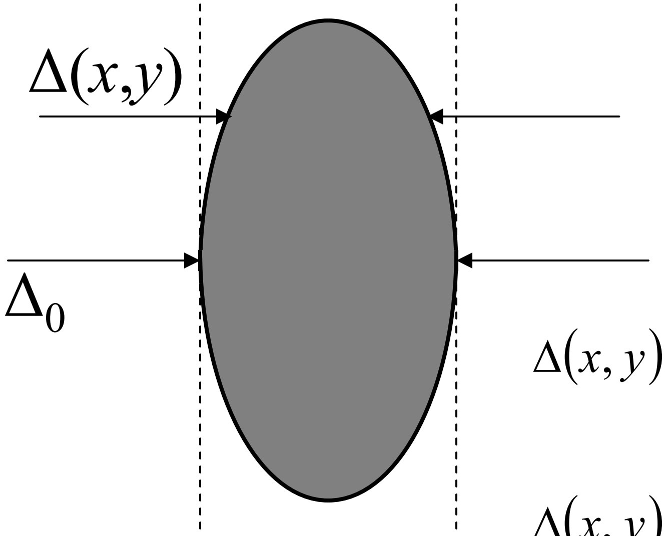
incoming wavefront

$a(x,y)$

outgoing wavefront

$$a(x,y)\ \underbrace{t(x,y)\mathrm{e}^{i\varphi(x,y)}}_{\text{(thin transparency approximation)}}$$

The thin lens transmission function



$$\Delta(x,y)=\Delta_0-R_1\left(1-\sqrt{1-\frac{x^2+y^2}{R_1}}\right)+R_2\left(1-\sqrt{1-\frac{x^2+y^2}{R_2}}\right)$$

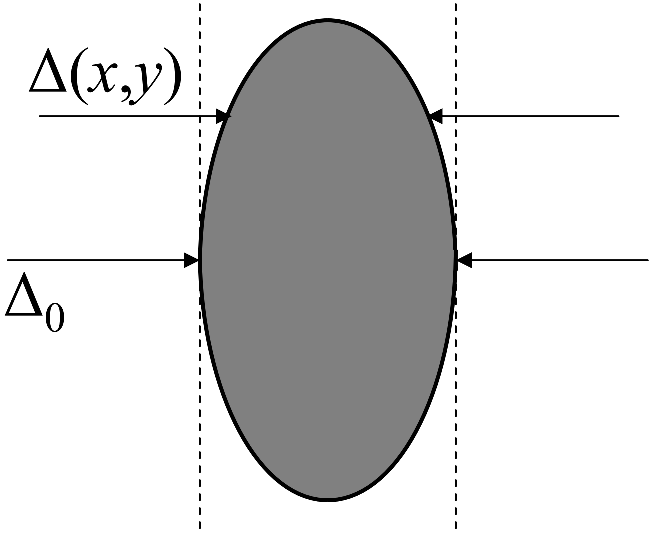
$$\Delta(x,y)\approx\Delta_0-R_1\left(1-\left[1-\frac{x^2+y^2}{2R_1}\right]\right)+R_2\left(1-\left[1-\frac{x^2+y^2}{2R_2}\right]\right)$$

$$\Delta(x,y)\approx\Delta_0-\frac{x^2+y^2}{2}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)$$

$$a_{\text{lens}}(x,y)=\exp\left\{i\frac{2\pi}{\lambda}\Delta_0+\frac{2\pi}{\lambda}(n-1)\Delta(x,y)\right\}$$

$$a_{\text{lens}}(x,y)\approx\exp\left\{i\frac{2\pi n}{\lambda}\Delta_0\right\}\exp\left\{-i\frac{2\pi}{\lambda}(n-1)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\frac{x^2+y^2}{2}\right\}$$

