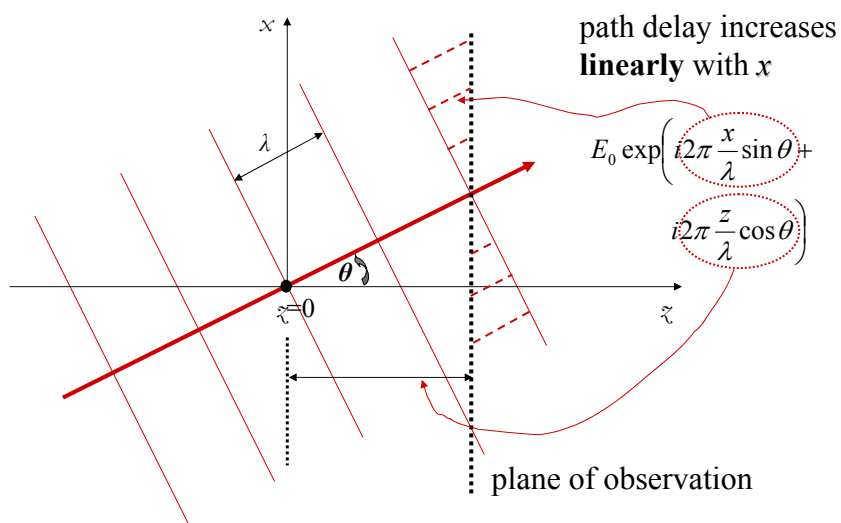


The spatial frequency domain

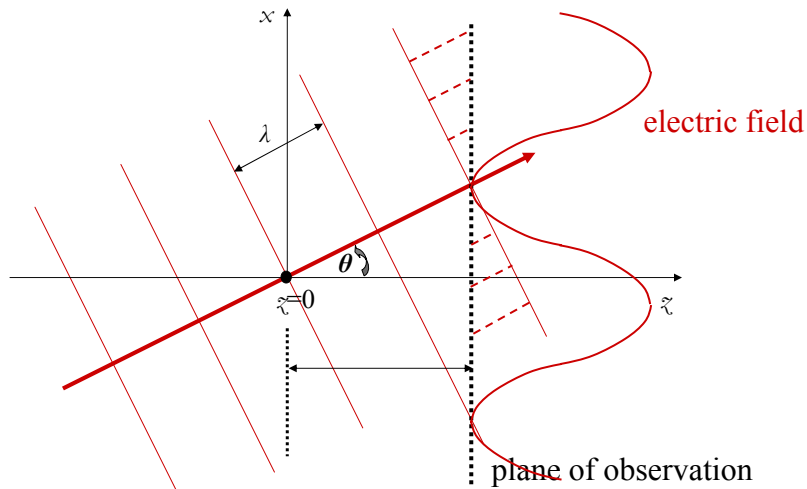
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Recall: plane wave propagation



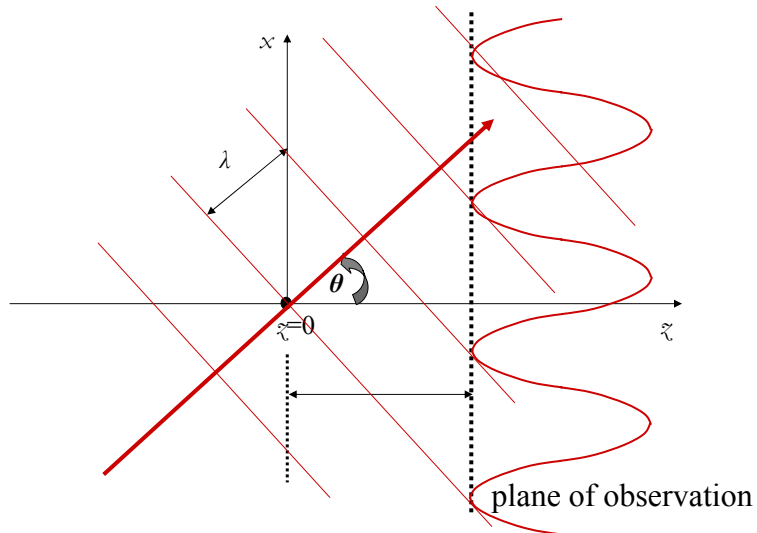
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Spatial frequency $\Leftrightarrow$ angle of propagation?



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Spatial frequency $\Leftrightarrow$ angle of propagation?



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Spatial frequency $\Leftrightarrow$ angle of propagation?

The cross-section of the optical field with the optical axis is a sinusoid of the form

$$E_0 \exp\left(i2\pi \frac{\sin \theta}{\lambda} x + \phi_0\right)$$

i.e. it looks like

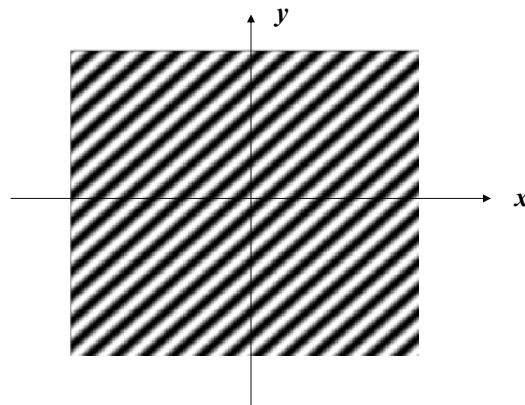
$$E_0 \exp(i2\pi u x + \phi_0) \quad \text{where} \quad \underline{u \equiv \frac{\sin \theta}{\lambda}}$$

