

So far

- Geometrical Optics
 - Reflection and refraction from planar and spherical interfaces
 - Imaging condition in the paraxial approximation
 - Apertures & stops
 - Aberrations (violations of the imaging condition due to terms of order higher than paraxial or due to dispersion)
- Limits of validity of geometrical optics: features of interest are much bigger than the wavelength λ
 - Problem: point objects/images are *smaller* than λ !!!
 - So light focusing at a single point is an artifact of our approximations
 - To understand light behavior at scales $\sim \lambda$ we need to take into account the *wave* nature of light.

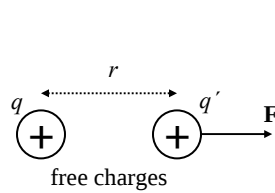
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Step #1 towards wave optics: electro-dynamics

- Electromagnetic fields (definitions and properties) *in vacuo*
- Electromagnetic fields in matter
- Maxwell's equations
 - Integral form
 - Differential form
 - Energy flux and the Poynting vector
- The electromagnetic wave equation

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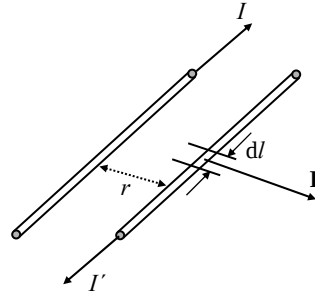
Electric and magnetic forces



Coulomb force

$$|\mathbf{F}| = \frac{1}{4\pi\epsilon_0} \frac{qq'}{r^2}$$

(dielectric) permittivity
of free space



Magnetic force

$$\frac{d|\mathbf{F}|}{dl} = \mu_0 \frac{II'}{2\pi r}$$

(magnetic) permeability
of free space

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Note the units...

$$\left(\begin{array}{c} \text{Electric} \\ \text{force} \end{array} \right) = \frac{1}{\epsilon_0} \left(\frac{\text{Charge}}{\text{Distance}} \right)^2$$

$$|\mathbf{F}| = \frac{1}{4\pi\epsilon_0} \frac{qq'}{r^2}$$

$$\left(\begin{array}{c} \text{Magnetic} \\ \text{force} \end{array} \right) = \mu_0 \left(\frac{\text{Charge}}{\text{Time}} \right)^2$$

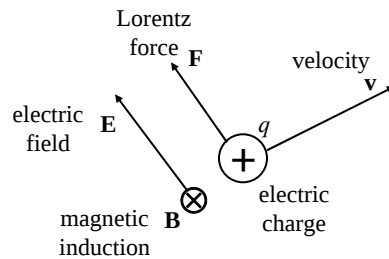
$$\frac{d|\mathbf{F}|}{dl} = \mu_0 \frac{II'}{2\pi r}$$

$$\Rightarrow (\epsilon_0 \mu_0)^{1/2} \equiv \left(\frac{\text{Distance}}{\text{Time}} \right) \equiv (\text{Speed})$$

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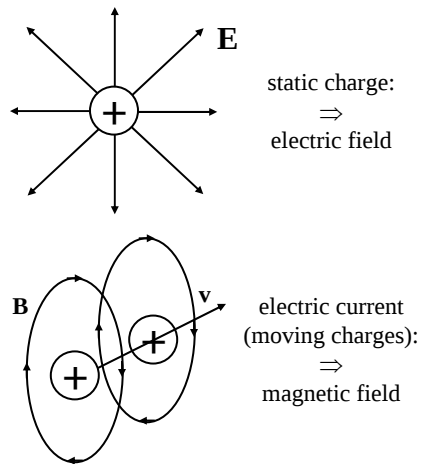
Electric and magnetic fields