The thin lens transmission function

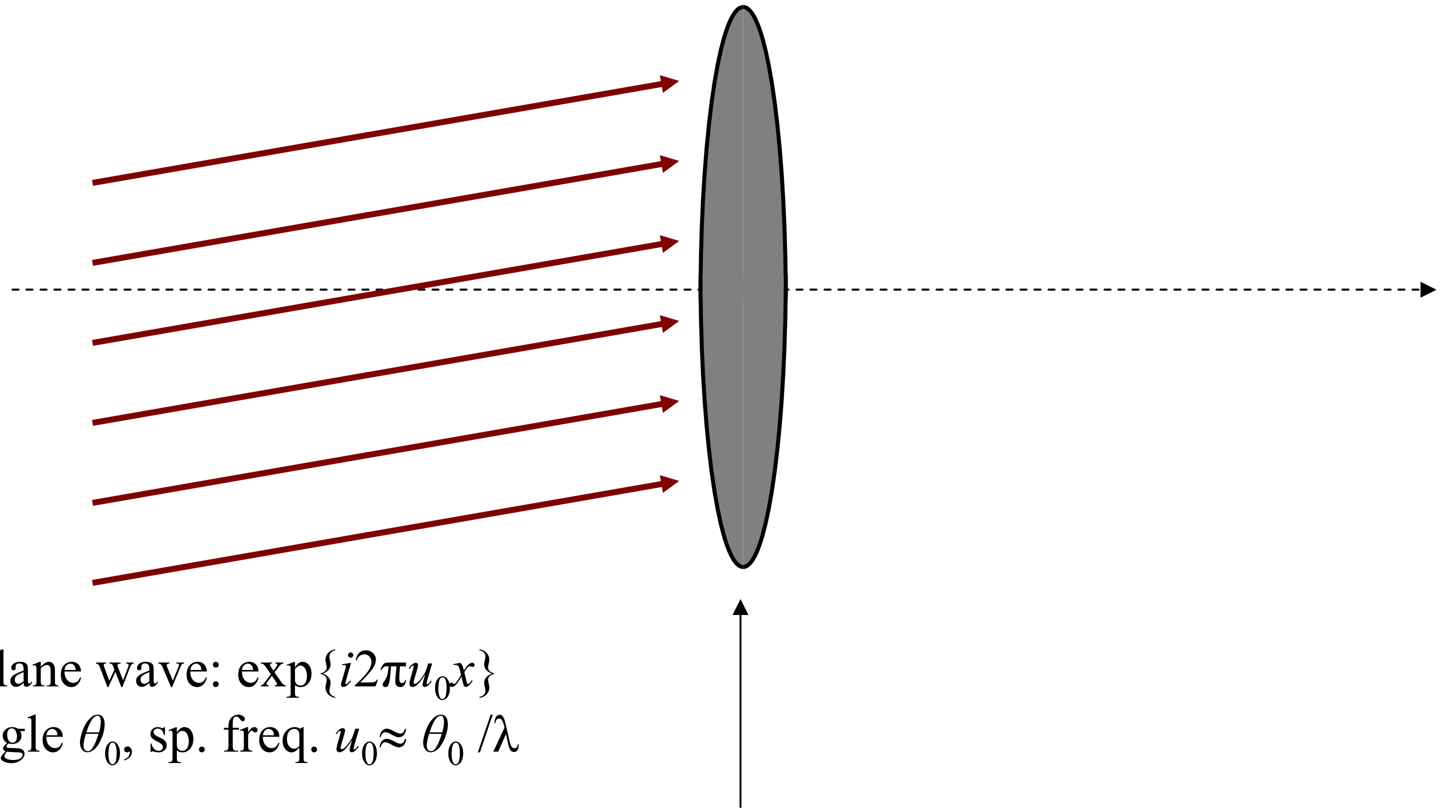


this constant-phase term can be omitted

$$a_{\text{lens}}(x,y) \approx \boxed{\exp\left\{i\frac{2\pi n}{\lambda}\Delta_0\right\}} \exp\left\{-i\pi\frac{x^2+y^2}{\lambda f}\right\}$$

where $\frac{1}{f} \equiv (n-1)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)$ is the focal length