is called the **spatial frequency**

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2D sinusoids

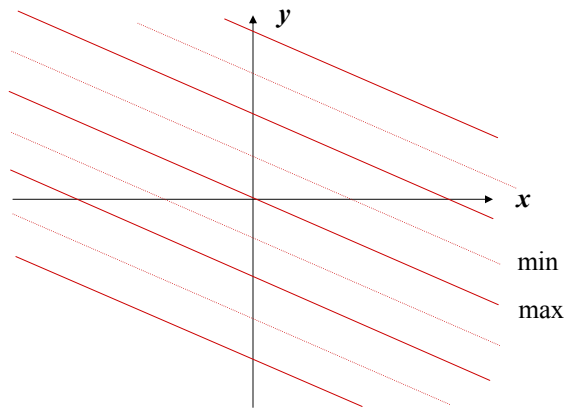
$$E_0 \cos[2\pi(ux + vy)] \quad \left(\begin{array}{c} \text{corresp. phasor} \\ E_0 \exp[i2\pi(ux + vy)] \end{array} \right)$$



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2D sinusoids

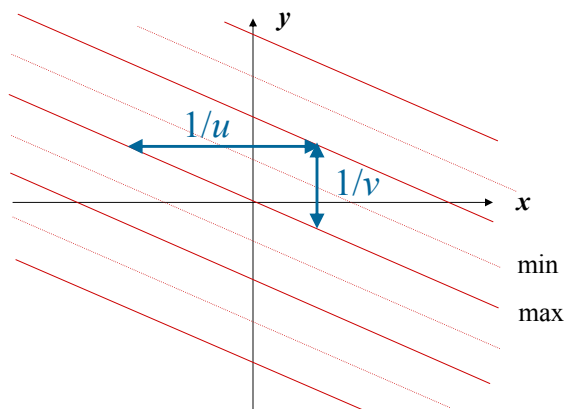
$$E_0 \cos[2\pi(ux + vy)] \quad \left(\begin{array}{c} \text{corresp. phasor} \\ E_0 \exp[i2\pi(ux + vy)] \end{array} \right)$$



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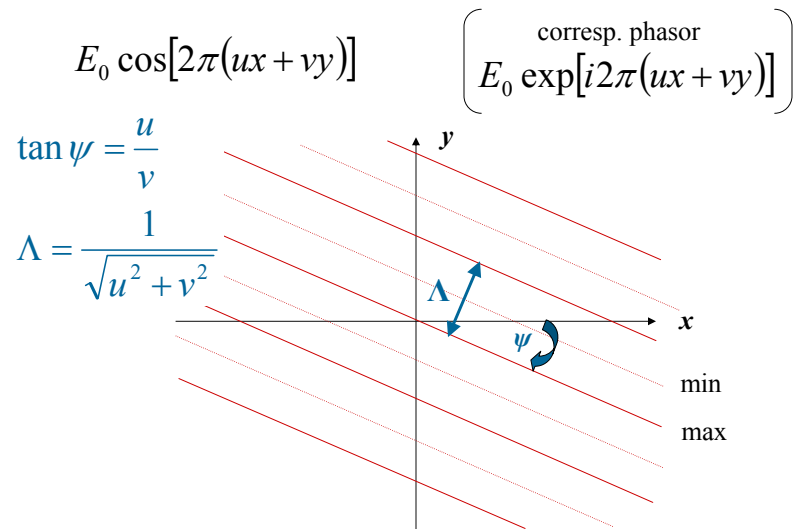
2D sinusoids

$$E_0 \cos[2\pi(ux + vy)] \quad \left(\begin{array}{c} \text{corresp. phasor} \\ E_0 \exp[i2\pi(ux + vy)] \end{array} \right)$$



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2D sinusoids



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Spatial (2D) Fourier Transforms

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The **2D** Fourier integral

(aka **inverse** Fourier transform)

$$g(x, y) = \int G(u, v) e^{+i2\pi(ux+vy)} du dv$$

superposition

sinusoids

complex weight,
expresses relative amplitude
(magnitude & phase)
of superposed sinusoids

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The **2D** Fourier transform

The complex weight coefficients $G(u, v)$,
aka **Fourier transform** of $g(x, y)$
are calculated from the integral

$$G(u, v) = \int g(x, y) e^{-i2\pi(ux+vy)} dx dy$$

(1D so we can draw it easily ...)

$$\text{Re}[G(u)] = \int \text{Re}[g(x) \text{Re}[e^{-i2\pi ux}]] dx$$

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2D Fourier transform *pairs*

TABLE 2.1
Transform pairs for some functions separable in rectangular coordinates.