Observation



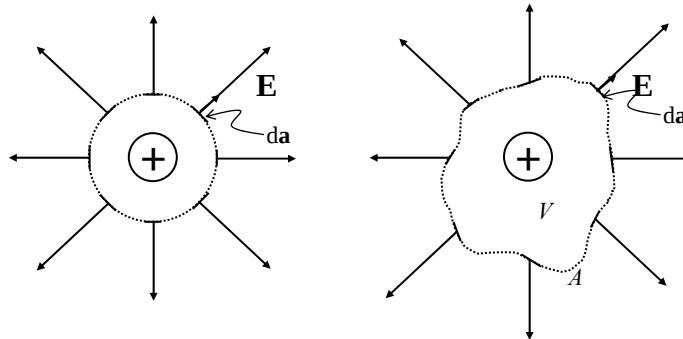
$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Generation



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Gauss Law: electric fields



$$\oint_A \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} \iiint_V \rho dV$$

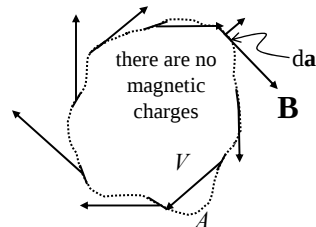
charge density

Gauss theorem

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

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Gauss Law: magnetic fields



$$\oint_A \mathbf{B} \cdot d\mathbf{a} = 0$$

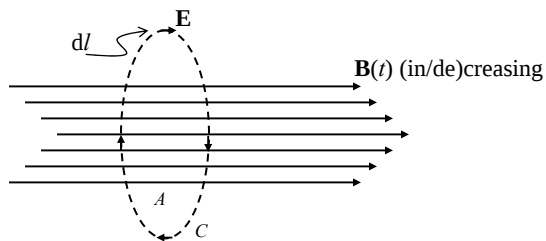
Gauss theorem

$$\nabla \cdot \mathbf{B} = 0$$

“magnetic charge” density

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Faraday's Law: electromotive force



$$\oint_C \mathbf{E} \cdot d\mathbf{l} = - \frac{d}{dt} \int_A \mathbf{B} \cdot d\mathbf{a}$$

Stokes theorem

$$\nabla \times \mathbf{E} = - \frac{\partial \mathbf{B}}{\partial t}$$

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Ampere's Law: magnetic induction

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 \left(\underbrace{\iint_A \mathbf{J} \cdot d\mathbf{a}}_{\text{current density}} + \underbrace{\iint_A \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \cdot d\mathbf{a}}_{\text{Maxwell's extension, Displacement current}} \right) \xLeftrightarrow[\text{Stokes theorem}] \nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right)$$

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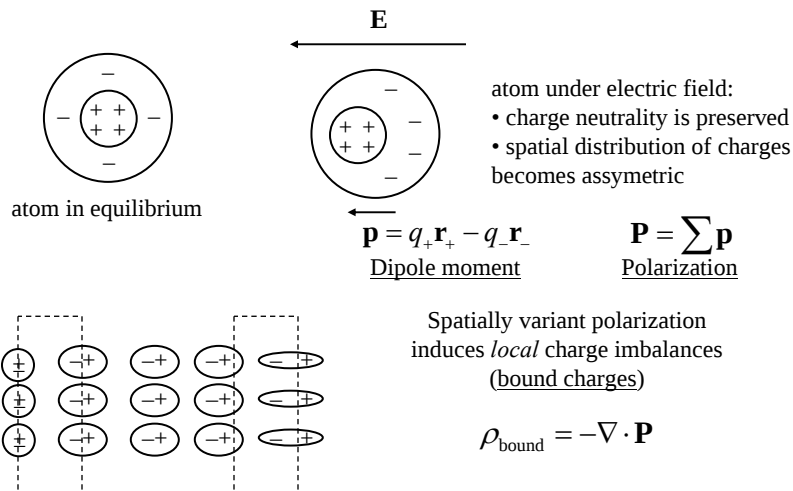
Maxwell's equations