Example: plane wave through lens

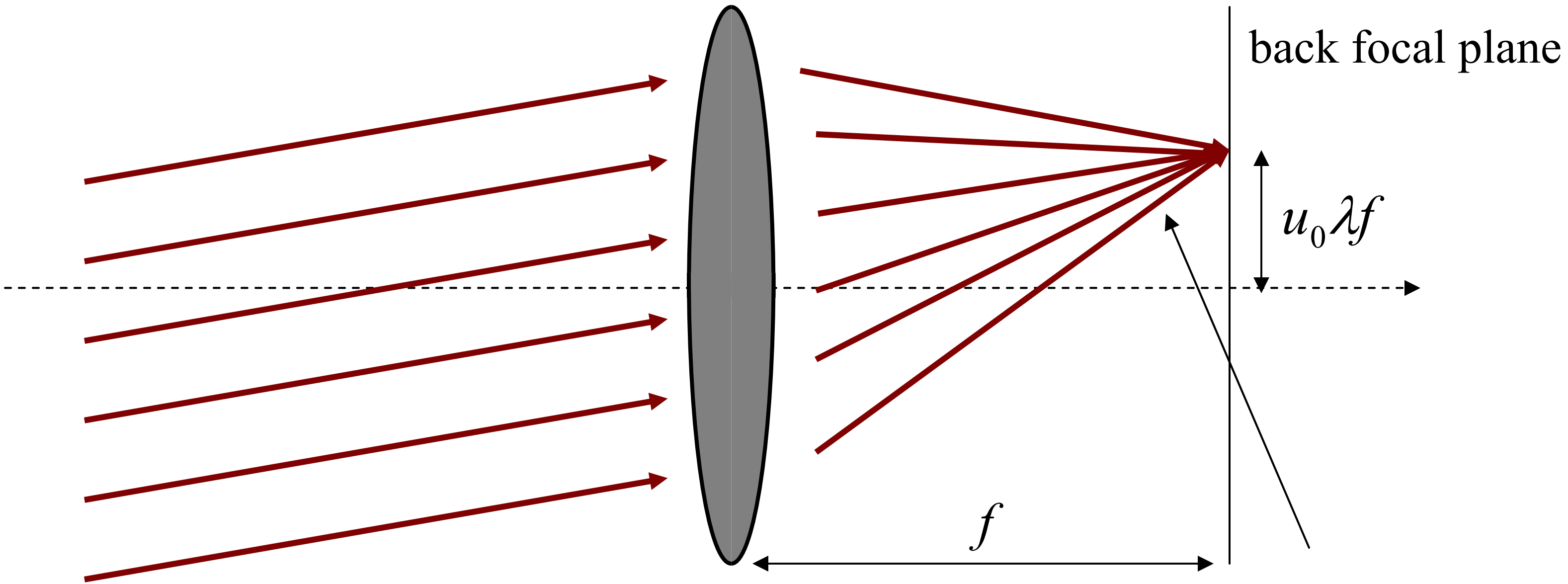


plane wave: $\exp\{i2\pi u_0 x\}$

angle θ_0, sp. freq. $u_0 \approx \theta_0 / \lambda$

lens, $a_{\text{lens}}(x,y) = \exp\left\{-i\pi \frac{x^2+y^2}{\lambda f}\right\}$

Example: plane wave through lens



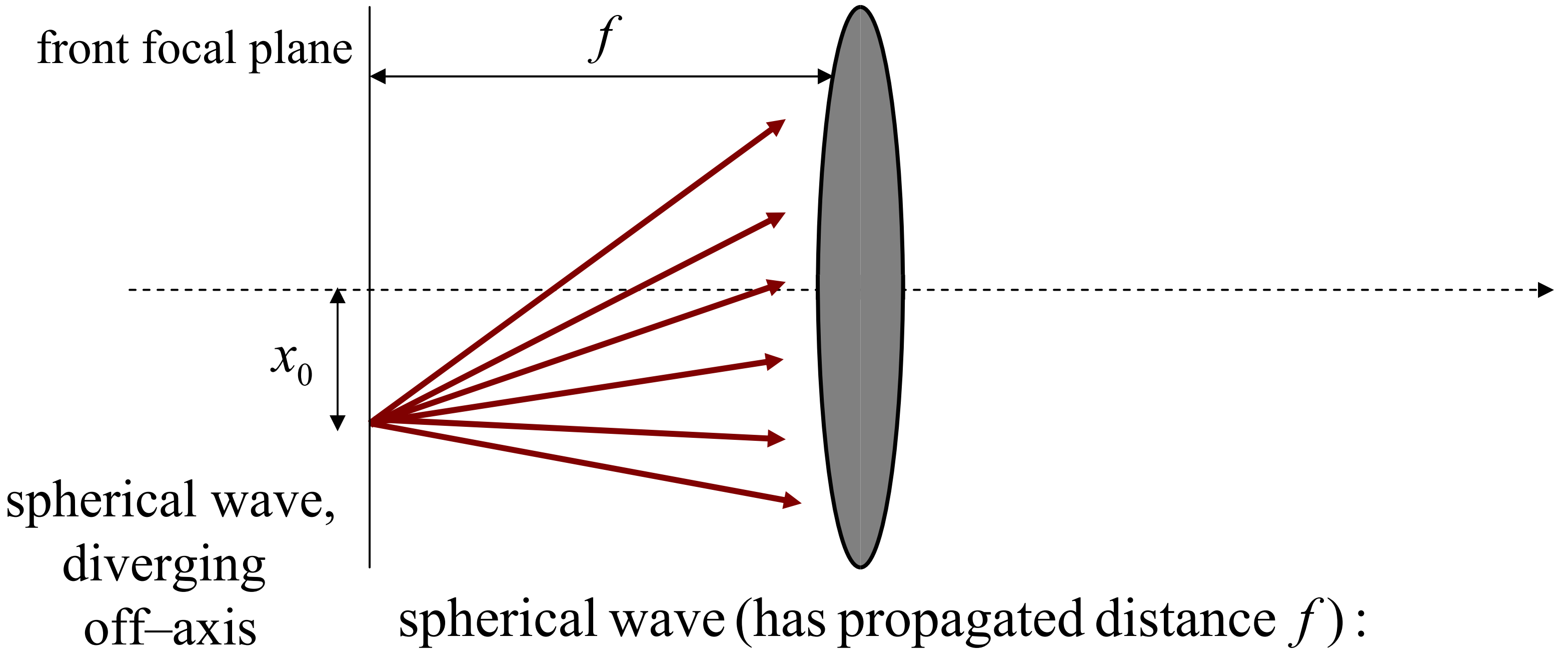
spherical wave,
converging
off–axis

wavefront after lens:

$$a_+(x,y)=\exp\{i2\pi u_0 x\}\exp\left\{-i\pi\frac{x^2+y^2}{\lambda f}\right\}$$

$$a_+(x,y)=\underset{\text{ignore}\uparrow}{\boxed{\exp\{i\pi u_0^2\lambda f\}}}\exp\left\{-i\pi\frac{(x-u_0\lambda f)^2+y^2}{\lambda f}\right\}$$

