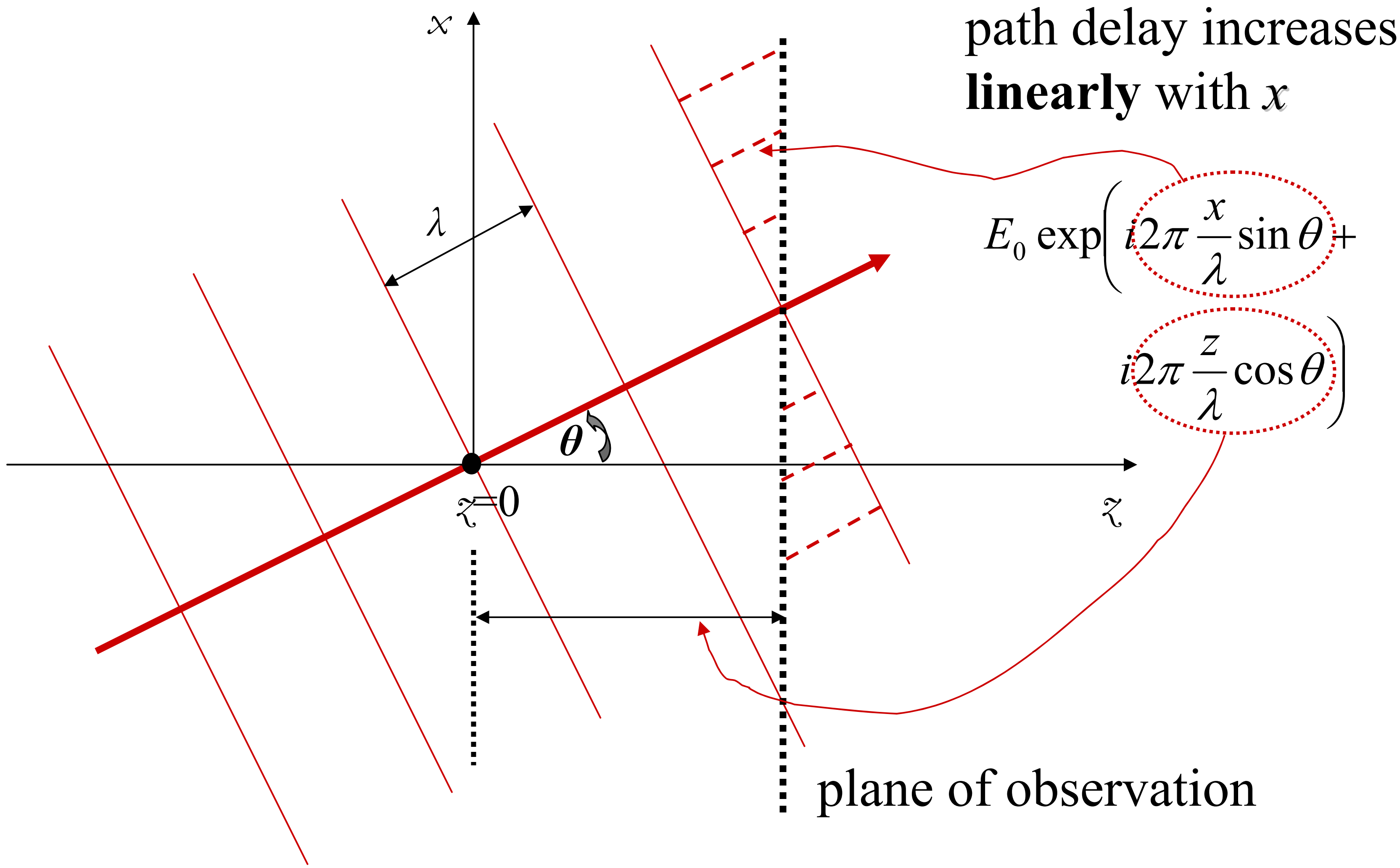
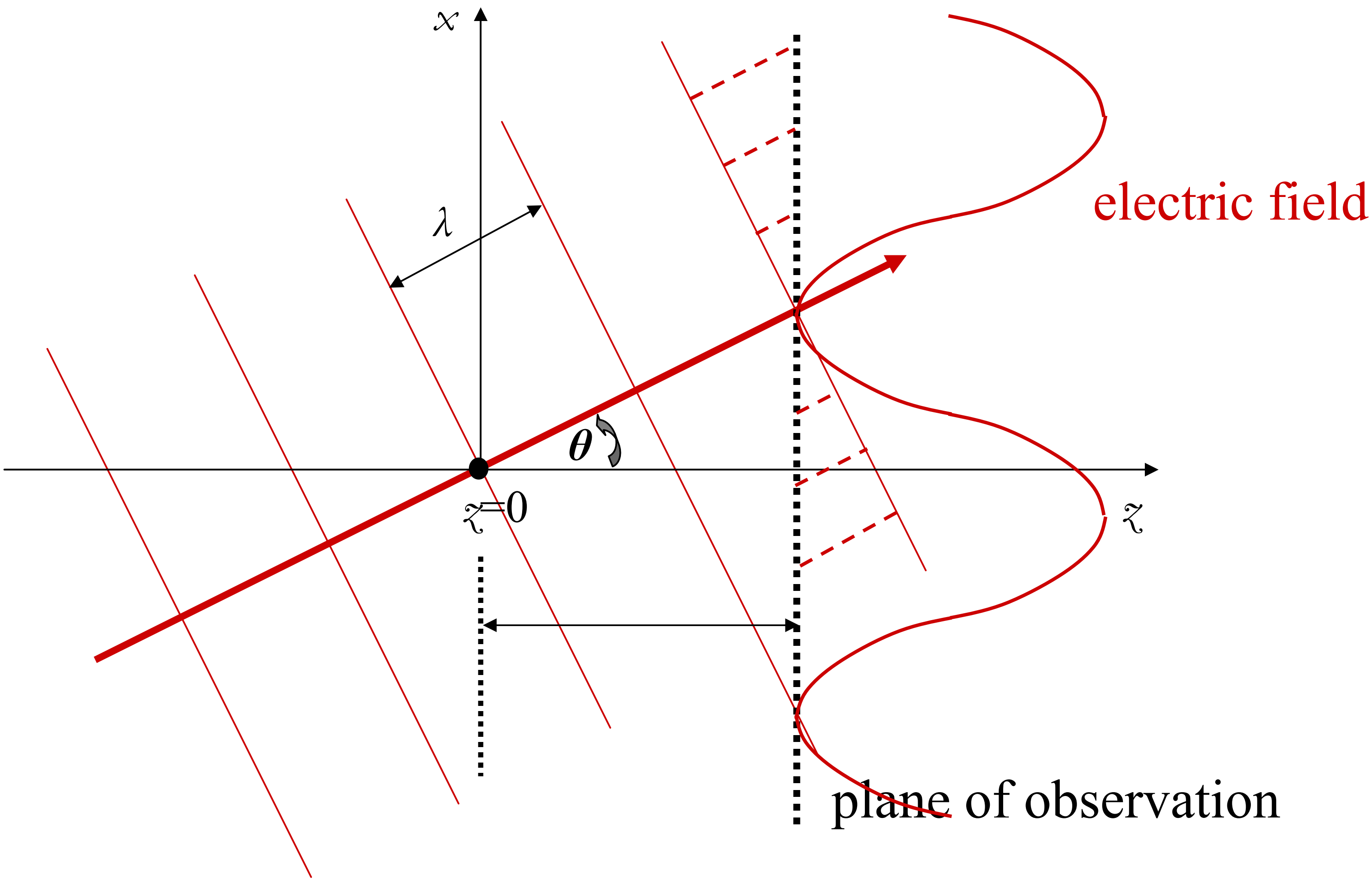
The spatial frequency domain

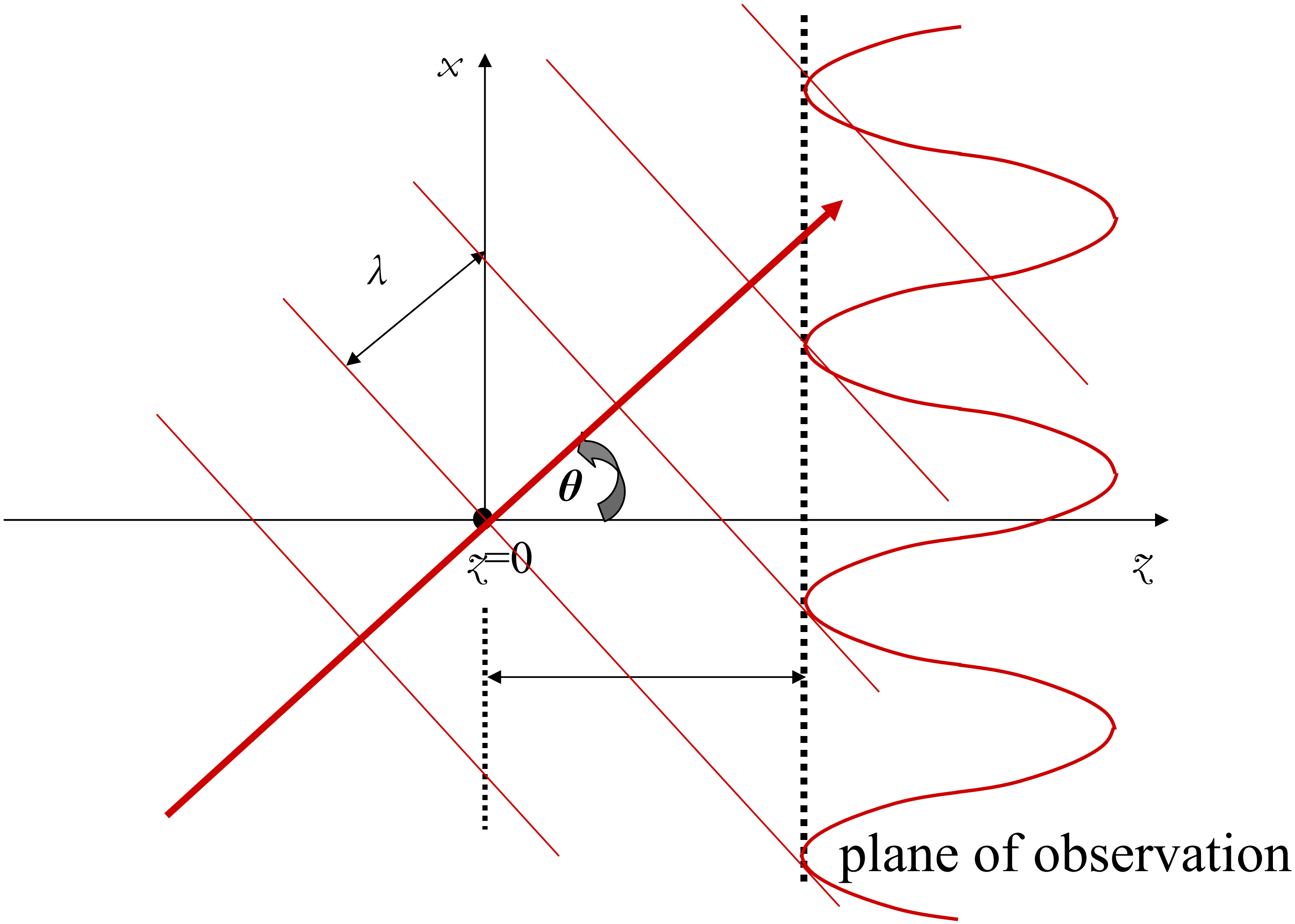
Recall: plane wave propagation



Spatial frequency ⇔ angle of propagation?



Spatial frequency ⇔ angle of propagation?



Spatial frequency ⇔ angle of propagation?