(in vacuo)

$$\begin{aligned} \oint_A \mathbf{E} \cdot d\mathbf{a} &= \frac{1}{\epsilon_0} \iiint_V \rho \, dV & \xLeftrightarrow & \nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} & \text{Gauss/electric} \\ \oint_A \mathbf{B} \cdot d\mathbf{a} &= 0 & \xLeftrightarrow & \nabla \cdot \mathbf{B} = 0 & \text{Gauss/magnetic} \\ \oint_C \mathbf{E} \cdot d\mathbf{l} &= -\frac{d}{dt} \int_A \mathbf{B} \cdot d\mathbf{a} & \xLeftrightarrow & \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} & \text{Faraday} \\ \oint_C \mathbf{B} \cdot d\mathbf{l} &= \mu_0 \iint_A \left(\mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) \cdot d\mathbf{a} & \xLeftrightarrow & \nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) & \text{Ampere-Maxwell} \end{aligned}$$

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Electric fields in dielectric media



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Electric displacement

Gauss Law:
$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0} \rho_{\text{total}} = \frac{1}{\epsilon_0} (\rho_{\text{free}} + \rho_{\text{bound}})$$

$$= \frac{1}{\epsilon_0} (\rho_{\text{free}} - \nabla \cdot \mathbf{P})$$

$$\nabla \cdot (\underbrace{\epsilon_0 \mathbf{E} + \mathbf{P}}) = \rho_{\text{free}}$$

Electric displacement field: $\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \implies \nabla \cdot \mathbf{D} = \rho_{\text{free}}$

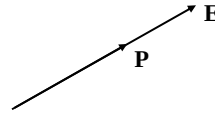
Linear, isotropic polarizability: $\mathbf{P} = \epsilon_0 \chi \mathbf{E}$

$$\implies \mathbf{D} = \epsilon_0 (1 + \chi) \mathbf{E} \equiv \epsilon \mathbf{E}$$

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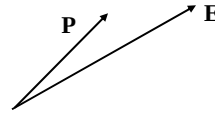
General cases of polarization

Linear, isotropic polarizability: $\mathbf{P} = \varepsilon_0 \chi \mathbf{E}$



Linear, anisotropic polarizability:

$$\mathbf{P} = \varepsilon_0 \begin{pmatrix} \chi_{11} & \chi_{12} & \chi_{13} \\ \chi_{21} & \chi_{22} & \chi_{23} \\ \chi_{31} & \chi_{32} & \chi_{33} \end{pmatrix} \mathbf{E}$$



Nonlinear, isotropic polarizability: $\mathbf{P} = \varepsilon_0 \chi \mathbf{E} + \varepsilon_0 \chi^{(2)} \mathbf{E} \mathbf{E} + \dots$

etc.

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Constitutive relationships

E: electric field

D: electric displacement

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}$$

polarization

B: magnetic induction

H: magnetic field

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M}$$

magnetization

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Maxwell's equations

(in matter)

$$\oint_A \mathbf{D} \cdot d\mathbf{a} = \iiint_V \rho_{\text{free}} dV \quad \longleftrightarrow \quad \nabla \cdot \mathbf{D} = \rho_{\text{free}} \quad \text{Gauss/electric}$$

$$\oint_A \mathbf{B} \cdot d\mathbf{a} = 0 \quad \longleftrightarrow \quad \nabla \cdot \mathbf{B} = 0 \quad \text{Gauss/magnetic}$$

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt} \int_A \mathbf{B} \cdot d\mathbf{a} \quad \longleftrightarrow \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \text{Faraday}$$

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \mu_0 \iiint_A \left(\mathbf{J}_{\text{free}} + \frac{\partial \mathbf{D}}{\partial t} \right) \cdot d\mathbf{a} \quad \longleftrightarrow \quad \nabla \times \mathbf{H} = \mu_0 \left(\mathbf{J}_{\text{free}} + \frac{\partial \mathbf{D}}{\partial t} \right) \quad \text{Ampere-Maxwell}$$

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Maxwell's equations $\Rightarrow$ wave equation