Example: spherical wave through lens



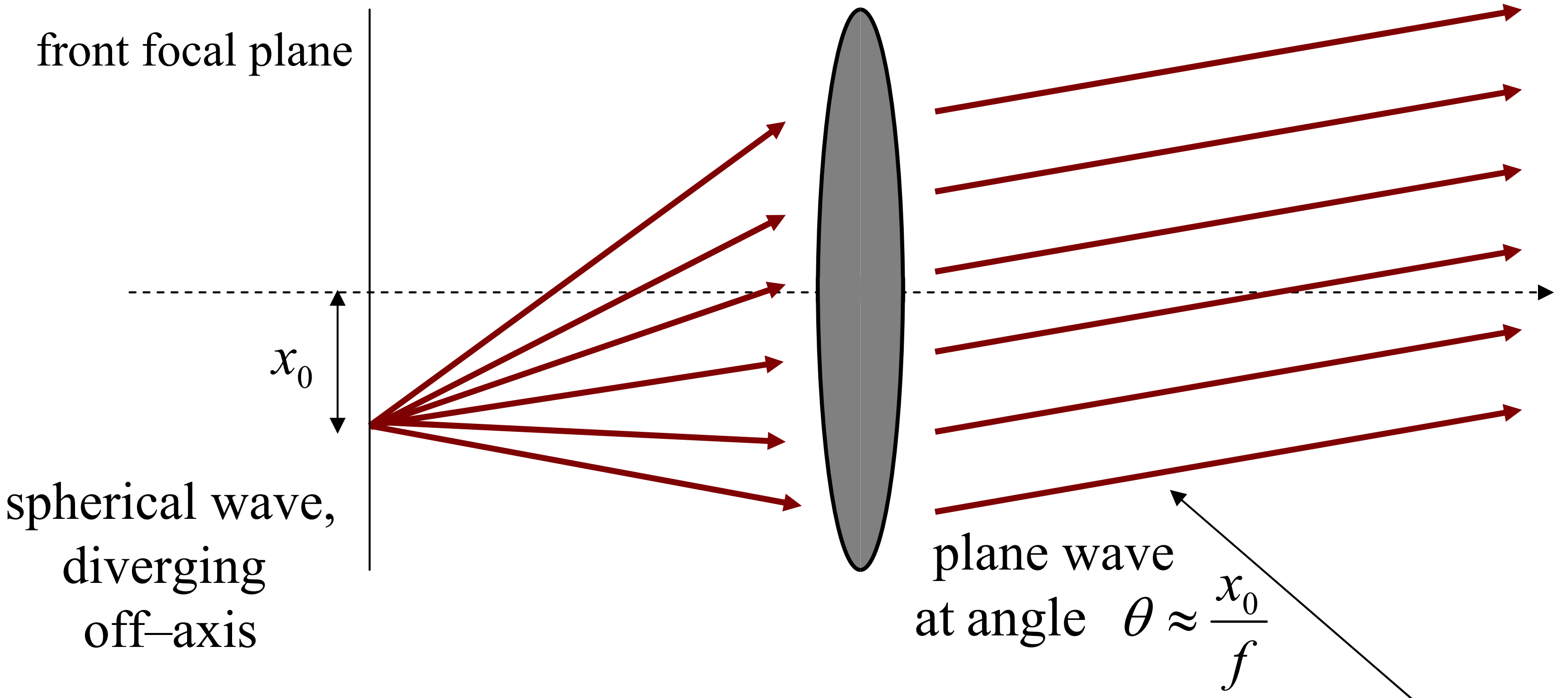
spherical wave (has propagated distance f) :

$$a_{-}(x,y)=\exp\left\{i2\pi\frac{f}{\lambda}\right\}\exp\left\{i\pi\frac{(x+x_0)^2+y^2}{\lambda f}\right\}$$

lens transmission function :

$$a_{\text{lens}}(x,y)=\exp\{i2\pi n\Delta_0\}\exp\left\{-i\pi\frac{x^2+y^2}{\lambda f}\right\}$$