The cross-section of the optical field with the optical axis is a sinusoid of the form

$$E_0 \exp\left(i2\pi \frac{\sin\theta}{\lambda} x + \phi_0 \right)$$

i.e. it looks like

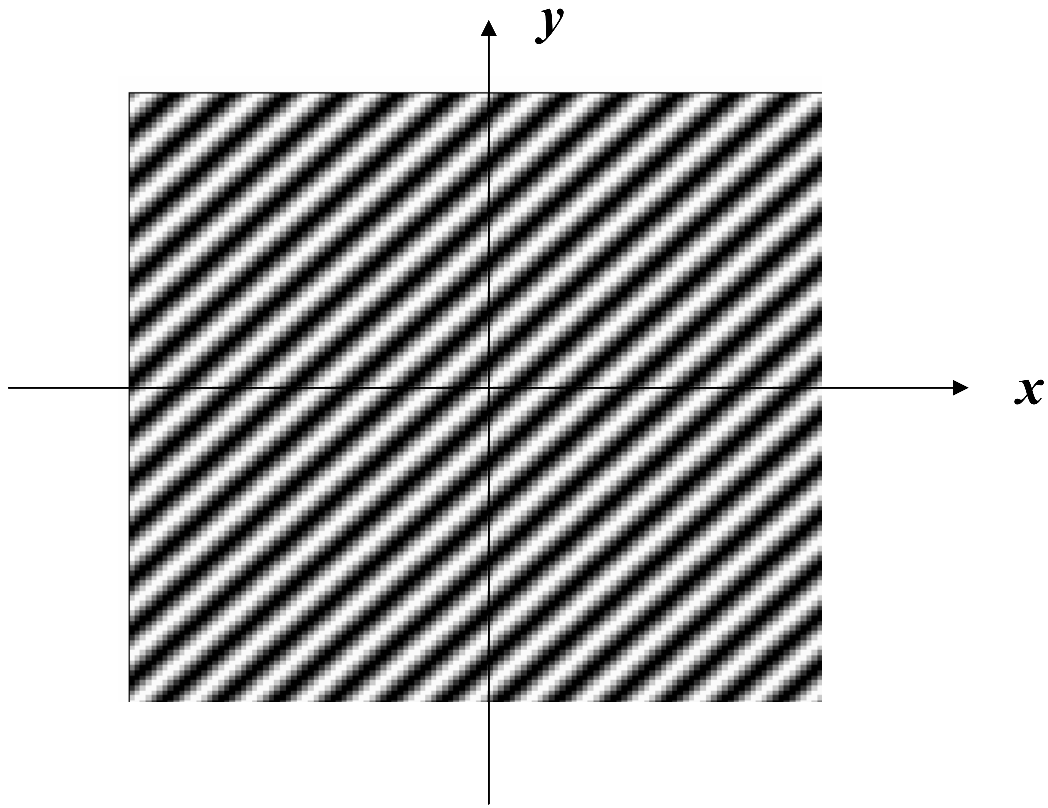
$$E_0 \exp\left(i2\pi\, u\, x + \phi_0 \right) \quad \text{where} \quad u \equiv \frac{\sin\theta}{\lambda}$$

is called the **spatial frequency**

2D sinusoids

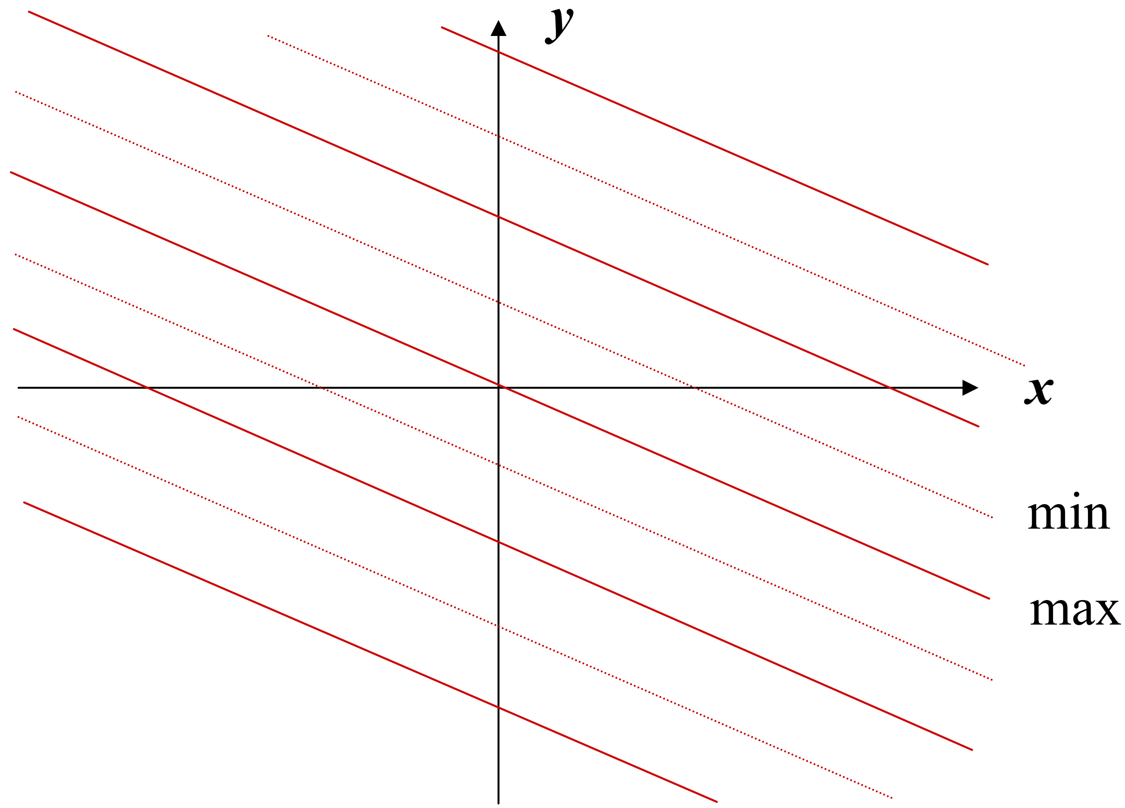
$$E_0 \cos[2\pi(ux + vy)]$$

$$\left(\begin{array}{c}\text{corresp. phasor}\\ E_0 \exp[i2\pi(ux+vy)]\end{array}\right)$$



2D sinusoids

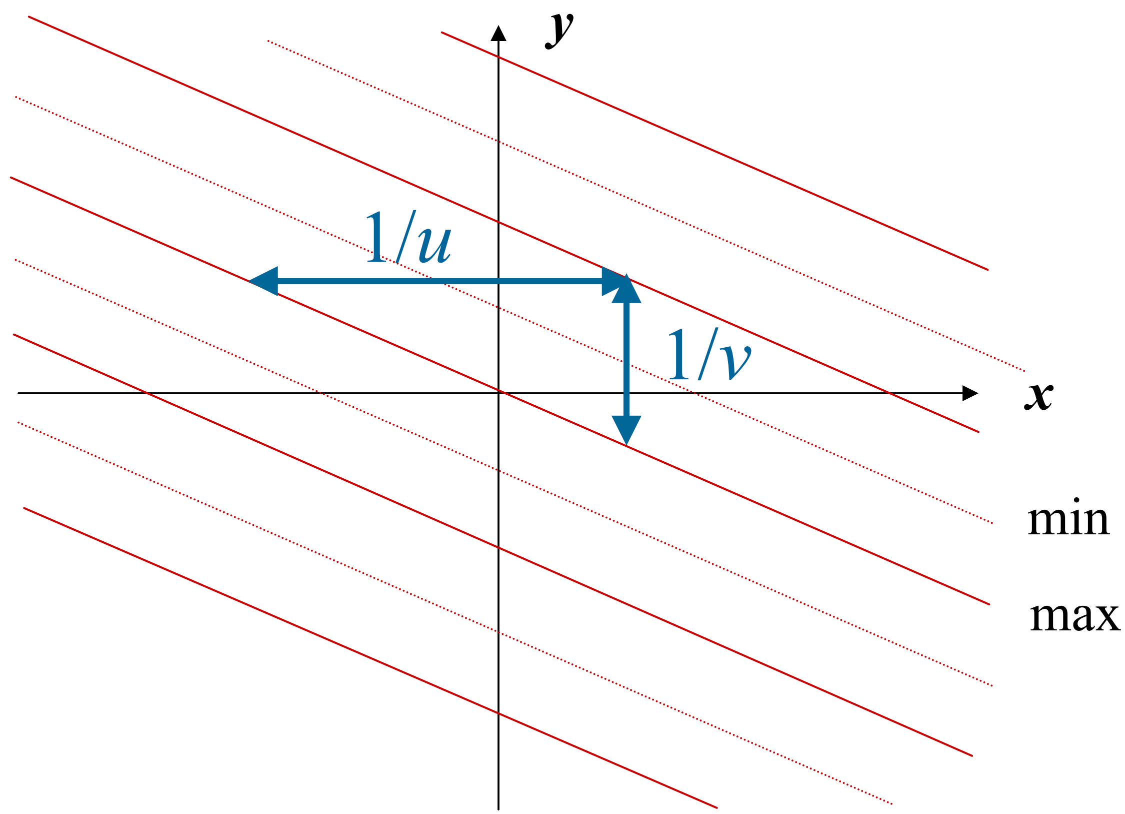
$$E_0 \cos\left[2\pi\left(ux+vy\right)\right] \qquad \left(\begin{array}{c}\text{corresp. phasor}\\ E_0 \exp\left[i2\pi\left(ux+vy\right)\right]\end{array}\right)$$



2D sinusoids

$$E_0 \cos\left[2\pi\left(ux + vy\right)\right]$$

$$\left(\begin{array}{c}\text{corresp. phasor}\\ E_0 \exp\left[i2\pi\left(ux+vy\right)\right]\end{array}\right)$$



2D sinusoids

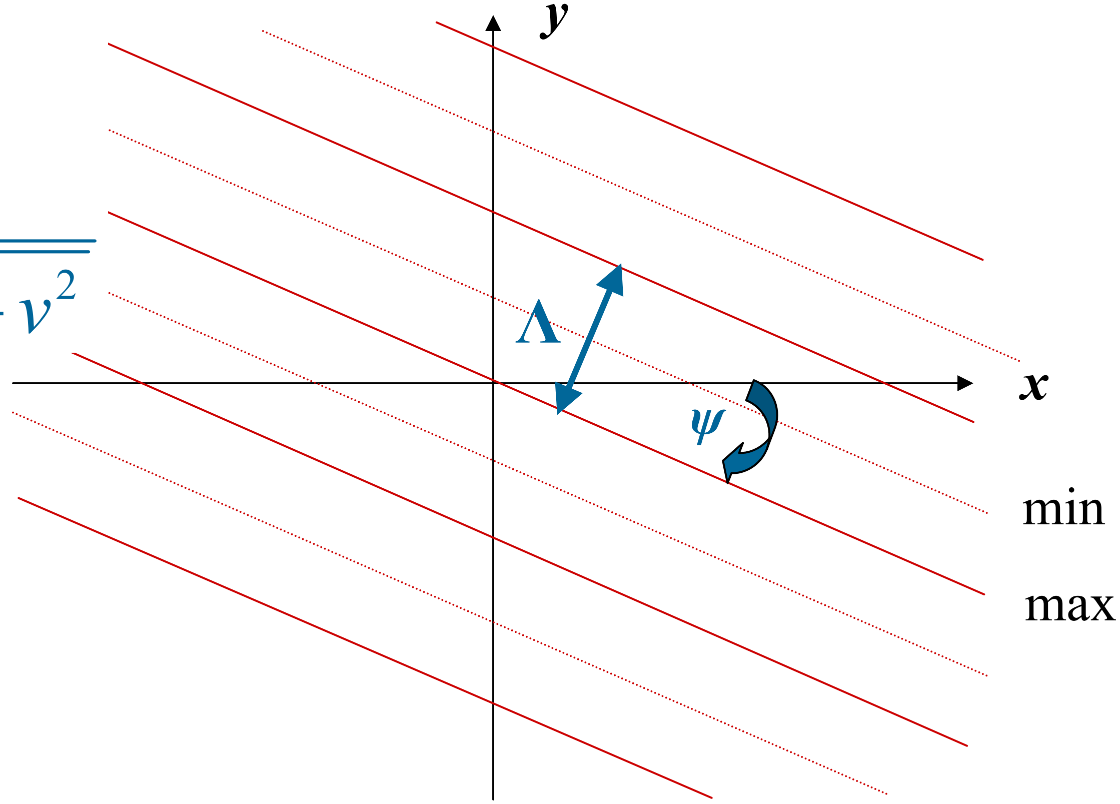
$$E_0 \cos\left[2\pi\left(ux + vy\right)\right]$$

corresp. phasor

$$E_0 \exp\left[i2\pi\left(ux + vy\right)\right]$$

$$\tan\psi = \frac{u}{v}$$

$$\Lambda = \frac{1}{\sqrt{u^2 + v^2}}$$



Spatial (2D) Fourier Transforms

The 2D Fourier integral

(aka **inverse Fourier transform**)

$$g(x,y) = \int G(u,v)\ \mathrm{e}^{+i2\pi(ux+vy)}\ \mathrm{d}u\mathrm{d}v$$

superposition

sinusoids

complex weight,
expresses relative amplitude
(magnitude & phase)
of superposed sinusoids

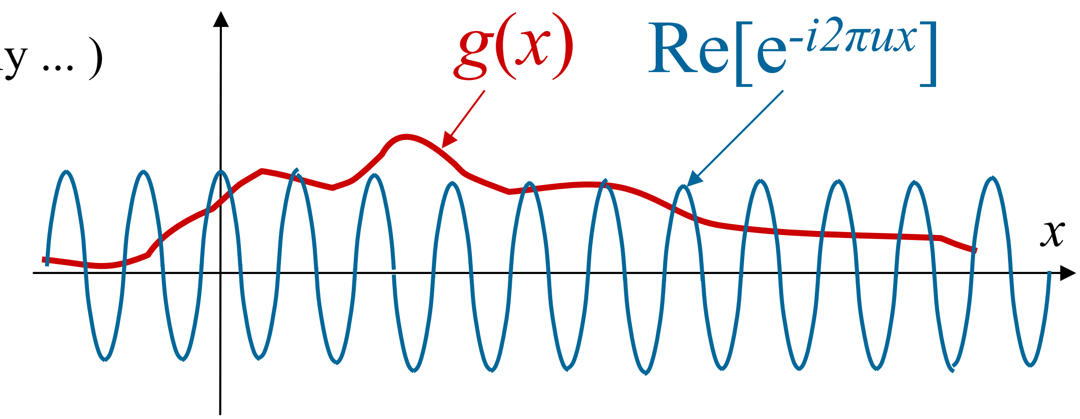
The 2D Fourier transform

The complex weight coefficients *G(u,v)*,
aka **Fourier transform** of *g(x,y)*
are calculated from the integral

$$G(u,v) = \int g(x,y)\, \mathrm{e}^{-i2\pi(ux+vy)}\, \mathrm{d}x\mathrm{d}y$$

(1D so we can draw it easily ...)

$$\mathrm{Re}[G(u)] = \int \quad \cdots \quad \mathrm{d}x$$



2D Fourier transform *pairs*

TABLE 2.1
Transform pairs for some functions separable in rectangular coordinates.