(in linear, anisotropic, non-magnetic matter, no free charges/currents)

$$\left. \begin{aligned} \nabla \cdot (\epsilon \mathbf{E}) &= 0 \\ \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \times \mathbf{B} &= \mu_0 \frac{\partial (\epsilon \mathbf{E})}{\partial t} \end{aligned} \right\} \xrightarrow{\text{matter spatially and temporally invariant}} \left. \begin{aligned} \nabla \cdot \mathbf{E} &= 0 \\ \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \times \mathbf{B} &= \mu_0 \epsilon \frac{\partial \mathbf{E}}{\partial t} \end{aligned} \right\}$$

$$\left. \begin{aligned} \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \Rightarrow \nabla \times (\nabla \times \mathbf{E}) = -\frac{\partial (\nabla \times \mathbf{B})}{\partial t} = -\mu_0 \epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} \\ \nabla \times (\nabla \times \mathbf{E}) &= \nabla (\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} \end{aligned} \right\} \xrightarrow{\substack{\uparrow \\ =0}} \boxed{\nabla^2 \mathbf{E} - \mu_0 \epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0}$$

electromagnetic wave equation

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Maxwell's equations $\Rightarrow$ wave equation

(in linear, anisotropic, non-magnetic matter, no free charges/currents)

$$\left. \begin{aligned} \nabla \cdot \mathbf{E} &= 0 \\ \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \times \mathbf{B} &= \mu_0 \epsilon \frac{\partial \mathbf{E}}{\partial t} \end{aligned} \right\} \Rightarrow \boxed{\nabla^2 \mathbf{E} - \mu_0 \epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0}$$

$$\mu_0 \epsilon \equiv \frac{1}{c^2} \quad \boxed{\nabla^2 \mathbf{E} - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0}$$

wave velocity

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Light velocity and refractive index

$$\mu_0 \epsilon_0 \equiv \frac{1}{c_{\text{vacuum}}^2}$$

c_{vacuum} : speed of light
in vacuum

$$\epsilon = (1 + \chi) \epsilon_0 \equiv n^2 \epsilon_0$$

n : index of refraction

$$\mu_0 \epsilon = \frac{n^2}{c_{\text{vacuum}}^2} \equiv \frac{1}{c^2}$$

$c \equiv c_{\text{vacuum}}/n$: speed of light
in medium of refr. index n

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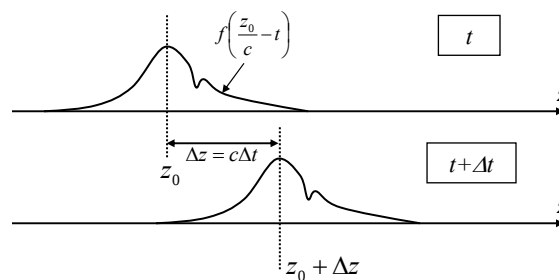
Simplified (1D, scalar) wave equation

$$\frac{\partial^2 E}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2} = 0$$

- E is a scalar quantity (e.g. the component E_y of an electric field $\mathbf{E}$)
- the geometry is symmetric in $x, y \Rightarrow$ the x, y derivatives are zero

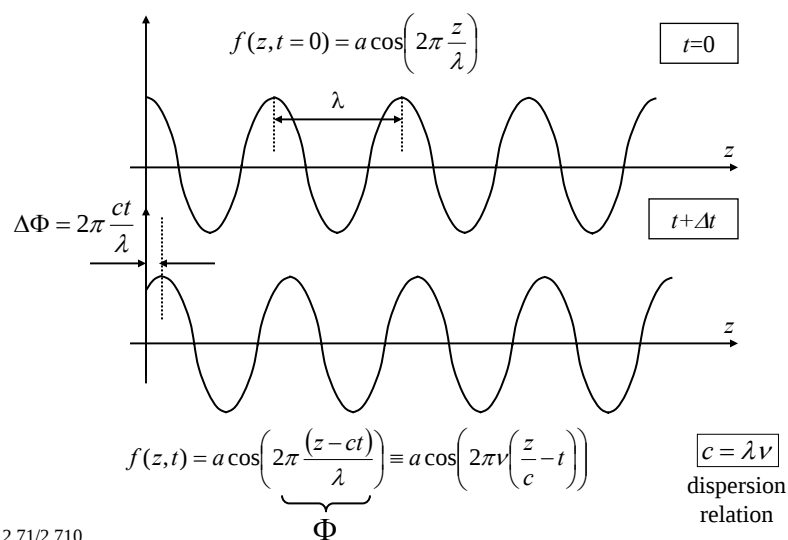
Solution:

$$E(z, t) = f\left(\frac{z}{c} - t\right) + g\left(\frac{z}{c} + t\right)$$



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Special case: harmonic solution