Function	Transform
$\exp[-\pi(a^2x^2 + b^2y^2)]$	$\frac{1}{ ab } \exp\left[-\pi\left(\frac{f_x^2}{a^2} + \frac{f_y^2}{b^2}\right)\right]$
$\text{rect}(ax) \text{rect}(by)$	$\frac{1}{ ab } \text{sinc}(f_x/a) \text{sinc}(f_y/b)$
$\Lambda(ax) \Lambda(by)$	$\frac{1}{ ab } \text{sinc}^2(f_x/a) \text{sinc}^2(f_y/b)$
$\delta(ax, by)$	$\frac{1}{ ab }$
$\exp[j\pi(ax + by)]$	$\delta(f_x - a/2, f_y - b/2)$
$\text{sgn}(ax) \text{sgn}(by)$	$\frac{ab}{ ab } \frac{1}{j\pi f_x} \frac{1}{j\pi f_y}$
$\text{comb}(ax) \text{comb}(by)$	$\frac{1}{ ab } \text{comb}(f_x/a) \text{comb}(f_y/b)$
$\exp[j\pi(a^2x^2 + b^2y^2)]$	$\frac{j}{ ab } \exp\left[-j\pi\left(\frac{f_x^2}{a^2} + \frac{f_y^2}{b^2}\right)\right]$
$\exp[-(a x + b y)]$	$\frac{1}{ ab } \frac{2}{1 + (2\pi f_x/a)^2} \frac{2}{1 + (2\pi f_y/b)^2}$

(from Goodman,
*Introduction to
Fourier Optics*,
page 14)

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Space and spatial frequency representations

SPACE DOMAIN $g(x,y)$

$$G(u,v) = \int g(x,y) e^{-i2\pi(ux+vy)} dx dy$$

2D Fourier transform

SPATIAL FREQUENCY
DOMAIN

$G(u,v)$

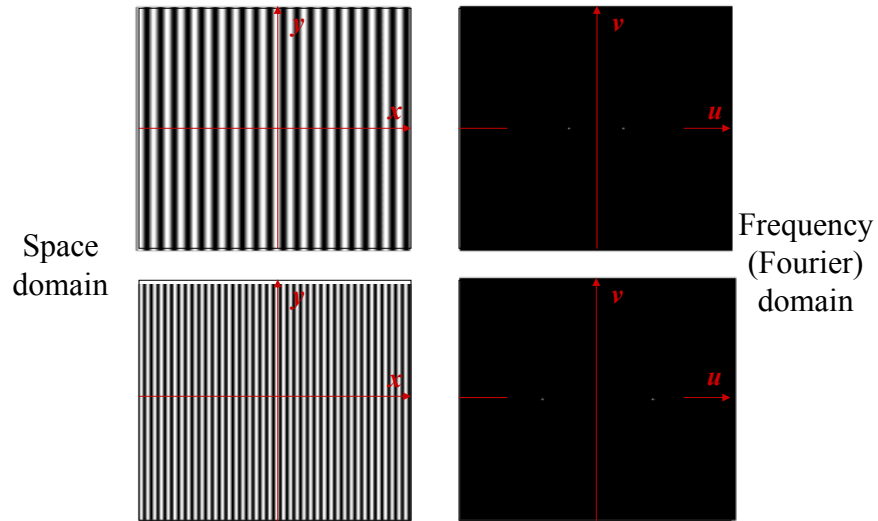
$$g(x,y) = \int G(u,v) e^{+i2\pi(ux+vy)} du dv$$

2D Fourier integral
aka

inverse 2D Fourier transform

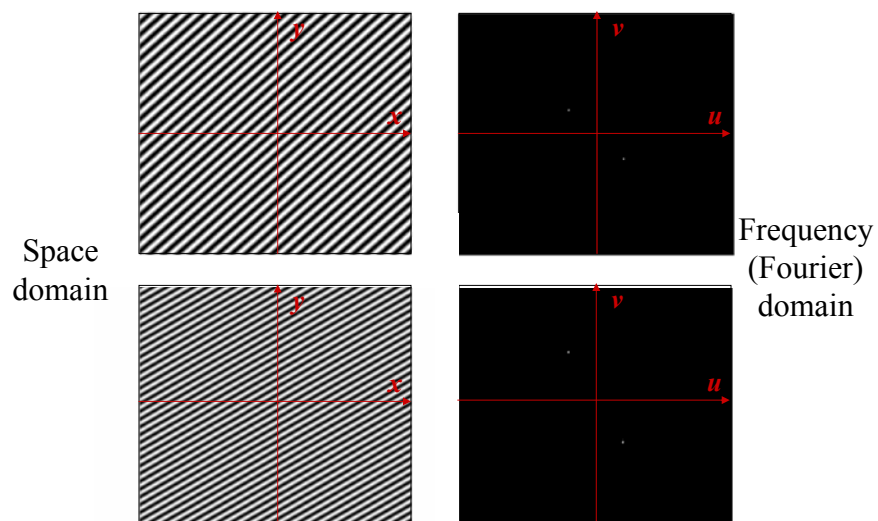
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Periodic Grating /1: vertical