Function	Transform
$\exp[-\pi(a^2x^2+b^2y^2)]$	$\frac{1}{\lvert ab\rvert}\exp\left[-\pi\left(\frac{f_X^2}{a^2}+\frac{f_Y^2}{b^2}\right)\right]$
$\mathrm{rect}(ax)\,\mathrm{rect}(by)$	$\frac{1}{\lvert ab\rvert}\,\mathrm{sinc}(f_X/a)\,\mathrm{sinc}(f_Y/b)$
$\Lambda(ax)\,\Lambda(by)$	$\frac{1}{\lvert ab\rvert}\,\mathrm{sinc}^2(f_X/a)\,\mathrm{sinc}^2(f_Y/b)$
$\delta(ax, by)$	$\frac{1}{\lvert ab\rvert}$
$\exp[j\pi(ax+by)]$	$\delta(f_X - a/2, f_Y - b/2)$
$\mathrm{sgn}(ax)\,\mathrm{sgn}(by)$	$\frac{ab}{\lvert ab\rvert}\,\frac{1}{j\pi f_X}\,\frac{1}{j\pi f_Y}$
$\mathrm{comb}(ax)\,\mathrm{comb}(by)$	$\frac{1}{\lvert ab\rvert}\,\mathrm{comb}(f_X/a)\,\mathrm{comb}(f_Y/b)$
$\exp[j\pi(a^2x^2+b^2y^2)]$	$\frac{j}{\lvert ab\rvert}\exp\left[-j\pi\left(\frac{f_X^2}{a^2}+\frac{f_Y^2}{b^2}\right)\right]$
$\exp[-(a\lvert x\rvert+b\lvert y\rvert)]$	$\frac{1}{\lvert ab\rvert}\,\frac{2}{1+(2\pi f_X/a)^2}\,\frac{2}{1+(2\pi f_Y/b)^2}$

(from Goodman, *Introduction to Fourier Optics*, page 14)

Space and spatial frequency representations

SPACE DOMAIN $g(x,y)$

$$G(u,v)=\int g(x,y)\ \mathrm{e}^{-i2\pi(ux+vy)}\,\mathrm{d}x\mathrm{d}y$$

2D Fourier transform

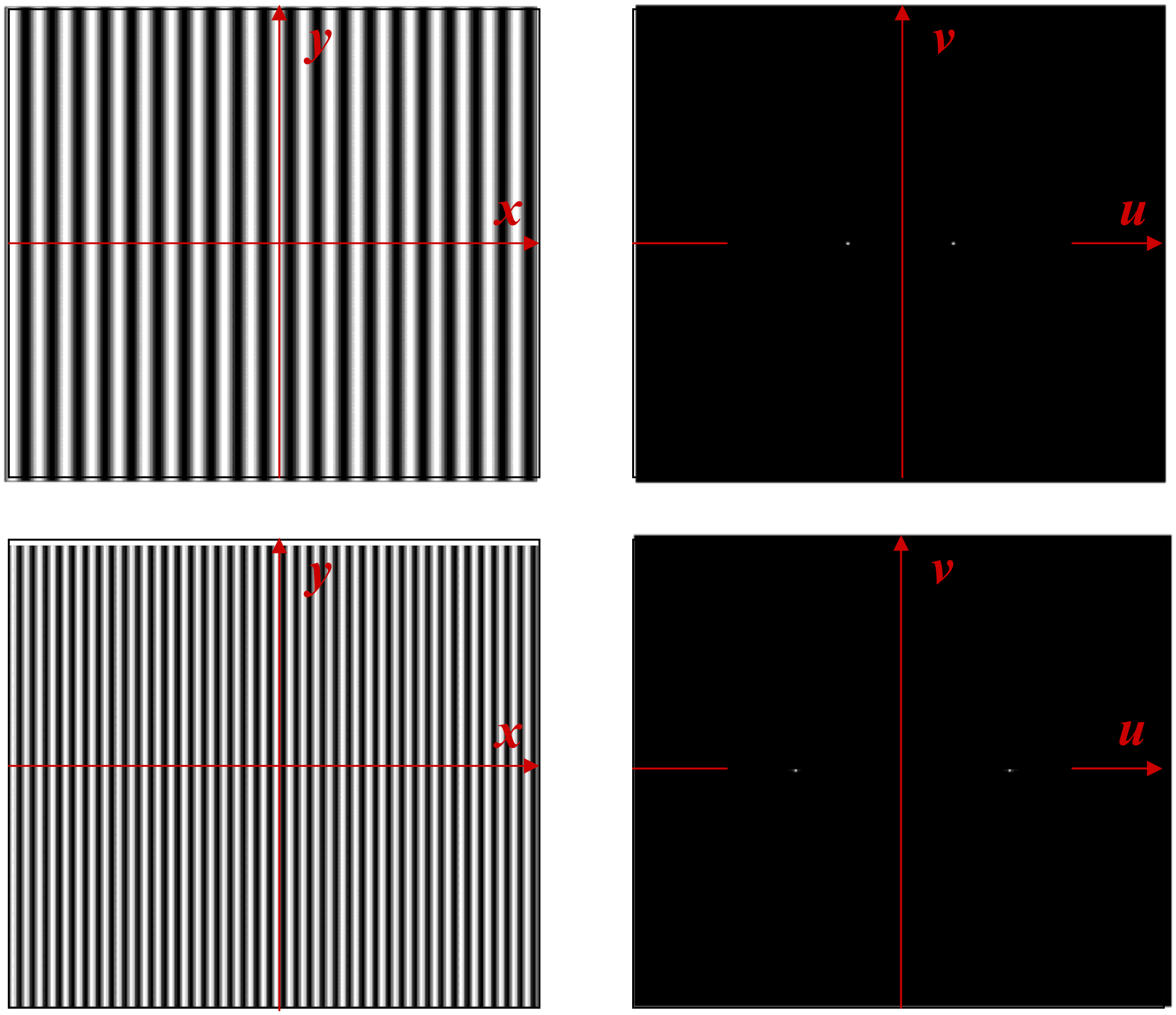
$$g(x,y)=\int G(u,v)\ \mathrm{e}^{+i2\pi(ux+vy)}\,\mathrm{d}u\mathrm{d}v$$