Example: spherical wave through lens



front focal plane

x_0

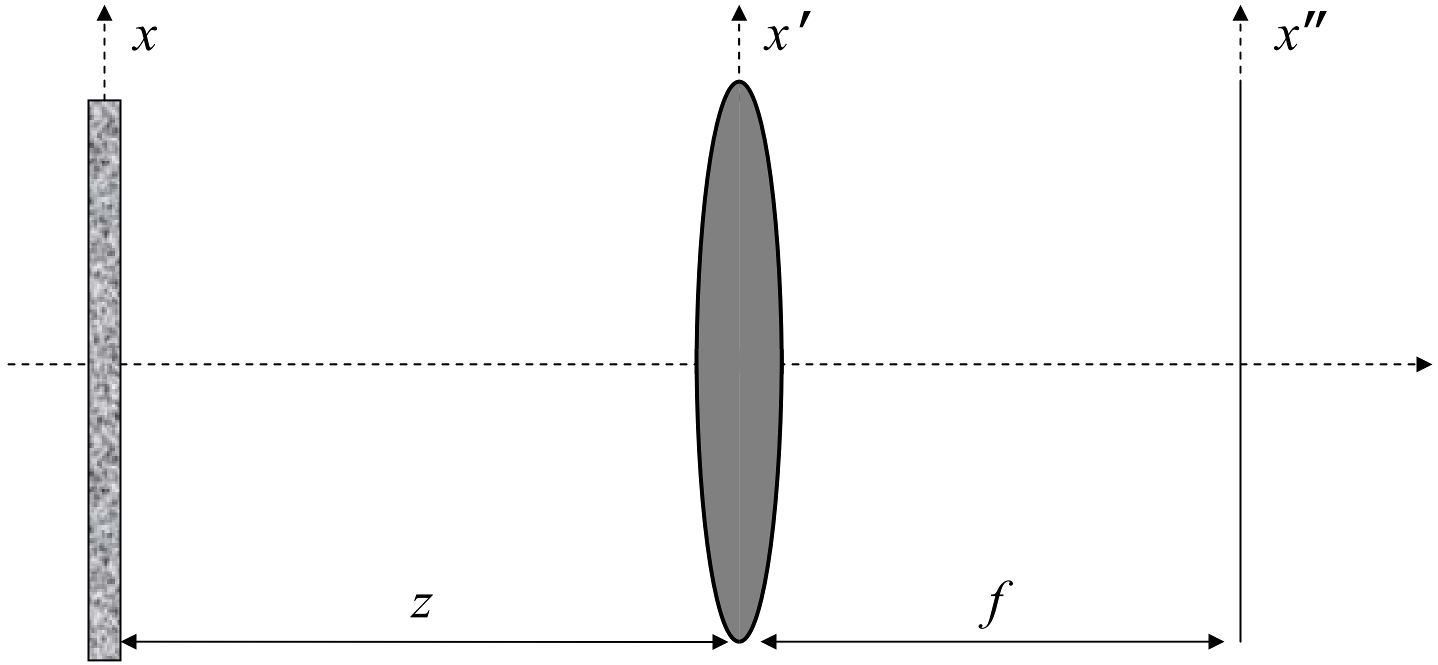
spherical wave,
diverging
off–axis

plane wave
at angle $\theta \approx \frac{x_0}{f}$

wavefront after lens

$$a_+(x,y)=a_-(x,y)\times a_{\text{lens}}(x,y)=\exp\left\{\underbrace{i2\pi\left(n\Delta_0+\frac{f}{\lambda}\right)+i\pi\frac{x_0^2}{\lambda f}}_{\text{ignore}}+i2\pi\frac{x_0 x}{\lambda f}\right\}$$