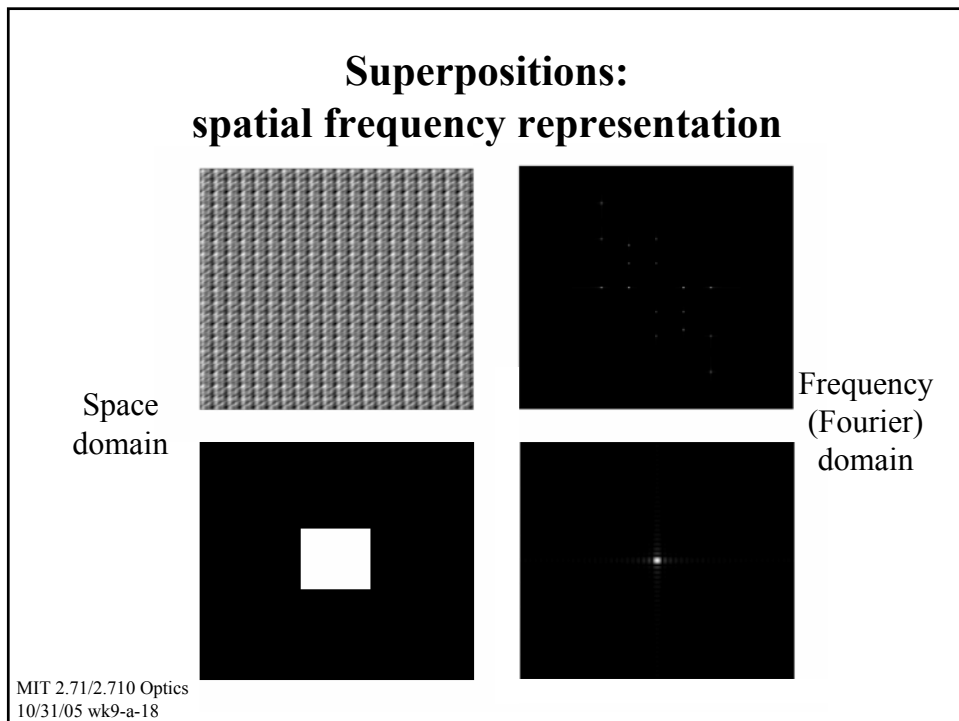
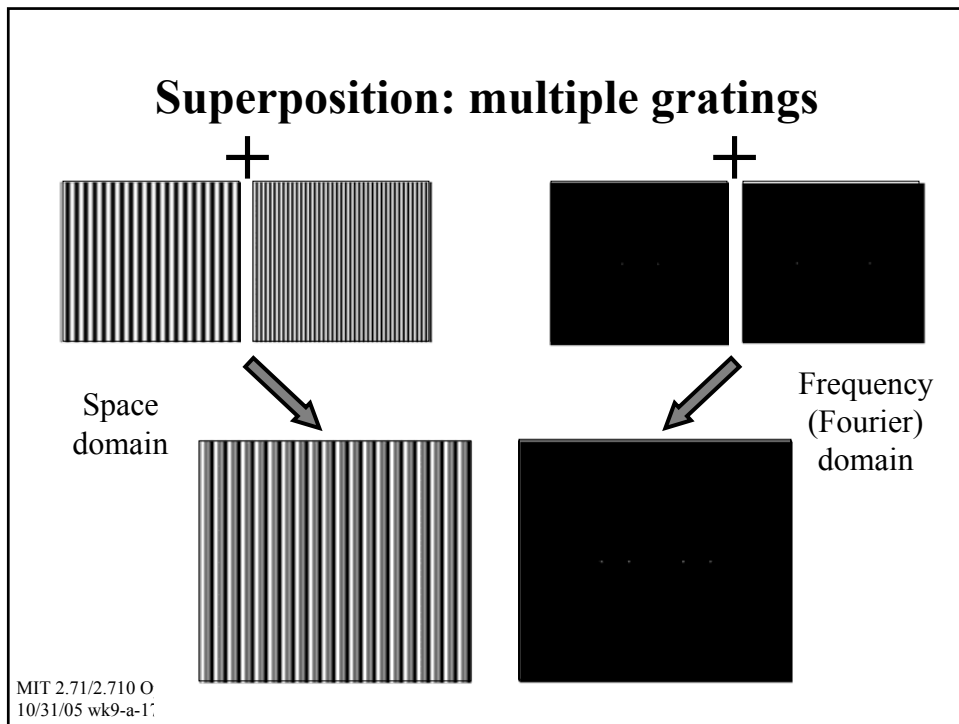


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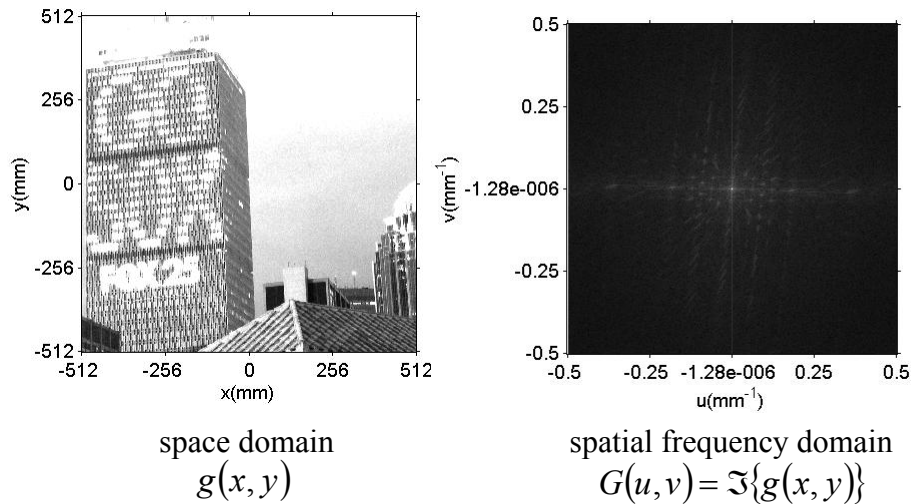
Periodic Grating /2: tilted



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Superpositions: spatial frequency representation



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Fourier transform properties /1

- Fourier transforms and the delta function

$$\mathfrak{F}\{\delta(x, y)\} = 1$$

$$\mathfrak{F}\{\exp[i2\pi(u_0x + v_0y)]\} = \delta(u - u_0)\delta(v - v_0)$$

- Linearity of Fourier transforms

$$\text{if } \mathfrak{F}\{g_1(x, y)\} = G_1(u, v) \text{ and } \mathfrak{F}\{g_2(x, y)\} = G_2(u, v)$$

$$\text{then } \mathfrak{F}\{a_1g_1(x, y) + a_2g_2(x, y)\} = a_1G_1(u, v) + a_2G_2(u, v)$$

for any pair of complex numbers a_1, a_2 .