2D Fourier integral
aka
inverse 2D Fourier transform

SPATIAL FREQUENCY DOMAIN $G(u,v)$

Periodic Grating /1: vertical

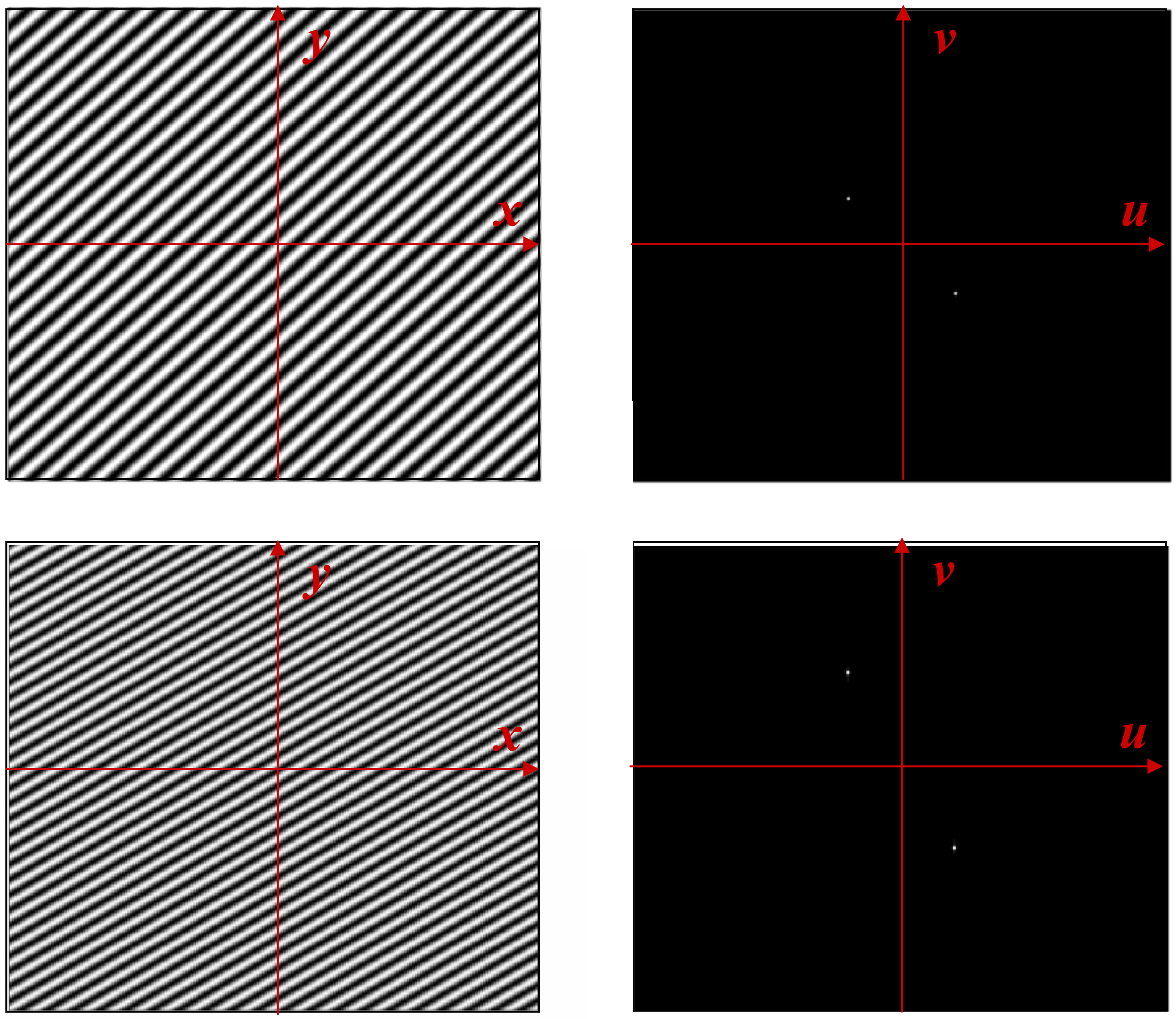
Space domain



Frequency (Fourier) domain

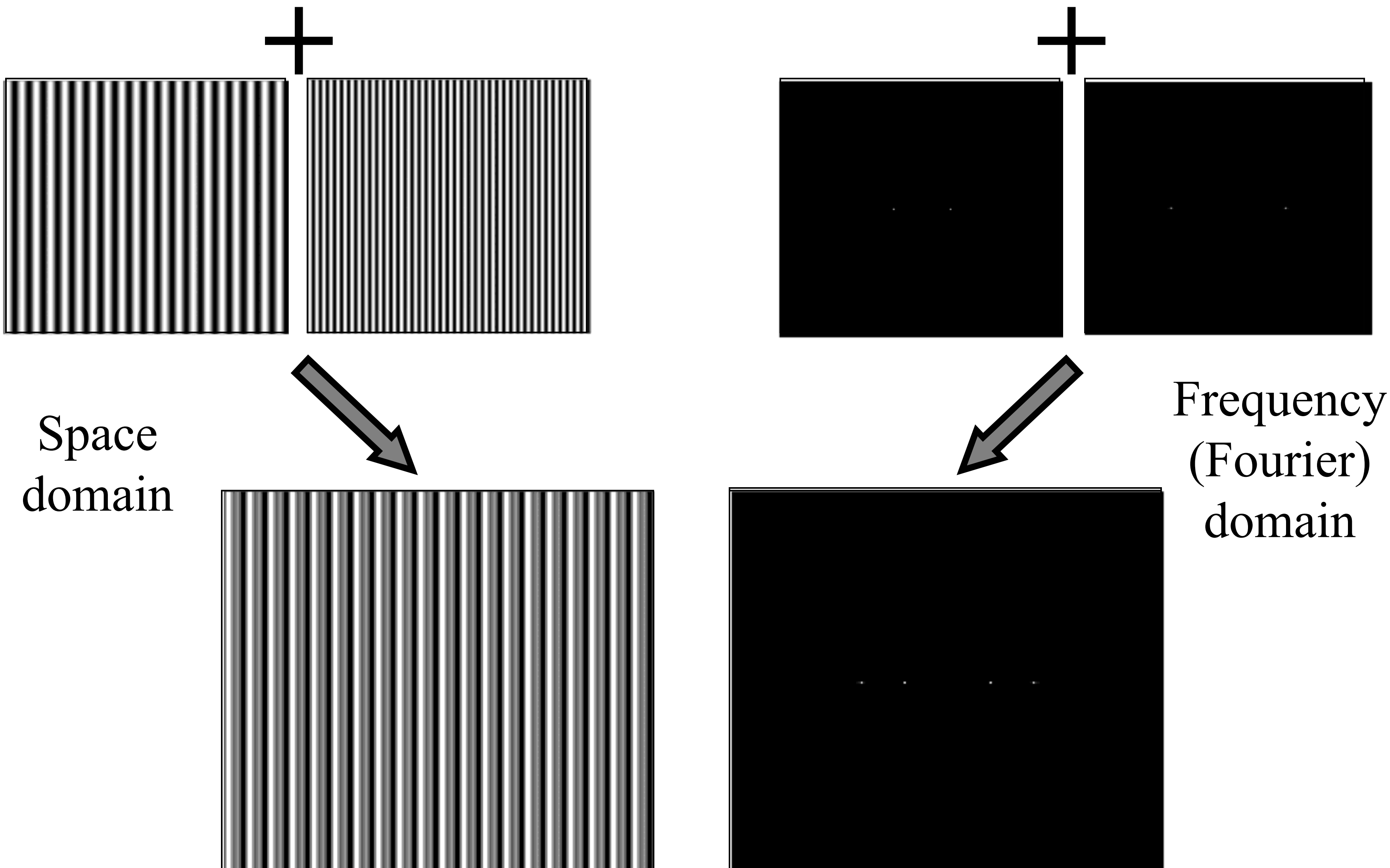
Periodic Grating /2: tilted

Space domain

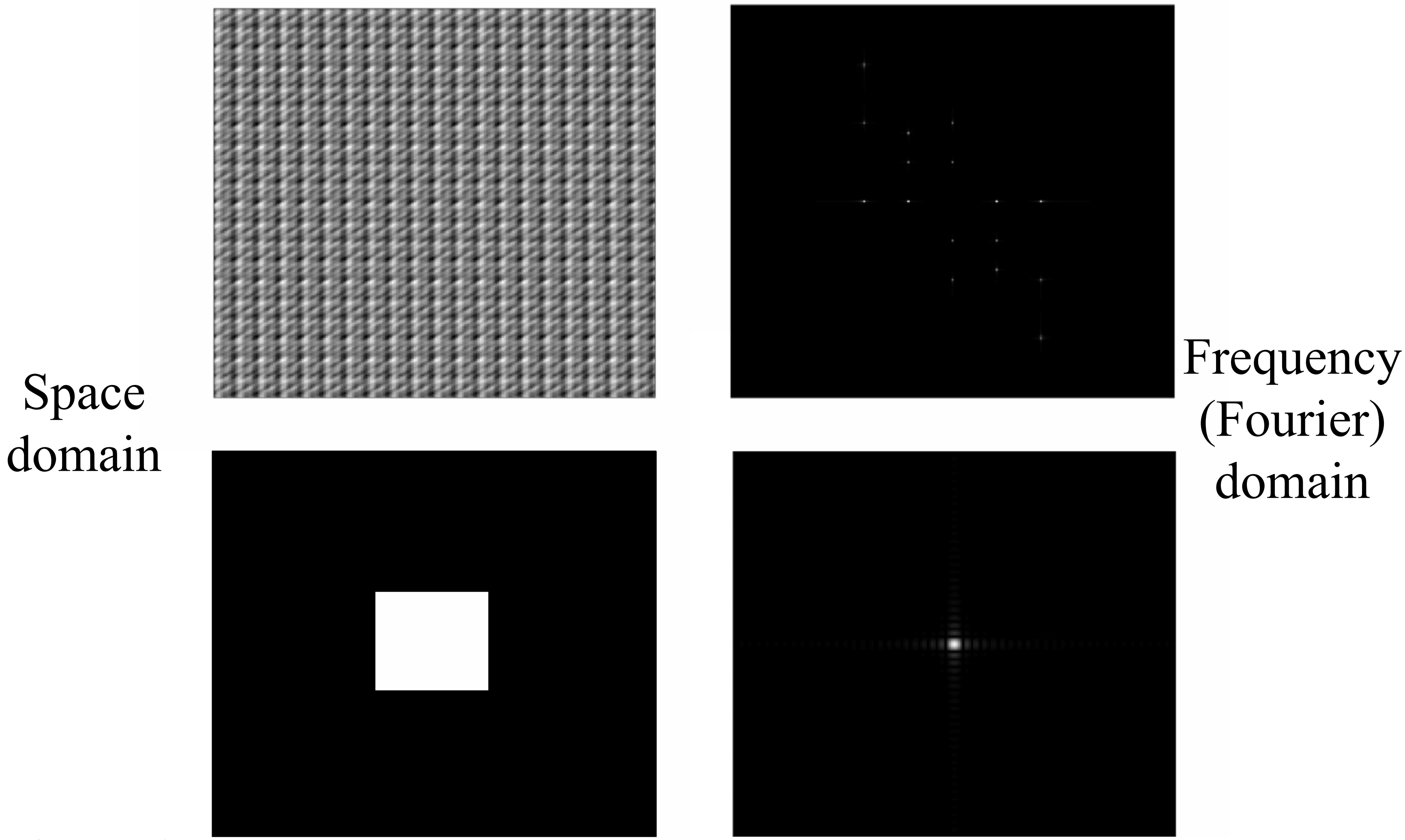


Frequency (Fourier) domain

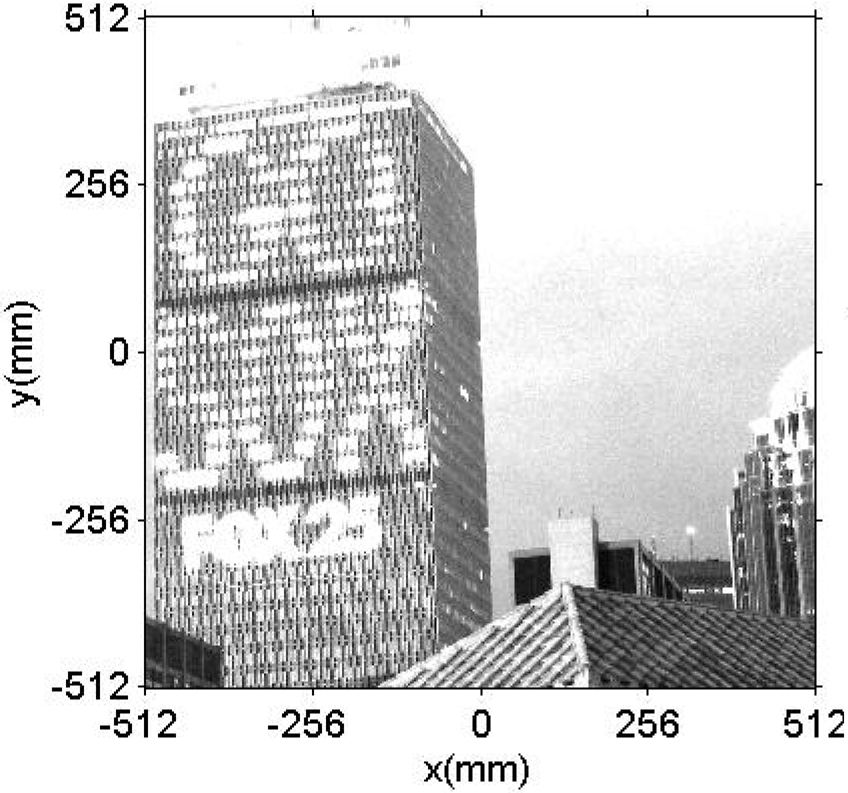
Superposition: multiple gratings



Superpositions: spatial frequency representation

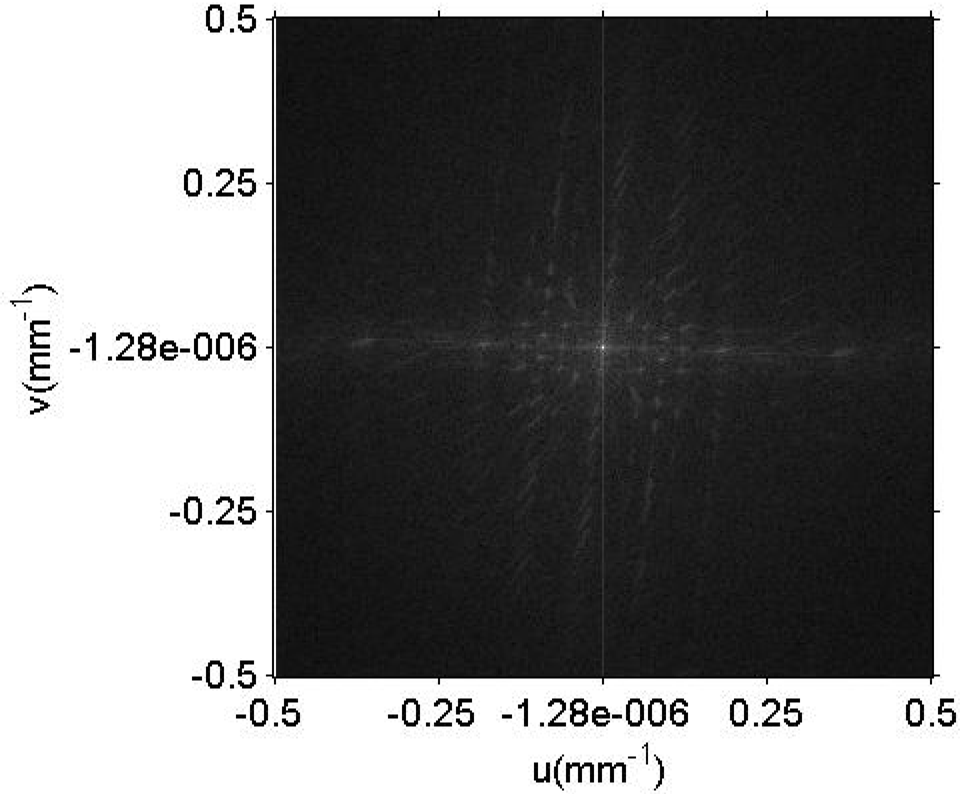


Superpositions: spatial frequency representation



space domain

$$g(x,y)$$



spatial frequency domain

$$G(u,v)=\Im\{g(x,y)\}$$

Fourier transform properties /1

- Fourier transforms and the delta function

$$\Im\{\delta(x,y)\}=1$$

$$\Im\{\exp[i2\pi(u_0x+v_0y)]\}=\delta(u-u_0)\delta(v-v_0)$$

- Linearity of Fourier transforms

if $\Im\{g_1(x,y)\}=G_1(u,v)$ and $\Im\{g_2(x,y)\}=G_2(u,v)$

then $\Im\{a_1g_1(x,y)+a_2g_2(x,y)\}=a_1G_1(u,v)+a_2G_2(u,v)$

for any pair of complex numbers a_1, a_2.

Fourier transform properties /2

Let $\Im\{g(x,y)\} = G(u,v)$

- Shift theorem (space $\rightarrow$ frequency)

$$\Im\{g(x-x_0, y-y_0)\} = G(u,v)\exp[-i2\pi(ux_0+vy_0)]$$

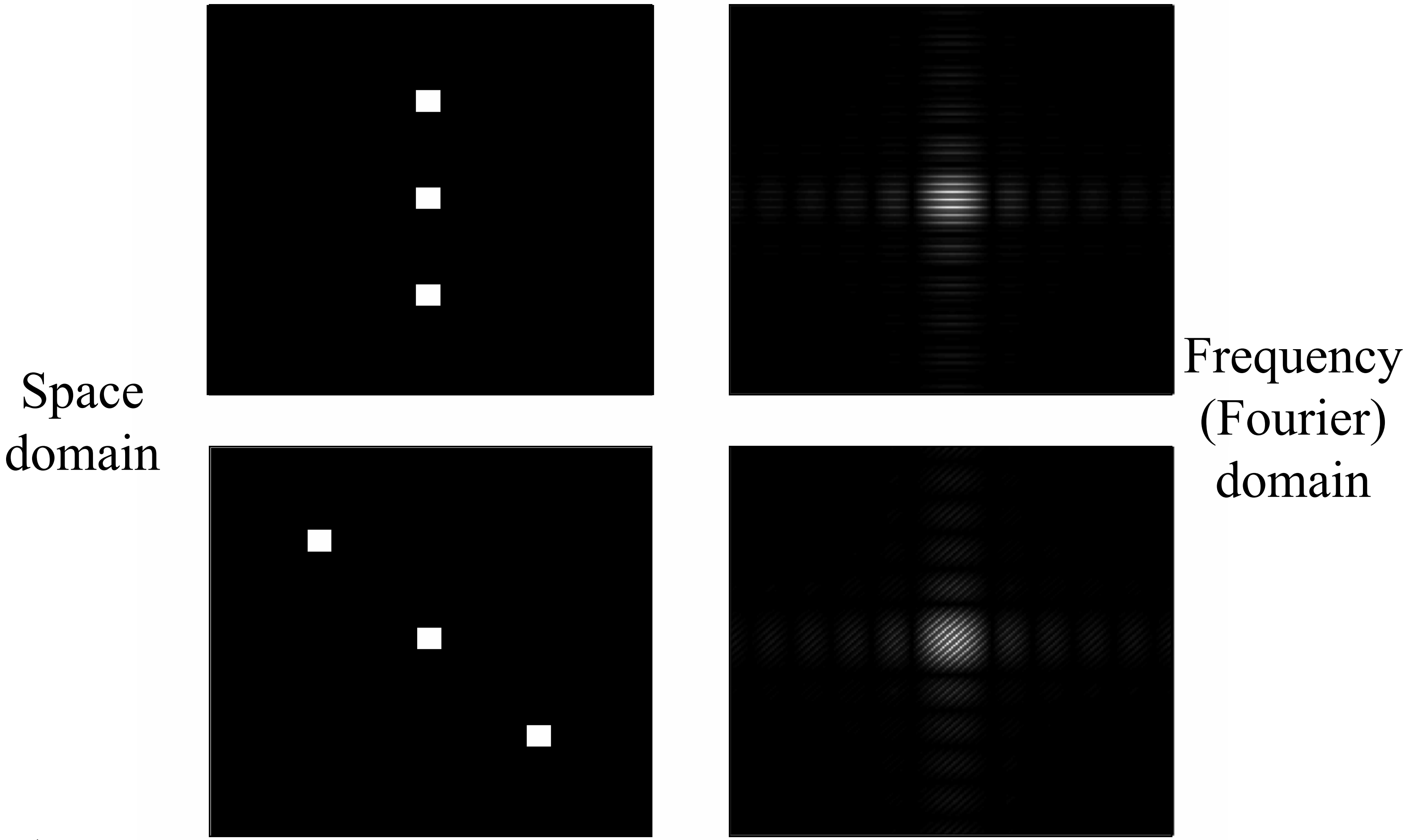
- Shift theorem (frequency $\rightarrow$ space)