Diffraction at the back focal plane

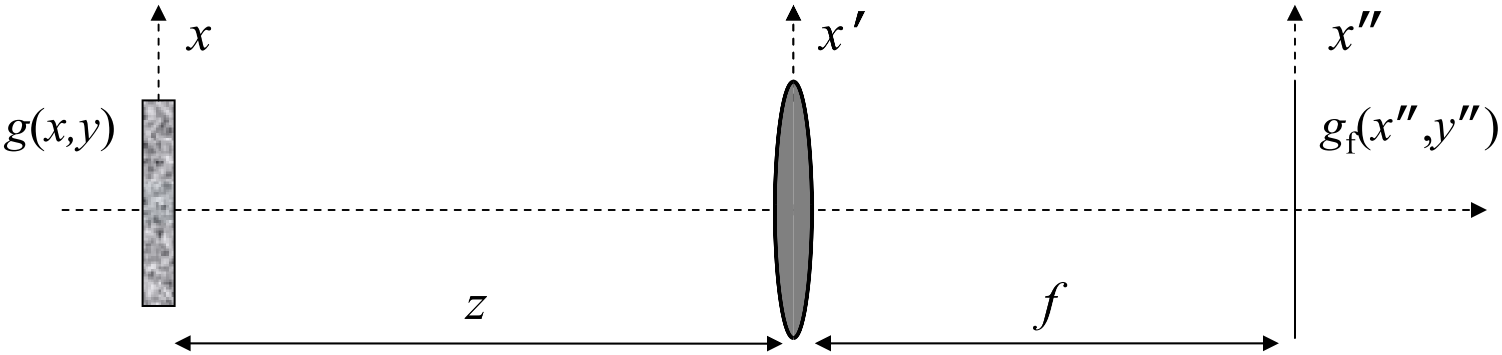


thin
transparency
$g(x,y)$

thin
lens

back focal plane
diffraction pattern
$g_f(x'',y'')$

Diffraction at the back focal plane



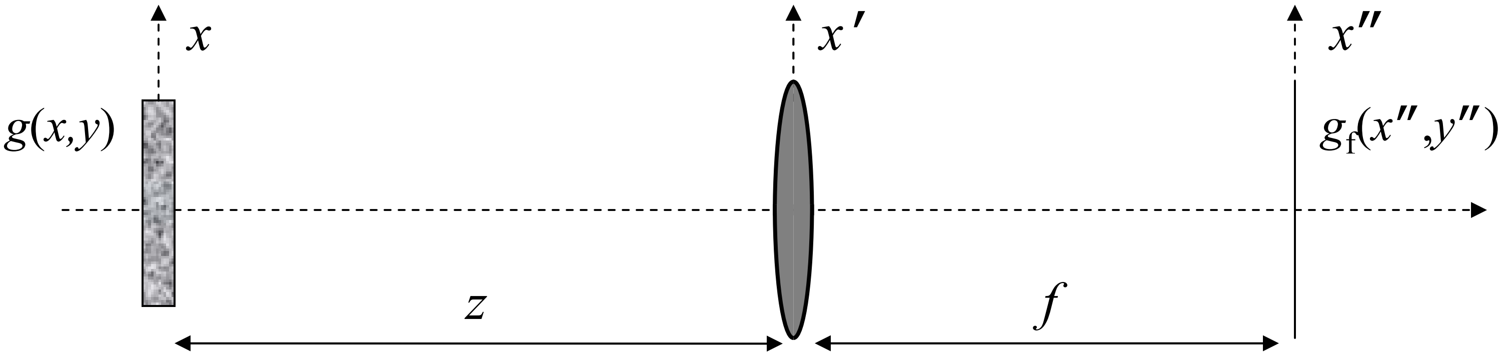
1D calculation

Field before lens $$g_{\text{lens}-}(x') = \int g(x) \exp\left\{ i\pi \frac{(x'-x)^2}{\lambda z} \right\} \mathrm{d}x$$

Field after lens $$g_{\text{lens}+}(x') = g_{\text{lens}-}(x') \exp\left\{ -i\pi \frac{x'^2}{\lambda f} \right\}$$

Field at back f.p. $$g_{\mathrm{f}}(x'') = \int g_{\text{lens}+}(x') \exp\left\{ i\pi \frac{(x''-x')^2}{\lambda f} \right\} \mathrm{d}x'$$

Diffraction at the back focal plane



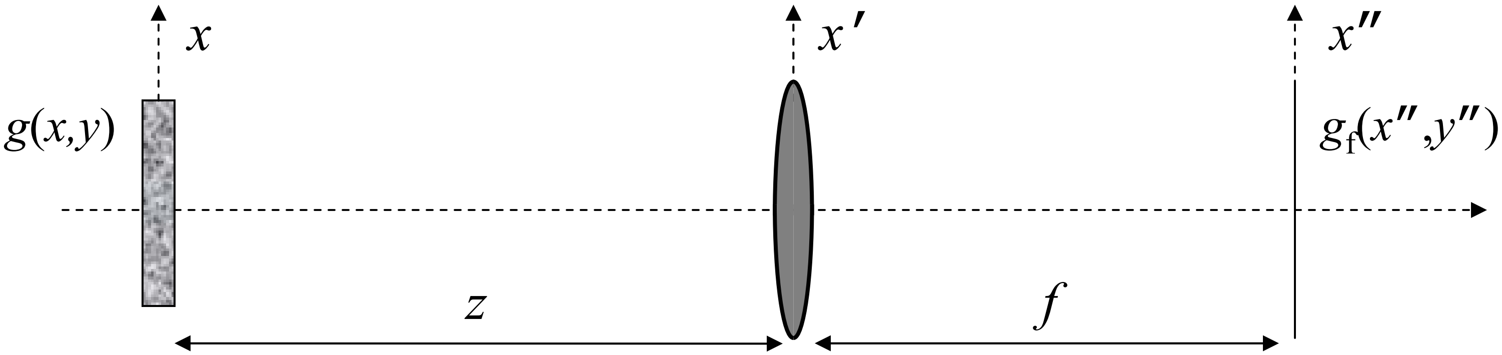
1D calculation

$$g_{\mathrm{f}}(x'') = \exp\left\{ i\pi \frac{x''^2}{\lambda f}\left(1-\frac{z}{f}\right)\right\} \int g(x) \exp\left\{-i2\pi \frac{xx''}{\lambda f}\right\} \mathrm{d}x$$