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Fourier transform properties /2

Let $\mathfrak{F}\{g(x, y)\} = G(u, v)$

- Shift theorem (space $\rightarrow$ frequency)

$$\mathfrak{F}\{g(x - x_0, y - y_0)\} = G(u, v) \exp[-i2\pi(ux_0 + vy_0)]$$

- Shift theorem (frequency $\rightarrow$ space)

$$\mathfrak{F}\{g(x, y) \exp[i2\pi(ux_0 + vy_0)]\} = G(u - u_0, v - v_0)$$

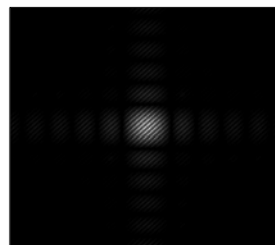
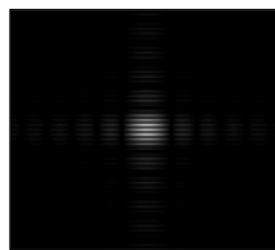
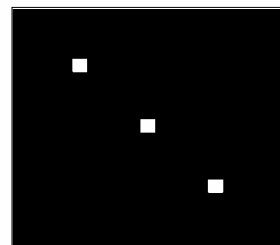
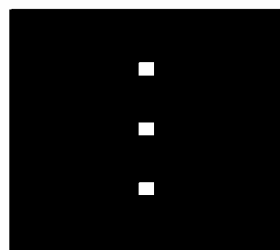
- Scaling theorem

$$\mathfrak{F}\{g(ax, by)\} = \frac{1}{|ab|} G\left(\frac{u}{a}, \frac{v}{b}\right)$$

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Modulation in the frequency domain: the shift theorem

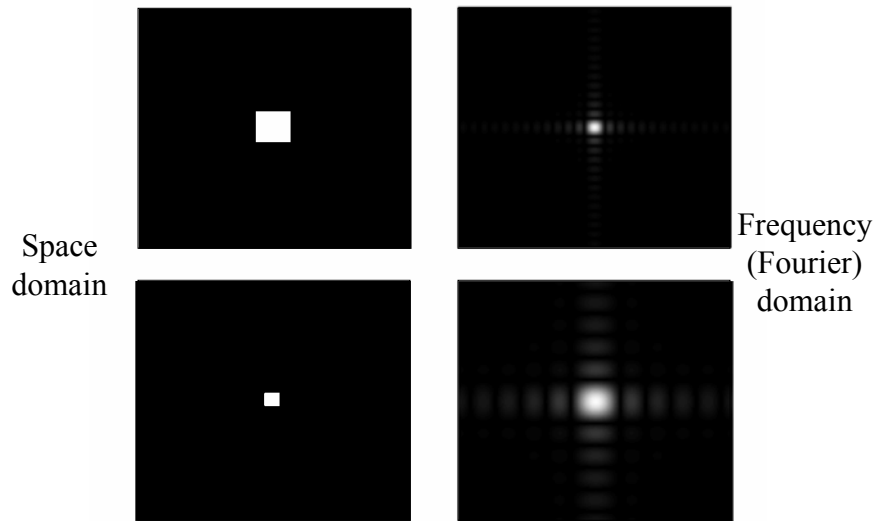
Space
domain



Frequency
(Fourier)
domain

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Size of object vs frequency content: the scaling theorem



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Fourier transform properties /3

Let $\mathfrak{F}\{f(x, y)\} = F(u, v)$ and $\mathfrak{F}\{h(x, y)\} = H(u, v)$

Let $g(x, y) = \int f(x', y') \cdot h(x - x', y - y') dx' dy'$

- Convolution theorem (space $\rightarrow$ frequency)

$$\mathfrak{F}\{g(x, y)\} = F(u, v) \cdot H(u, v)$$

Let $Q(u, v) = \int F(u', v') \cdot H(u - u', v - v') du' dv'$

- Convolution theorem (frequency $\rightarrow$ space)

$$Q(u, v) = \mathfrak{F}\{f(x, y) \cdot h(x, y)\}$$

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Fourier transform properties /4

Let $\mathfrak{F}\{f(x, y)\} = F(u, v)$ and $\mathfrak{F}\{h(x, y)\} = H(u, v)$

Let $g(x, y) = \int f(x', y') \cdot h(x' - x, y' - y) dx' dy'$

- Correlation theorem (space $\rightarrow$ frequency)

$$\mathfrak{F}\{g(x, y)\} = F(u, v) \cdot H^*(u, v)$$

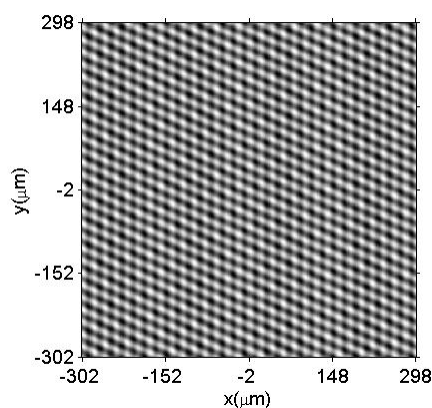
Let $Q(u, v) = \int F(u', v') \cdot H(u' - u, v' - v) du' dv'$