$$\Im\{g(x,y)\exp[i2\pi(ux_0+vy_0)]\} = G(u-u_0, v-v_0)$$

- Scaling theorem

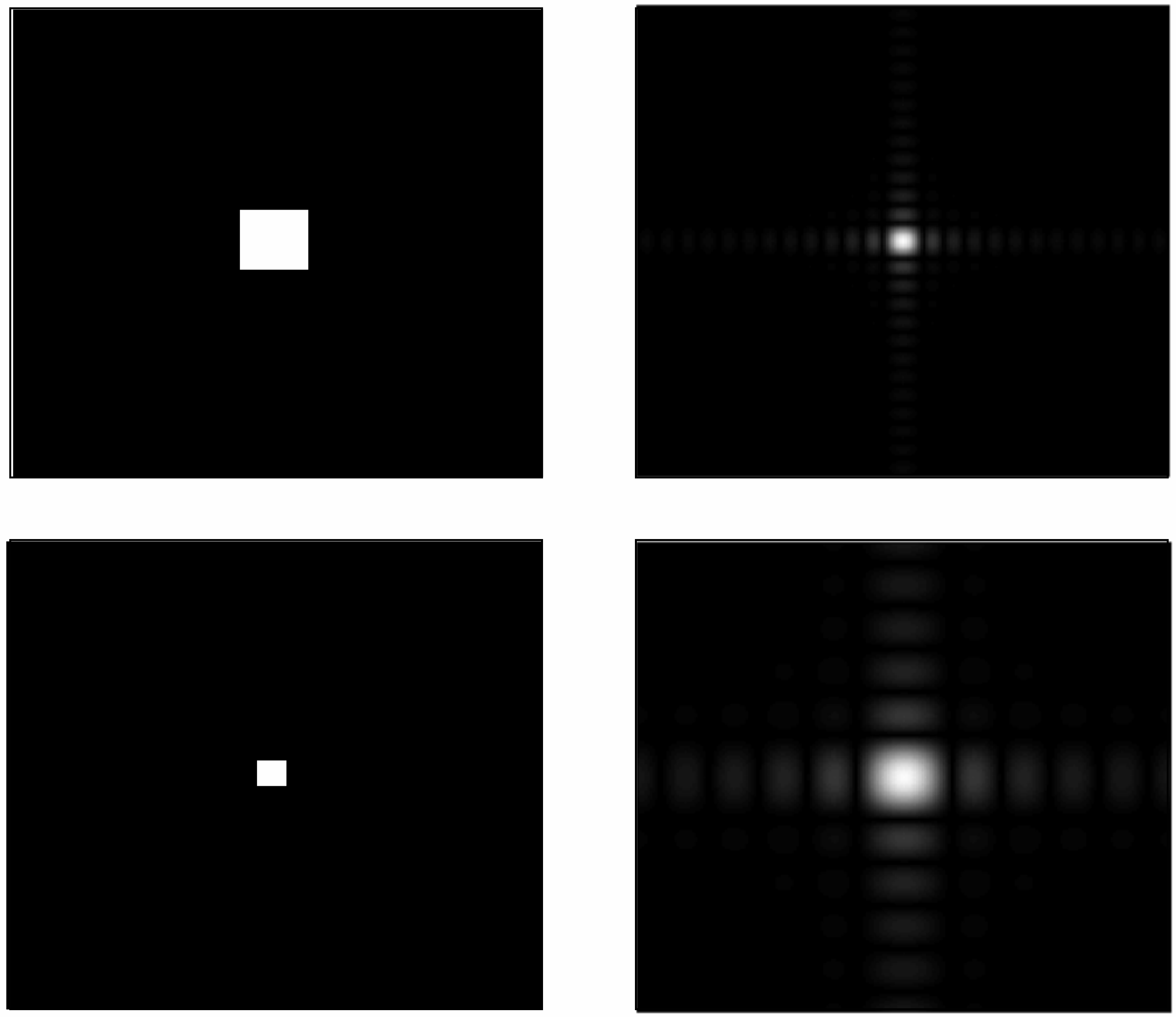
$$\Im\{g(ax,by)\} = \frac{1}{|ab|}G\left(\frac{u}{a},\frac{v}{b}\right)$$

Modulation in the frequency domain: the shift theorem



Size of object *vs* frequency content: the scaling theorem

Space domain



Frequency (Fourier) domain

Fourier transform properties /3

Let $\Im\{f(x,y)\} = F(u,v)$ and $\Im\{h(x,y)\} = H(u,v)$

Let $g(x,y) = \int f(x',y') \cdot h(x-x',y-y')\,\mathrm{d}x'\mathrm{d}y'$

- Convolution theorem (space $\rightarrow$ frequency)

$$\Im\{g(x,y)\} = F(u,v) \cdot H(u,v)$$

Let $Q(u,v) = \int F(u',v') \cdot H(u-u',v-v')\,\mathrm{d}u'\mathrm{d}v'$

- Convolution theorem (frequency $\rightarrow$ space)

$$Q(u,v) = \Im\{f(x,y) \cdot h(x,y)\}$$

Fourier transform properties /4

Let $\Im\{f(x,y)\} = F(u,v)$ and $\Im\{h(x,y)\} = H(u,v)$

Let $g(x,y) = \int f(x',y') \cdot h(x'-x, y'-y)\,\mathrm{d}x'\mathrm{d}y'$

- Correlation theorem (space $\rightarrow$ frequency)

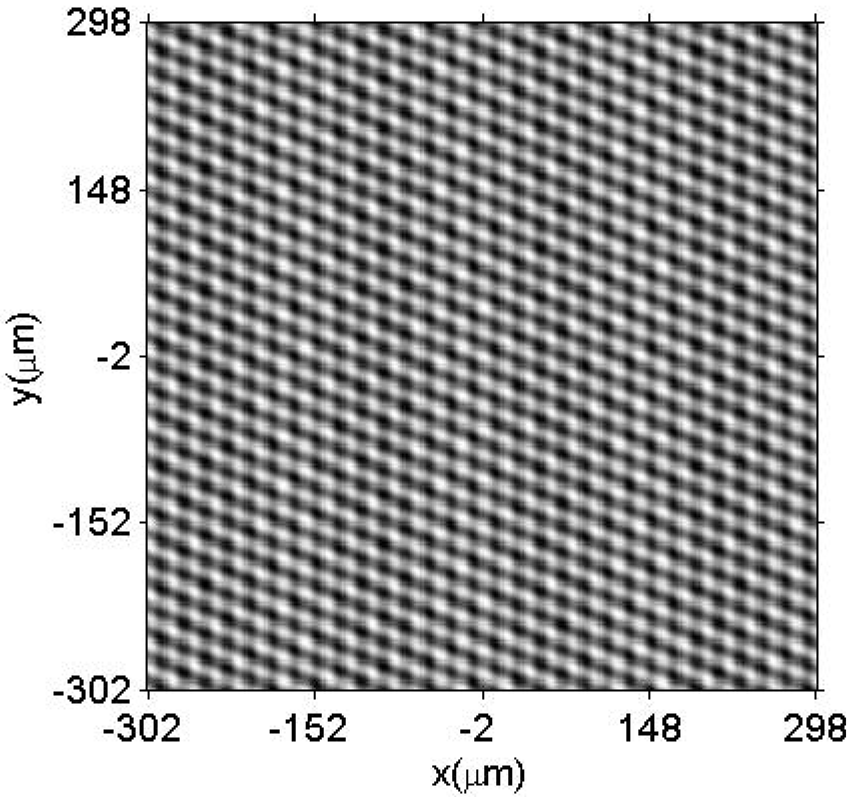
$$\Im\{g(x,y)\} = F(u,v) \cdot H^*(u,v)$$

Let $Q(u,v) = \int F(u',v') \cdot H(u'-u, v'-v)\,\mathrm{d}u'\mathrm{d}v'$

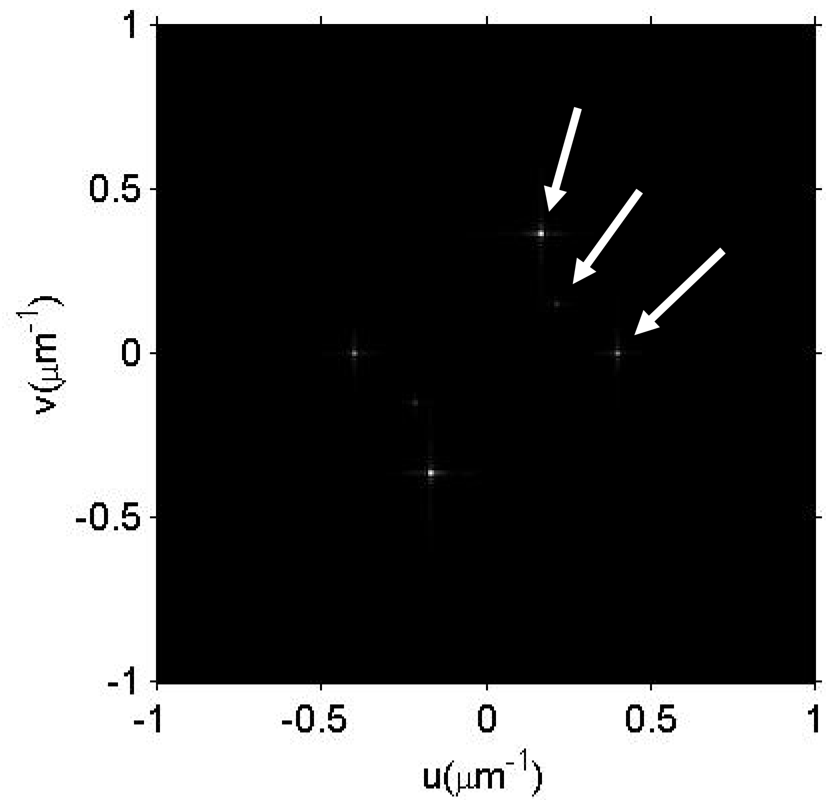
- Correlation theorem (frequency $\rightarrow$ space)

$$Q(u,v) = \Im\{f(x,y) \cdot h^*(x,y)\}$$

Spatial frequency representation

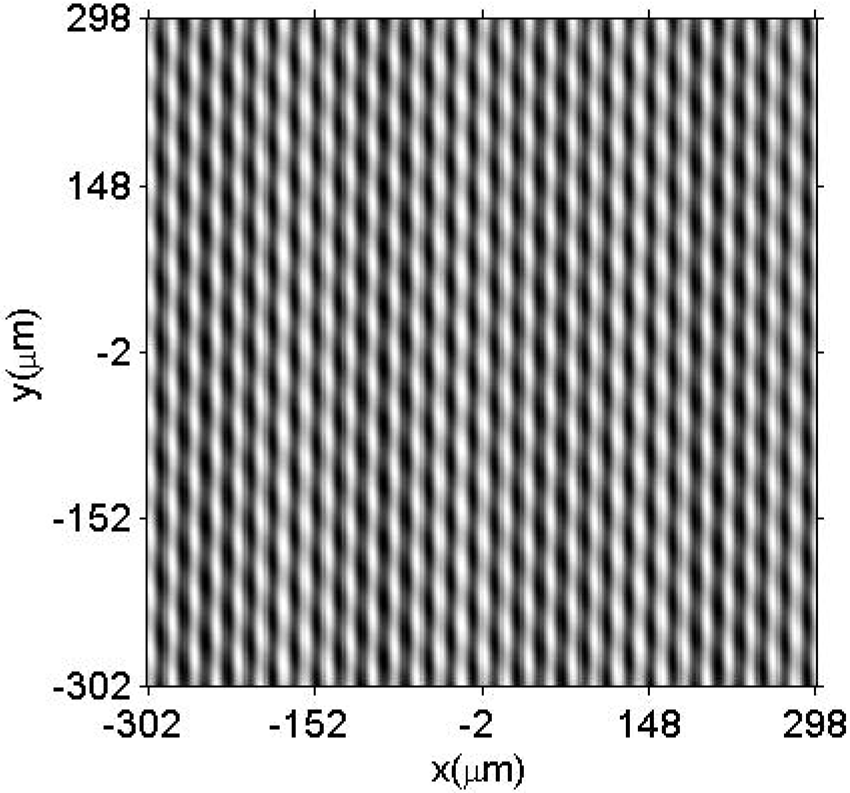


space domain
3 sinusoids

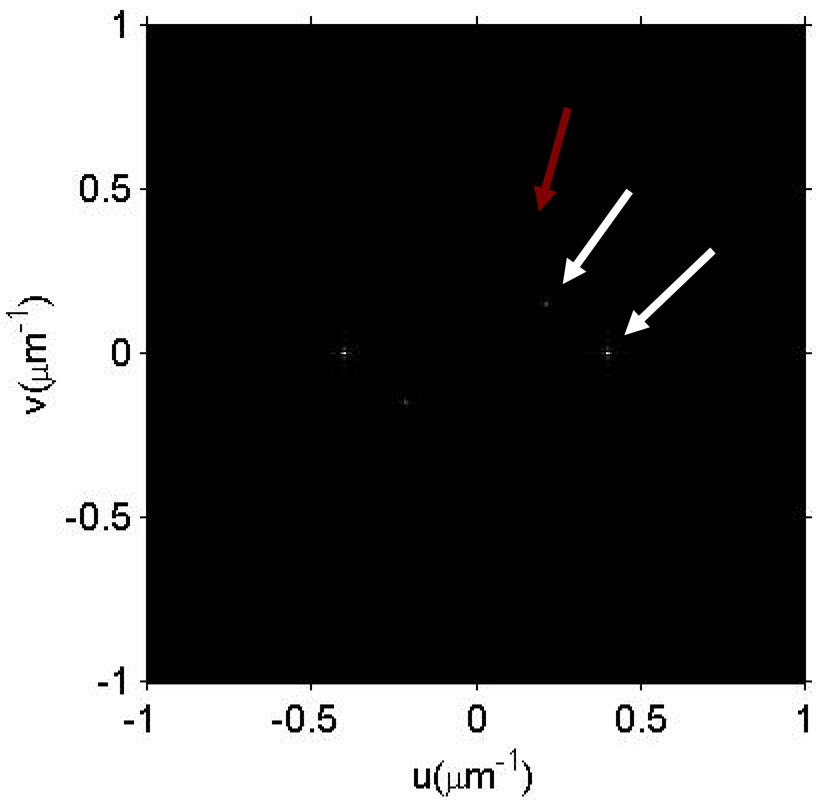


Fourier domain
(aka spatial frequency domain)