2D version

$$g_{\mathrm{f}}(x'', y'') = \exp\left\{ i\pi \frac{x''^2 + y''^2}{\lambda f}\left(1-\frac{z}{f}\right)\right\} \iint g(x,y) \exp\left\{-i2\pi \frac{xx'' + yy''}{\lambda f}\right\} \mathrm{d}x\,\mathrm{d}y$$

Diffraction at the back focal plane

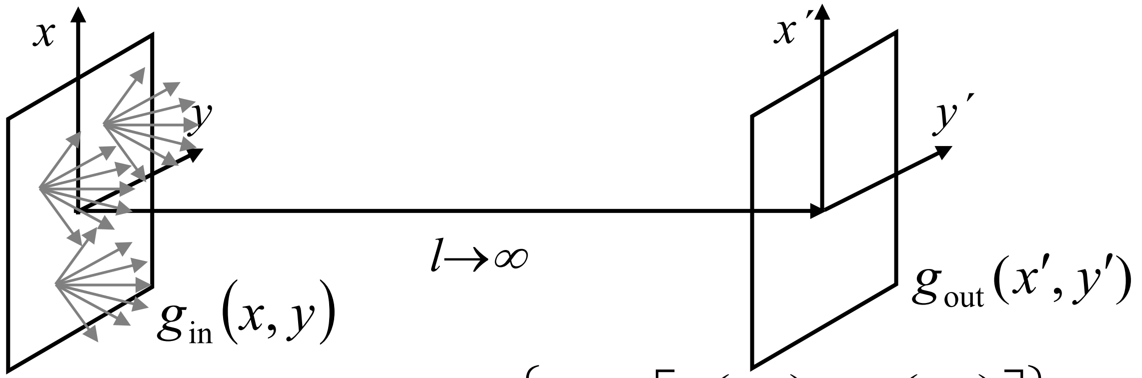


$$g_{\mathrm{f}}(x'',y'') = \exp\left\{i\pi\frac{x''^2+y''^2}{\lambda f}\left(1-\frac{z}{f}\right)\right\}\iint g(x,y)\exp\left\{-i2\pi\frac{xx''+yy''}{\lambda f}\right\}\mathrm{d}x\,\mathrm{d}y$$

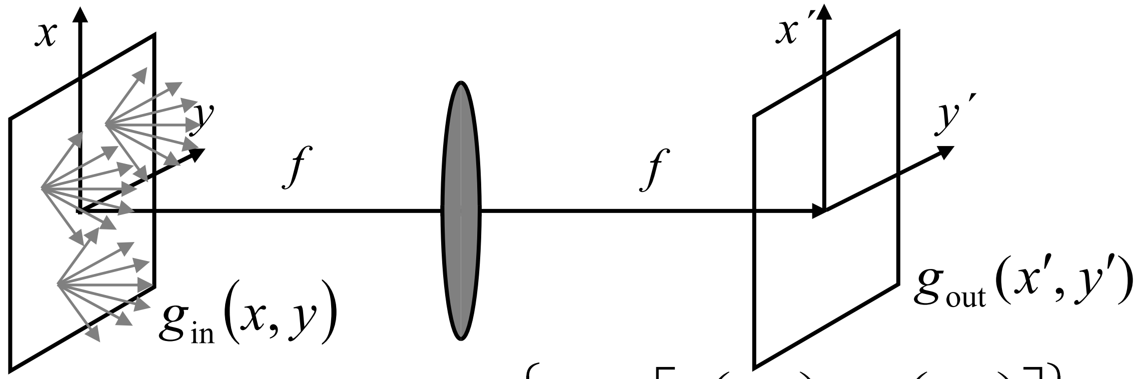
$$\therefore\quad g_{\mathrm{f}}(x'',y'') = \boxed{\exp\left\{i\pi\frac{x''^2+y''^2}{\lambda f}\left(1-\frac{z}{f}\right)\right\}}\ G\underbrace{\left(\frac{x''}{\lambda f},\frac{y''}{\lambda f}\right)}_{\text{Fourier transform of } g(x,y)}$$