- Correlation theorem (frequency $\rightarrow$ space)

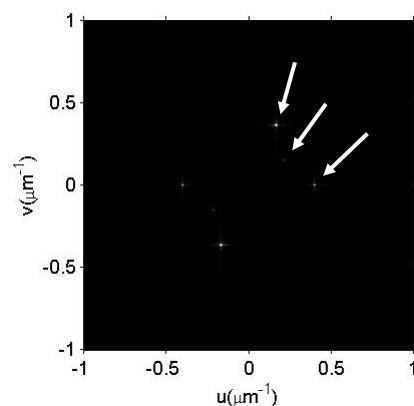
$$Q(u, v) = \mathfrak{F}\{f(x, y) \cdot h^*(x, y)\}$$

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Spatial frequency representation



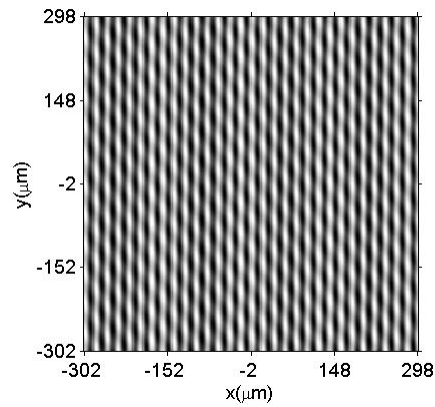
space domain
3 sinusoids



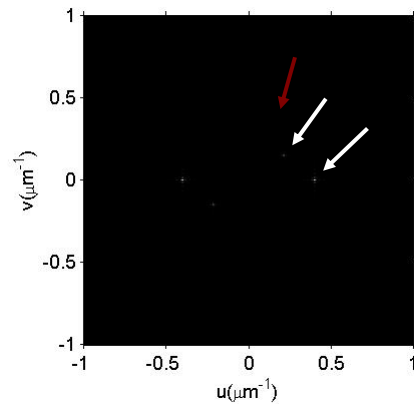
Fourier domain
(aka spatial frequency domain)

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Spatial filtering



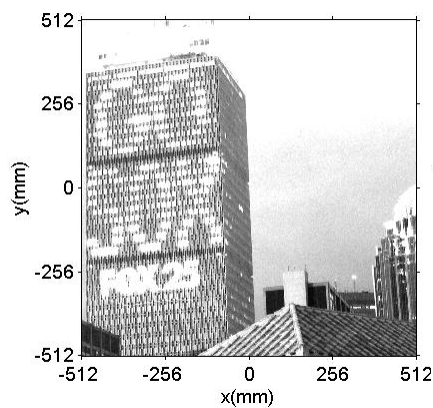
space domain
2 sinusoids (1 removed)



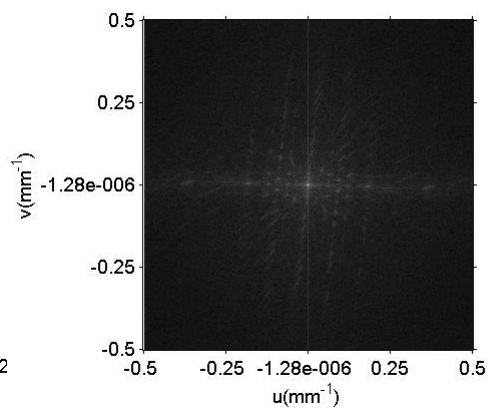
Fourier domain
(aka spatial frequency domain)

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Spatial frequency representation



space domain
 $g(x, y)$

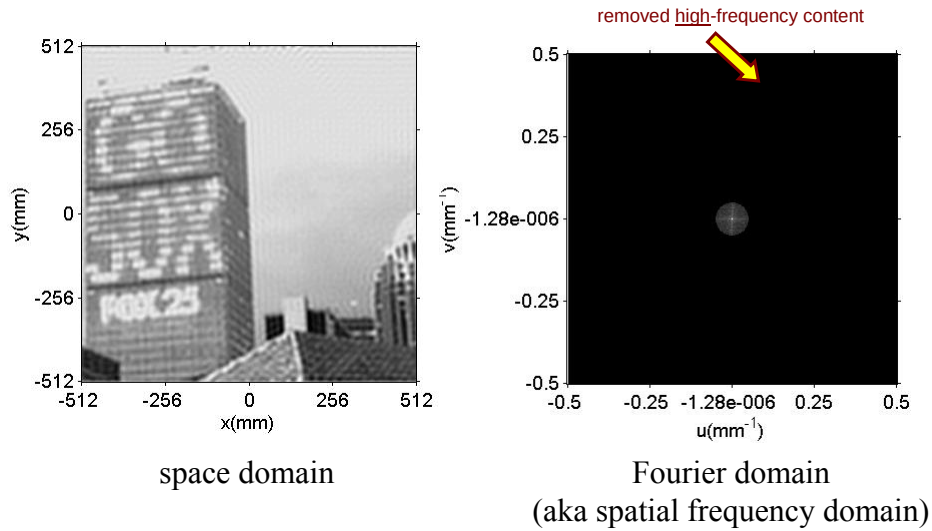


Fourier domain
(aka spatial frequency domain)