Spatial filtering

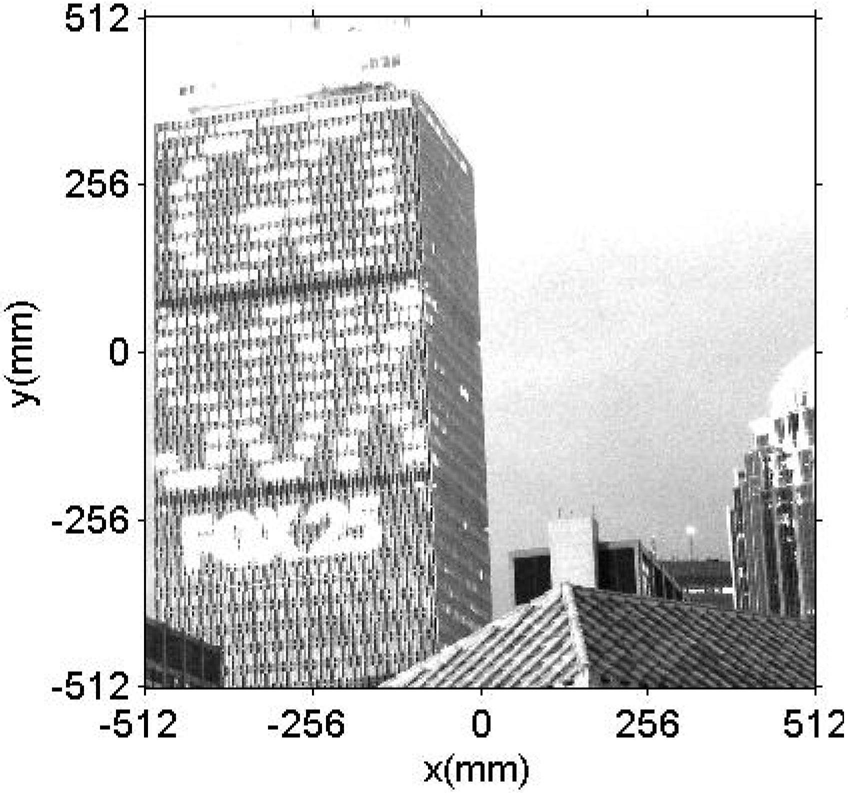


space domain
2 sinusoids (1 removed)



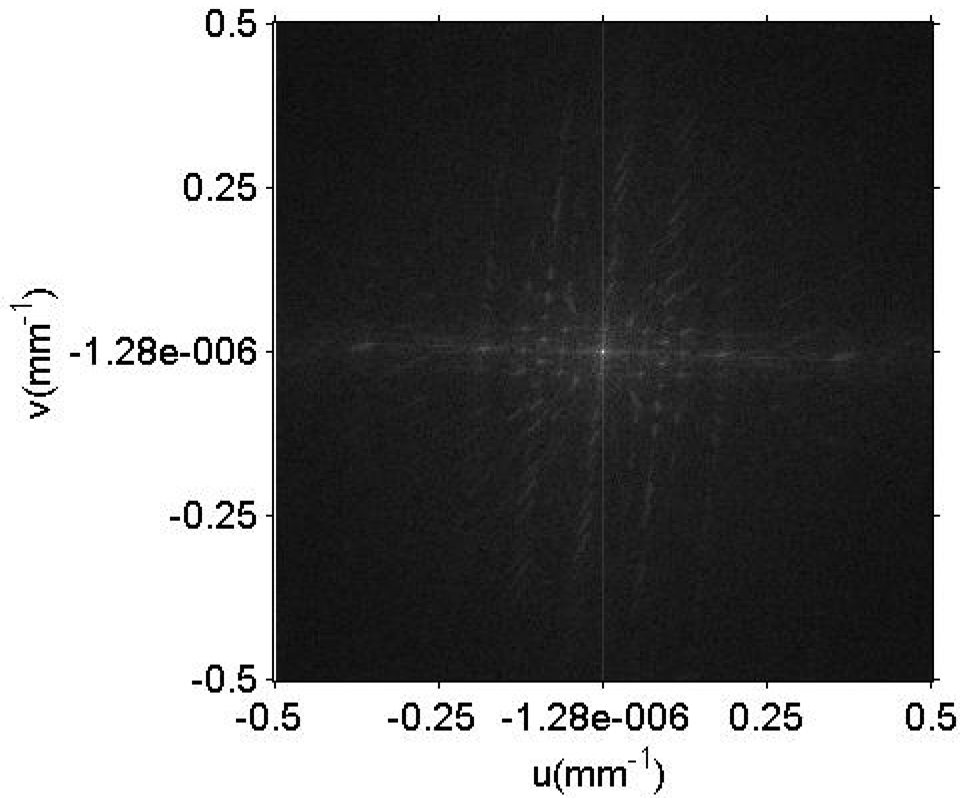
Fourier domain
(aka spatial frequency domain)

Spatial frequency representation



space domain

$$g(x,y)$$

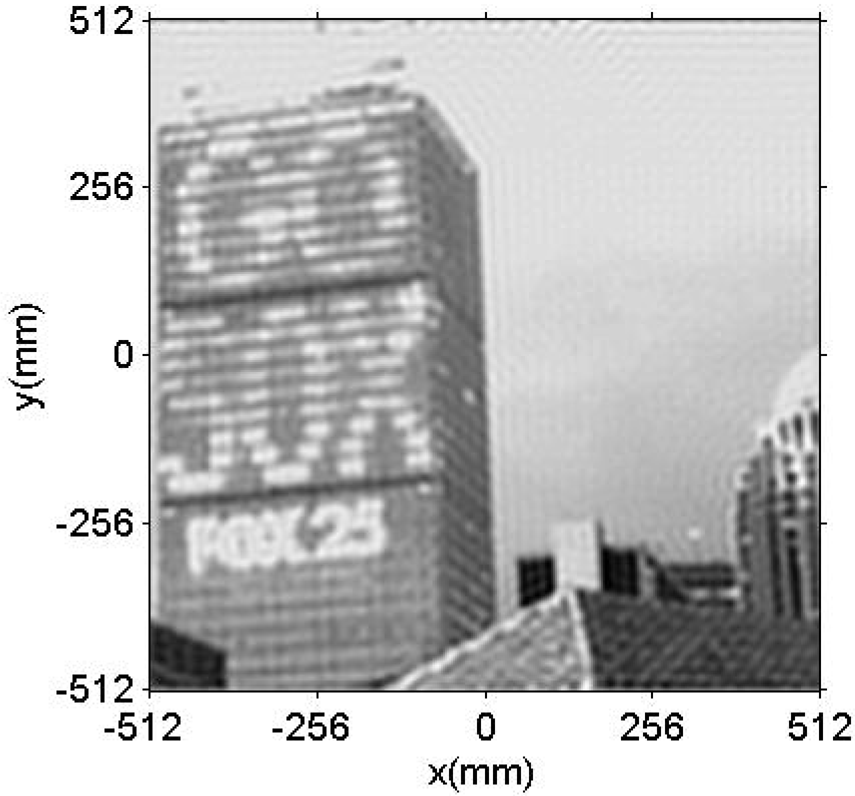


Fourier domain

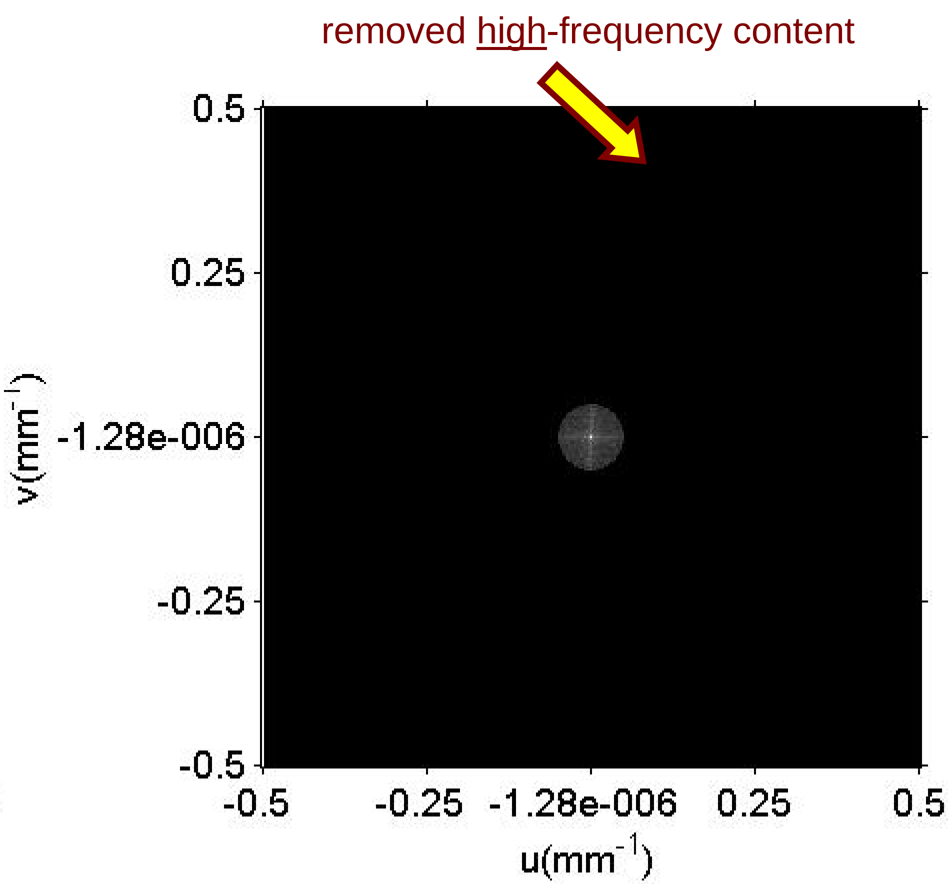
(aka spatial frequency domain)

$$G(u,v) = \Im\{g(x,y)\}$$

Spatial filtering (low–pass)