spherical wave-front

Fraunhofer diffraction *vis-á-vis* a lens

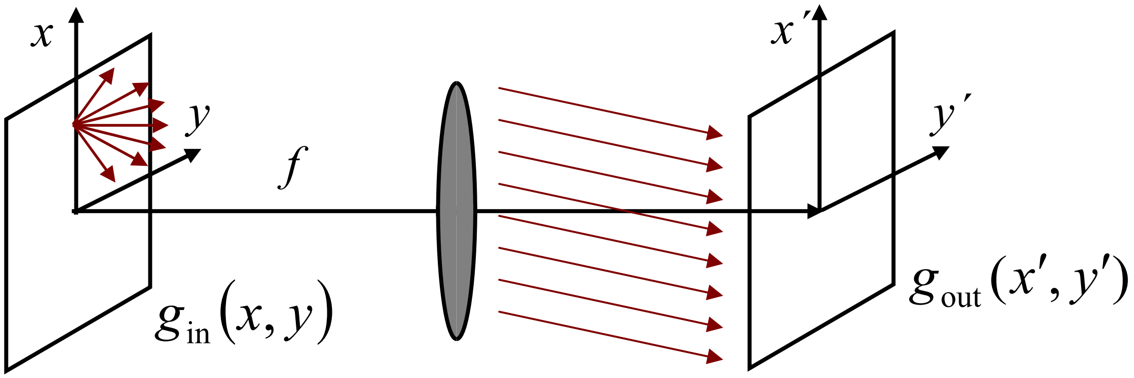


$$g_{\text{out}}(x', y'; l) \propto \iint g_{\text{in}}(x, y) \exp\left\{ -i2\pi \left[x\left(\frac{x'}{\lambda l}\right) + y\left(\frac{y'}{\lambda l}\right) \right] \right\} \mathrm{d}x\mathrm{d}y$$



$$g_{\text{out}}(x', y'; f) \propto \iint g_{\text{in}}(x, y) \exp\left\{ -i2\pi \left[x\left(\frac{x'}{\lambda f}\right) + y\left(\frac{y'}{\lambda f}\right) \right] \right\} \mathrm{d}x\mathrm{d}y$$

Spherical – plane wave duality



point source at (x,y)
amplitude $g_{\text{in}}(x,y)$

plane wave oriented

towards $\left(-\frac{x}{\lambda f},-\frac{y}{\lambda f}\right)$

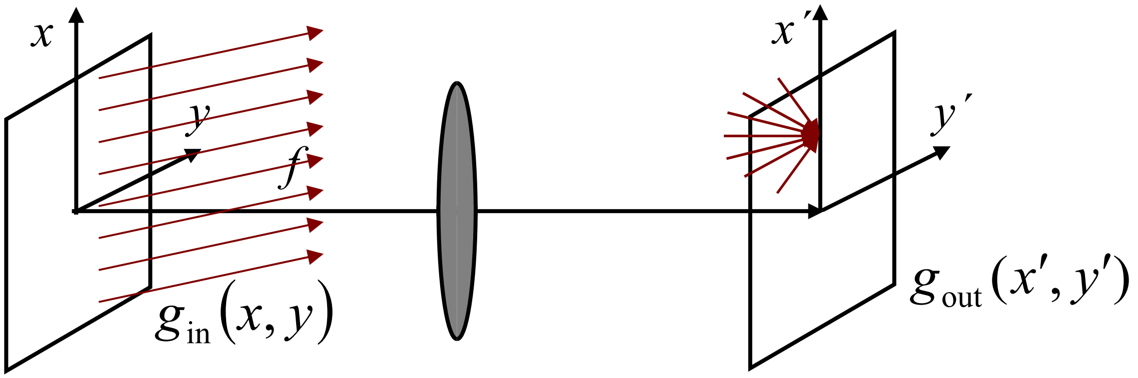
... a superposition ...

... of plane waves corresponding to point sources in the object

each output coordinate (x',y') receives ...

$$g_{\text{out}}(x',y') \propto \iint g_{\text{in}}(x,y)\exp\left\{ i2\pi\left[\left(-\frac{x}{\lambda f}\right)x' + \left(-\frac{y}{\lambda f}\right)y'\right]\right\}\mathrm{d}x\mathrm{d}y$$

Spherical – plane wave duality



a plane wave departing from the transparency at angle (θ_x, θ_y) has amplitude equal to the Fourier coefficient at frequency (θ_x/λ, θ_y/λ) of $g_{\text{in}}(x,y)$

produces a spherical wave converging towards $\left(\frac{\theta_x}{\lambda}\times(\lambda f), \frac{\theta_y}{\lambda}\times(\lambda f)\right) = (\theta_x f, \theta_y f)$

each output coordinate (x',y') receives amplitude equal to that of the corresponding Fourier component

$$g_{\text{out}}(x',y') \propto \overbrace{\iint g_{\text{in}}(x,y)\exp\left\{ -i2\pi\left[x\left(\frac{x'}{\lambda f}\right)+y\left(\frac{y'}{\lambda f}\right)\right]\right\}\mathrm{d}x\mathrm{d}y}$$

Conclusions

- When a thin transparency is illuminated coherently by a monochromatic plane wave and the light passes through a lens, the field at the focal plane is the Fourier transform of the transparency times a spherical wavefront
- The lens produces at its focal plane the Fraunhofer diffraction pattern of the transparency
- When the transparency is placed exactly one focal distance behind the lens (*i.e., $z=f$*), the Fourier transform relationship is exact.