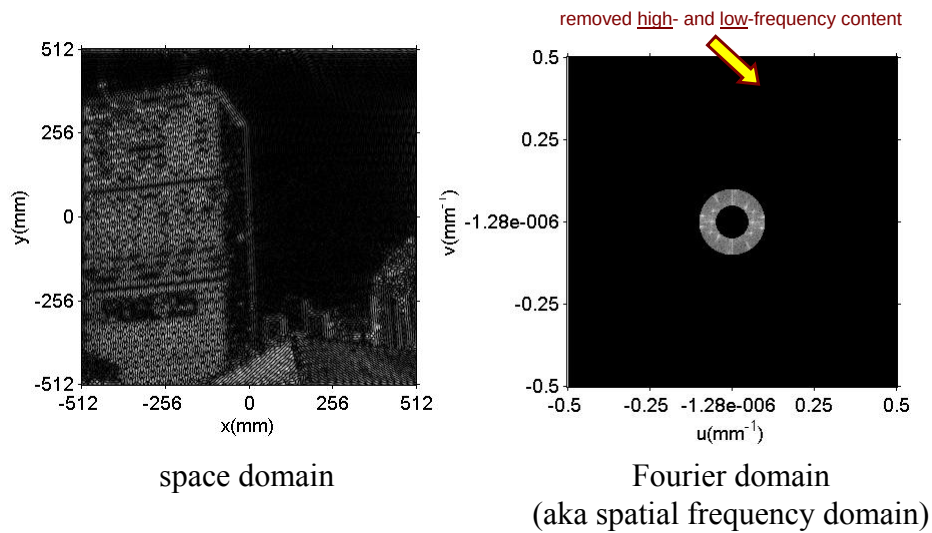
$$G(u, v) = \mathfrak{T}\{g(x, y)\}$$

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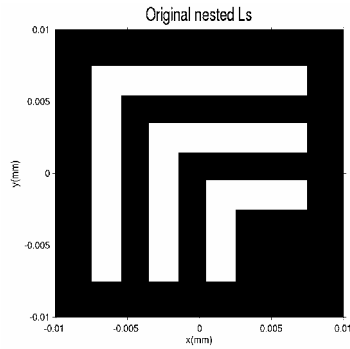
Spatial filtering (low-pass)



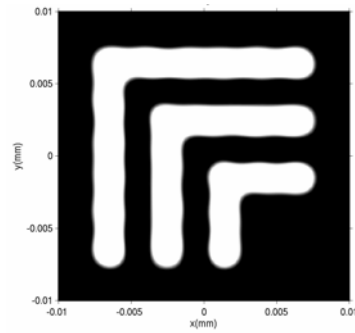
Spatial filtering (band-pass)



Example: optical lithography



original pattern
("nested L's")

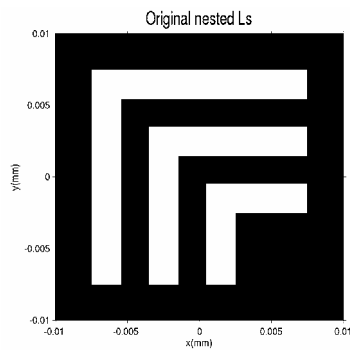


mild
low-pass filtering

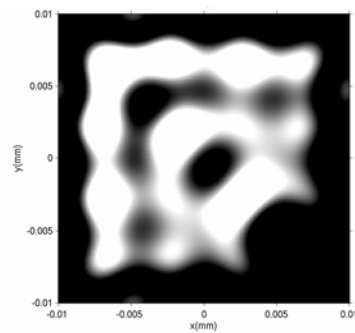
Notice:
(i) blurring at the edges
(ii) ringing

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Example: optical lithography



original pattern
("nested L's")

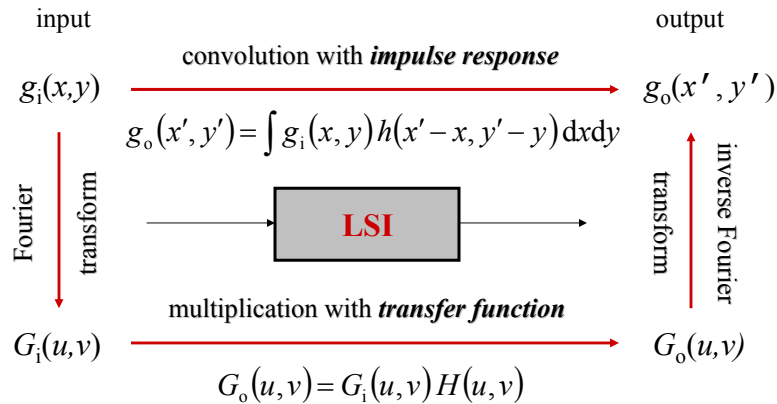


severe
low-pass filtering

Notice:
(i) blurring at the edges
(ii) ringing

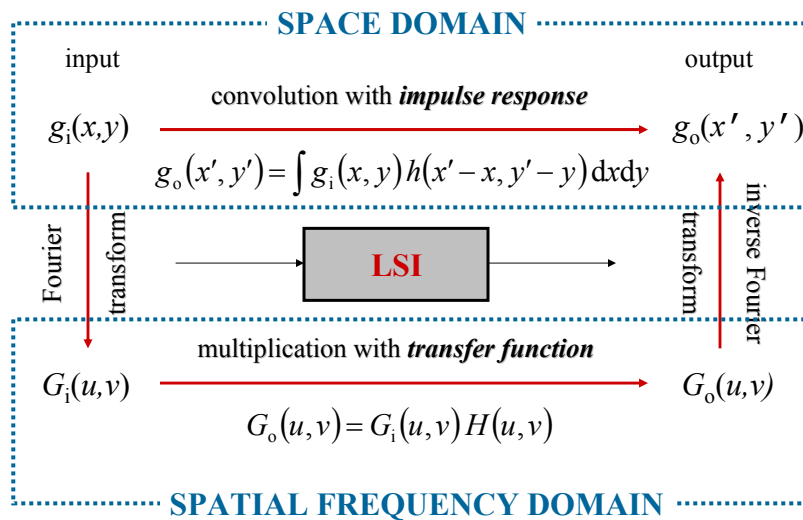
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2D linear shift invariant systems