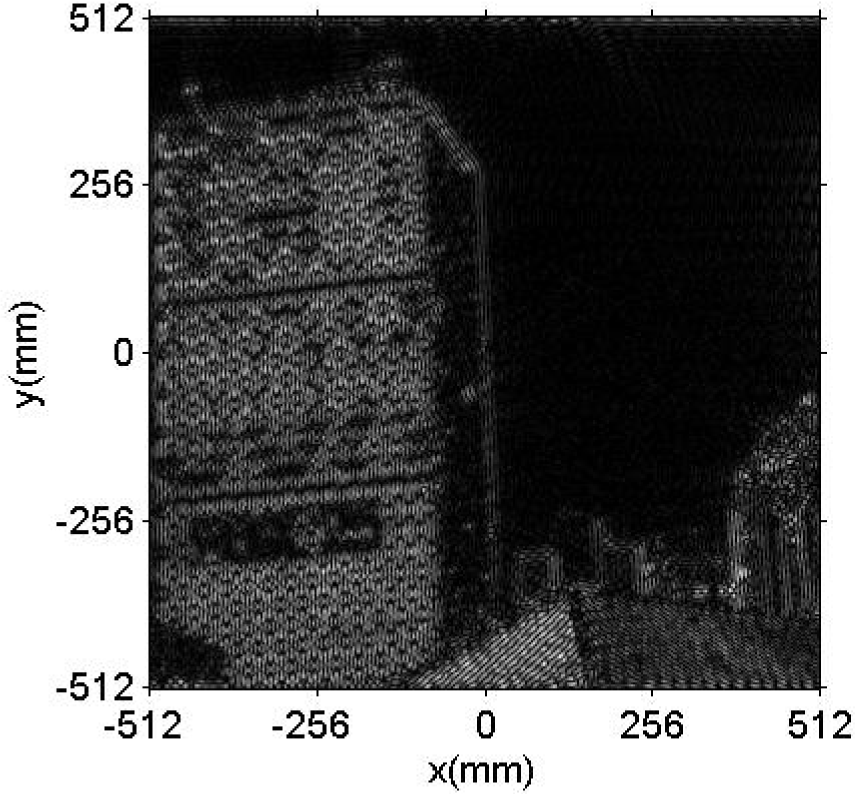


space domain



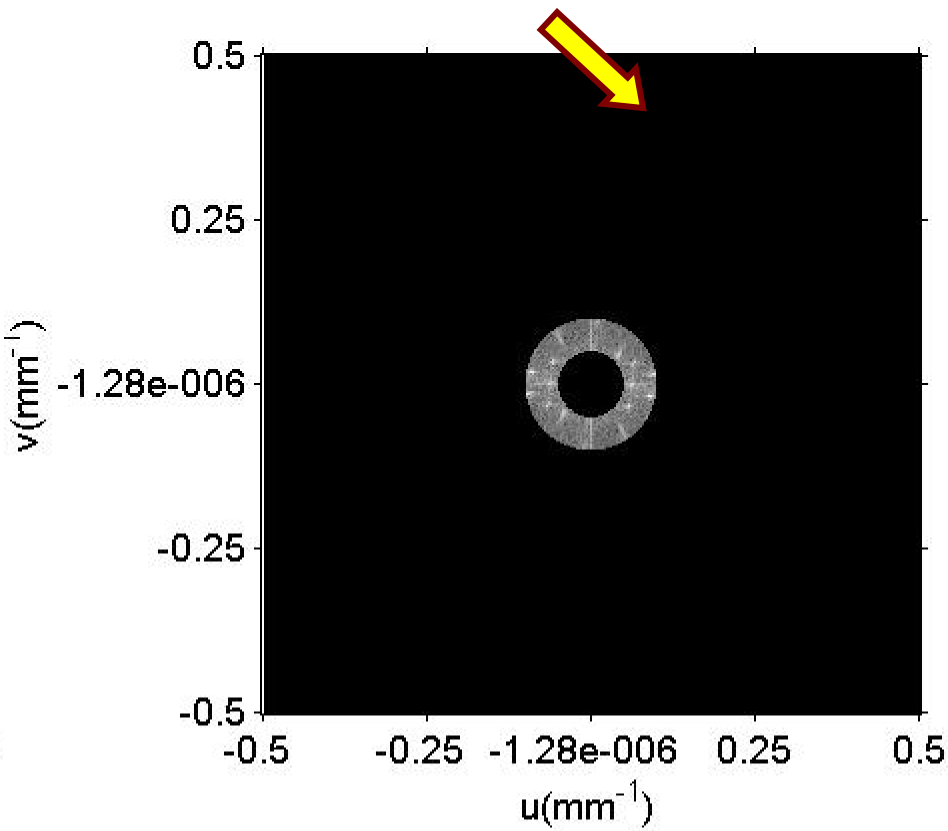
Fourier domain
(aka spatial frequency domain)

Spatial filtering (band–pass)



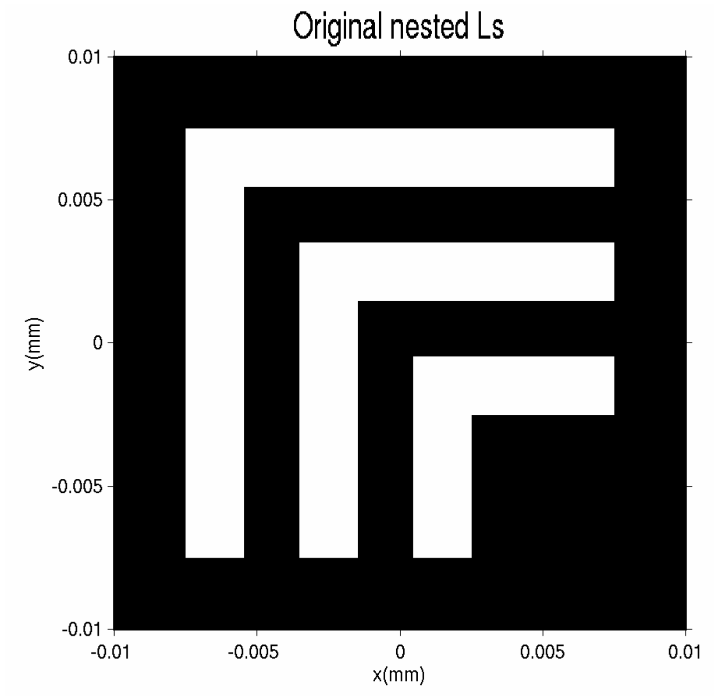
space domain

removed high- and low-frequency content

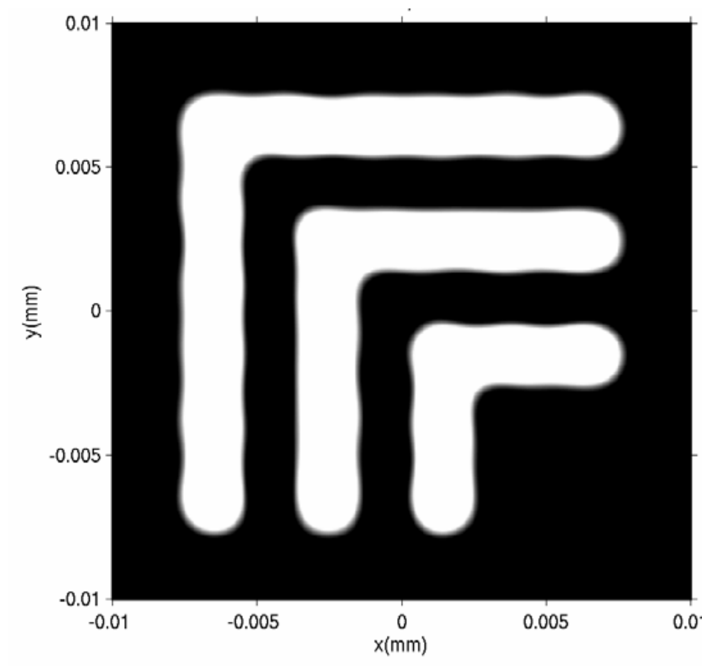


Fourier domain
(aka spatial frequency domain)

Example: optical lithography



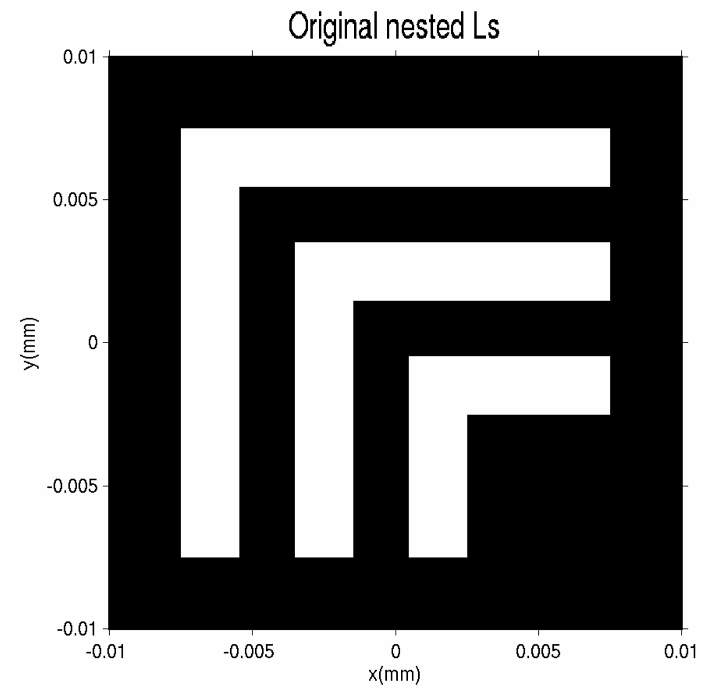
original pattern
("nested L's")



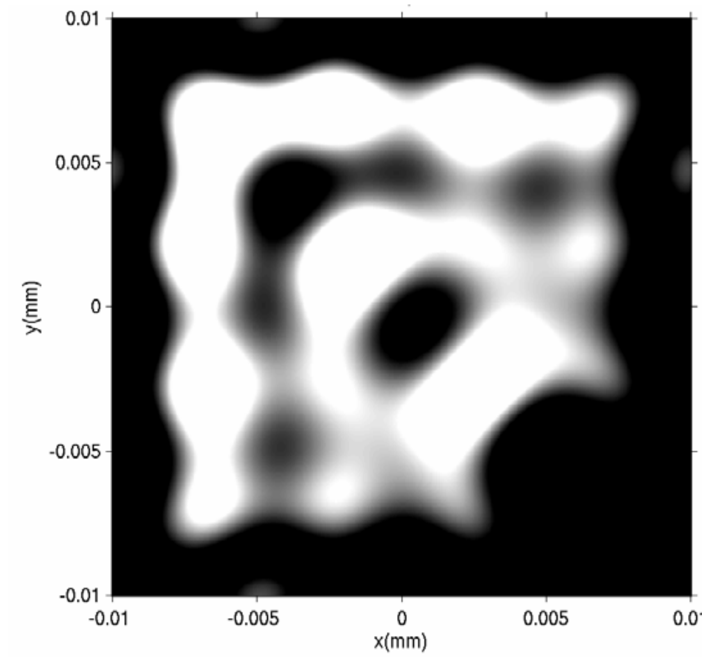
mild
low-pass filtering

Notice:
(i) blurring at the edges
(ii) ringing

Example: optical lithography



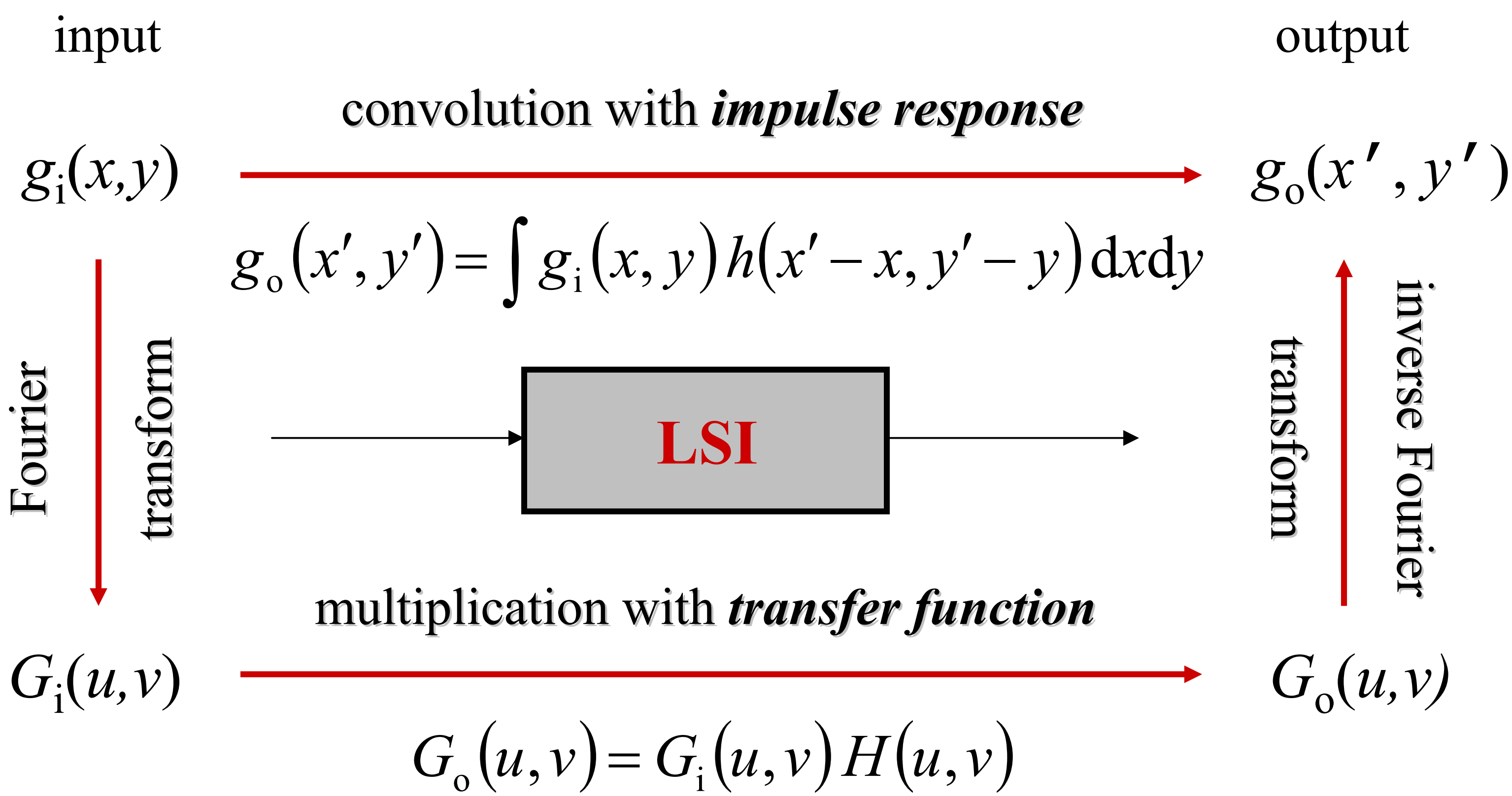
original pattern
("nested L's")