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Complex representation of waves

$$f(z, t) = A \cos\left(\frac{2\pi}{\lambda} z - 2\pi\nu t - \phi\right)$$

angular frequency

$$f(z, t) = A \cos(kz - \omega t - \phi) \quad k = \frac{2\pi}{\lambda}, \quad \omega = 2\pi\nu$$

wave-number

$$f(z, t) = A \cos(kz - \omega t - \phi)$$

$$\hat{f}(z, t) = A \cos(kz - \omega t - \phi) + iA \sin(kz - \omega t - \phi)$$

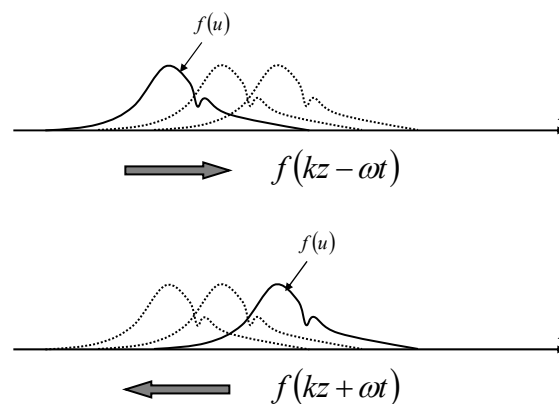
i.e. $f(z, t) = \text{Re}\{\hat{f}(z, t)\}$

$$\hat{f}(z, t) \text{ aka } \hat{f}(z, t) = A e^{i(kz - \omega t - \phi)} \quad \text{complex representation}$$

$$A e^{-i\phi} \quad \text{complex amplitude or "phasor"}$$

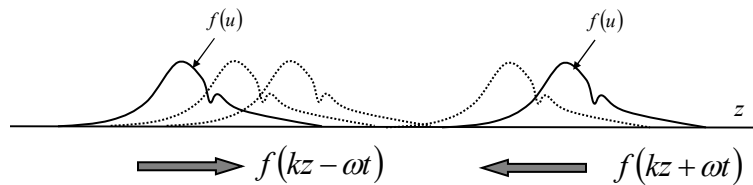
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Time reversal



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Superposition



$\Rightarrow f(kz - \omega t) + f(kz + \omega t)$ is also a solution

More generally, $f(kz - \omega t) + g(kz + \omega t)$ is a solution

Even more generally,

$f_1(kz - \omega t) + f_2(kz - \omega t) + \dots$
 $+ g_1(kz + \omega t) + g_2(kz + \omega t) + \dots$ is a solution

LINEARITY



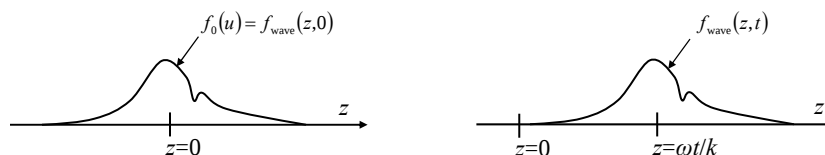
SUPERPOSITION

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What is the solution to the wave equation?

- In general: the solution is an (arbitrary) superposition of propagating waves
- Usually, we have to impose
 - initial conditions (as in any differential equation)
 - boundary condition (as in most partial differential equations)

Example: initial value problem



$$f_{\text{wave}}(z, 0) \equiv f_0(u) \Rightarrow f_{\text{wave}}(z, t) = f_0(kz - \omega t)$$

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What is the solution to the wave equation?

- In general: the solution is an (arbitrary) superposition of propagating waves
- Usually, we have to impose
 - initial conditions (as in any differential equation)
 - boundary condition (as in most partial differential equations)
- Boundary conditions: we will not deal much with them in this class, but it is worth noting that physically they explain interesting phenomena such as waveguiding from the wave point of view (we saw already one explanation as TIR).

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