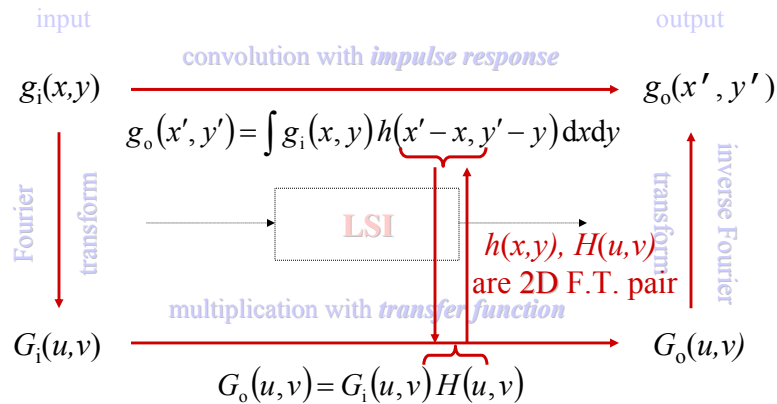
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2D linear shift invariant systems



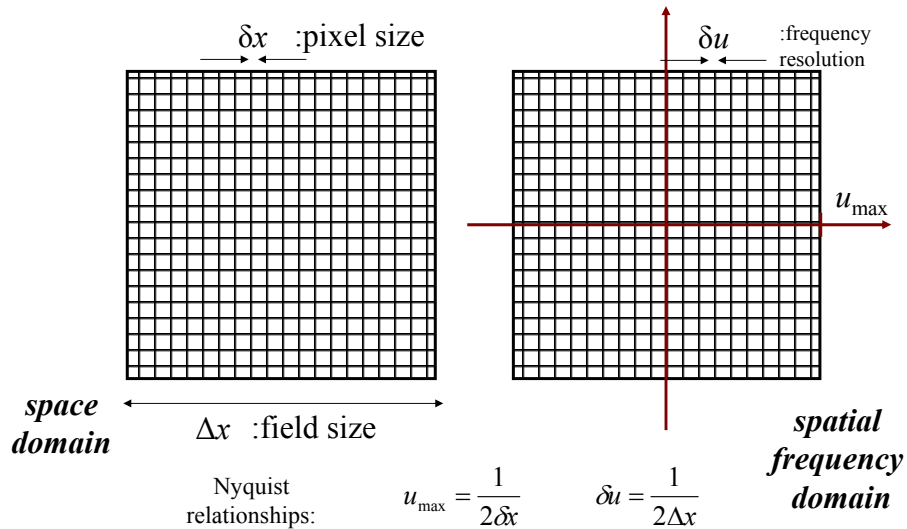
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2D linear shift invariant systems



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Sampling space and frequency



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The Space–Bandwidth Product

Nyquist relationships:

from space $\rightarrow$ spatial frequency domain: $u_{\max} = \frac{1}{2\delta x}$

from spatial frequency $\rightarrow$ space domain: $\frac{\Delta x}{2} = \frac{1}{2\delta u}$

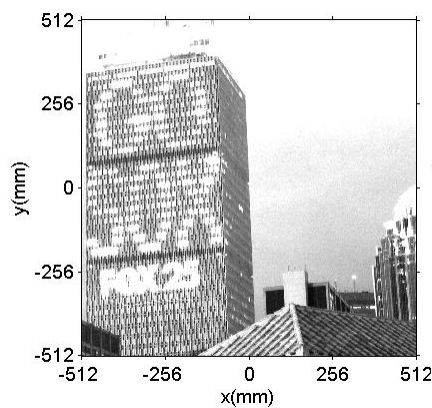
$$\frac{\Delta x}{\delta x} = \frac{2u_{\max}}{\delta u} \equiv N \quad : \text{1D Space–Bandwidth Product (SBP)}$$

aka number of pixels in the space domain

$$2\text{D SBP} \sim N^2$$

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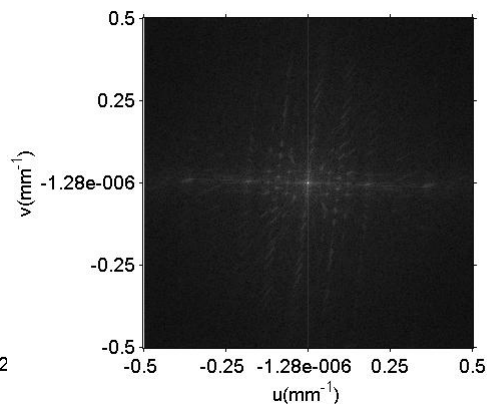
SBP: example



space domain

$$\delta x = 1 \quad \Delta x = 1024$$

$$u_{\max} = 0.5$$



Fourier domain

(aka spatial frequency domain)

$$\delta u = 1/1024 = 9.765625 \times 10^{-4}$$

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