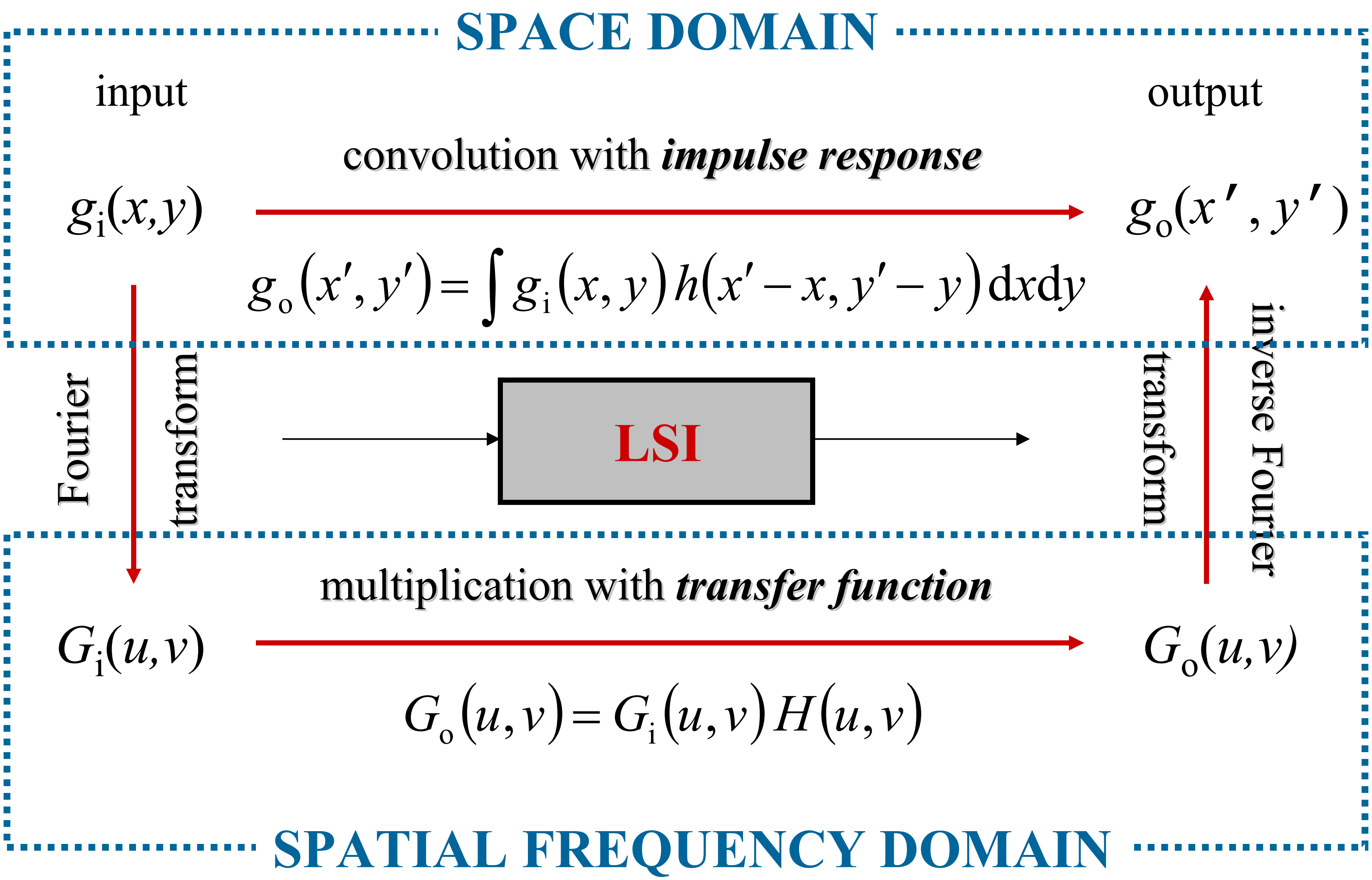
severe
low-pass filtering

Notice:
(i) blurring at the edges
(ii) ringing

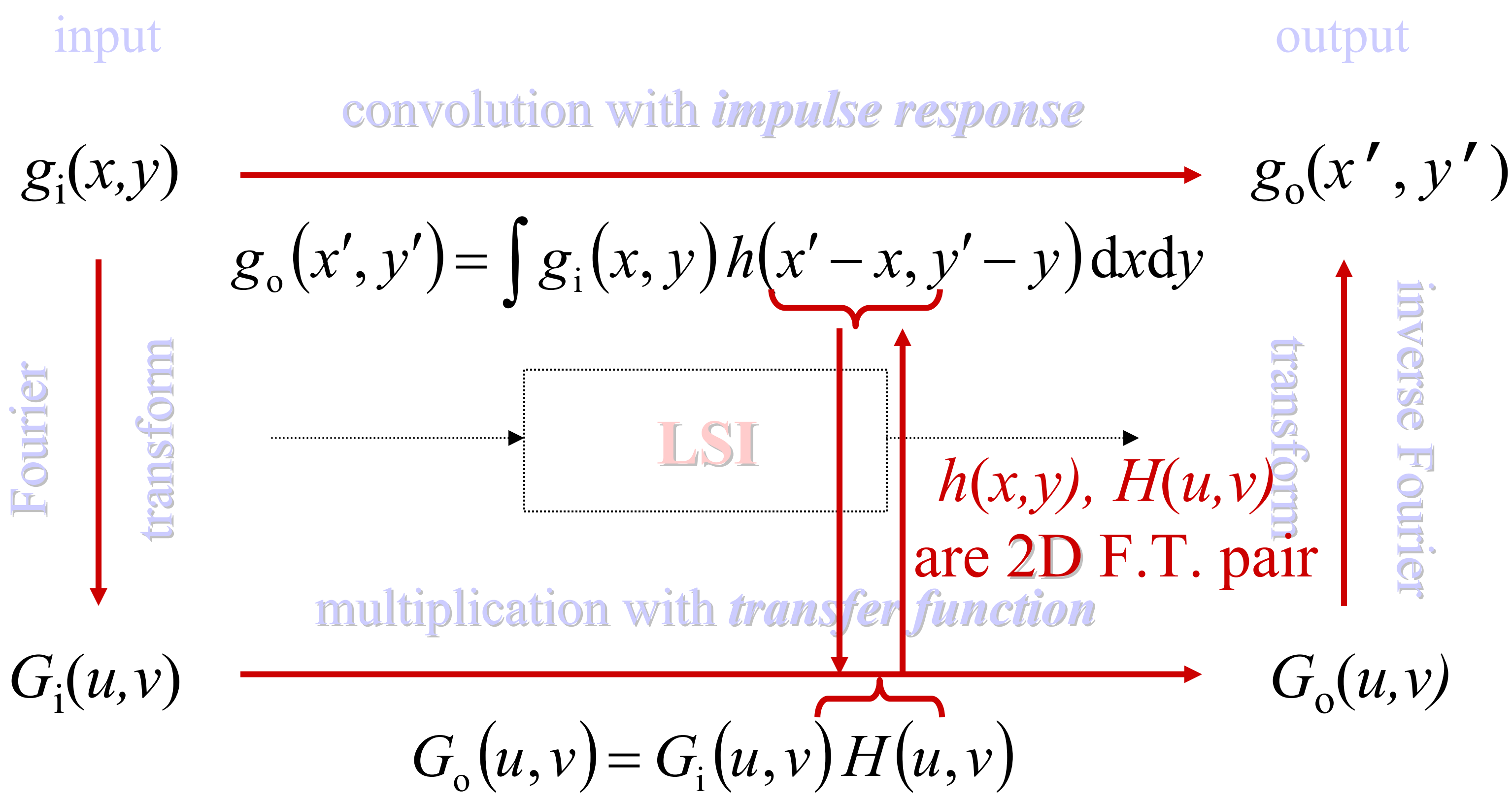
2D linear shift invariant systems



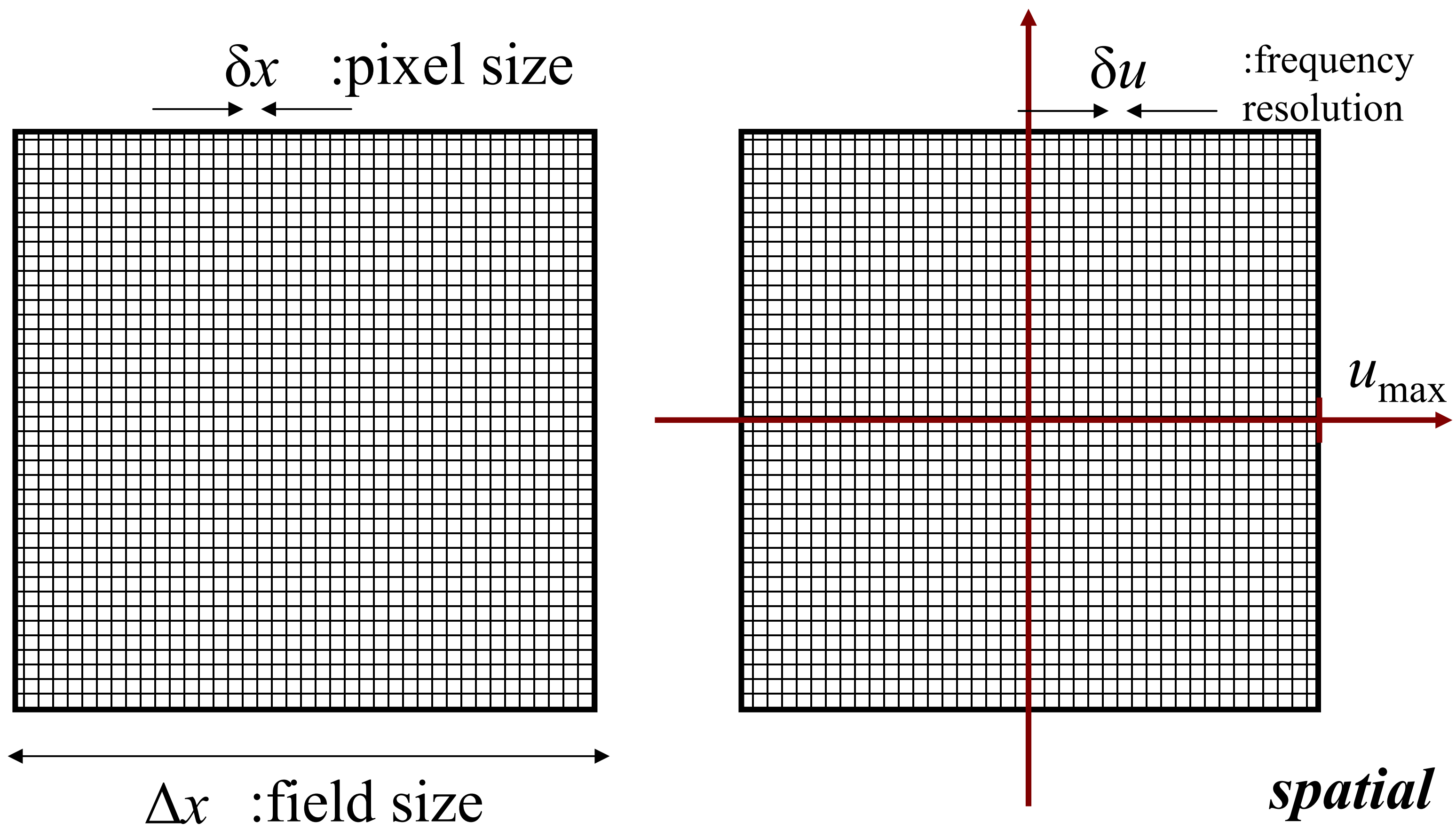
2D linear shift invariant systems



2D linear shift invariant systems



Sampling space *and* frequency



space domain

spatial frequency domain

Nyquist relationships:

$$u_{max} = \frac{1}{2\delta x} \qquad \delta u = \frac{1}{2\Delta x}$$

The Space–Bandwidth Product

Nyquist relationships:

from space $\rightarrow$ spatial frequency domain: $$u_{\max} = \frac{1}{2\delta x}$$

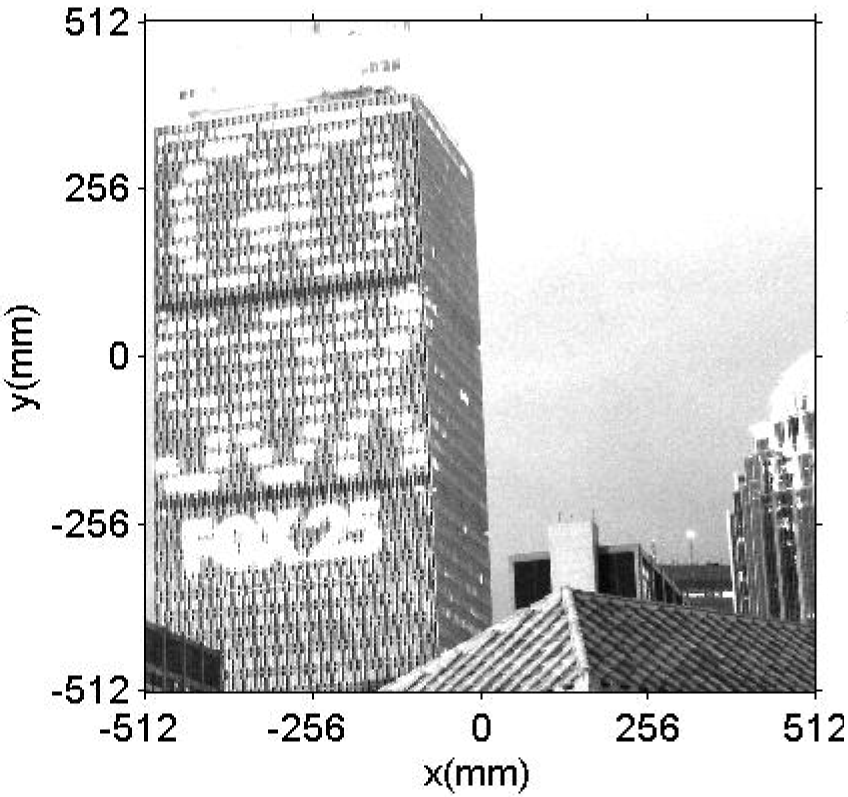
from spatial frequency $\rightarrow$ space domain: $$\frac{\Delta x}{2} = \frac{1}{2\delta u}$$

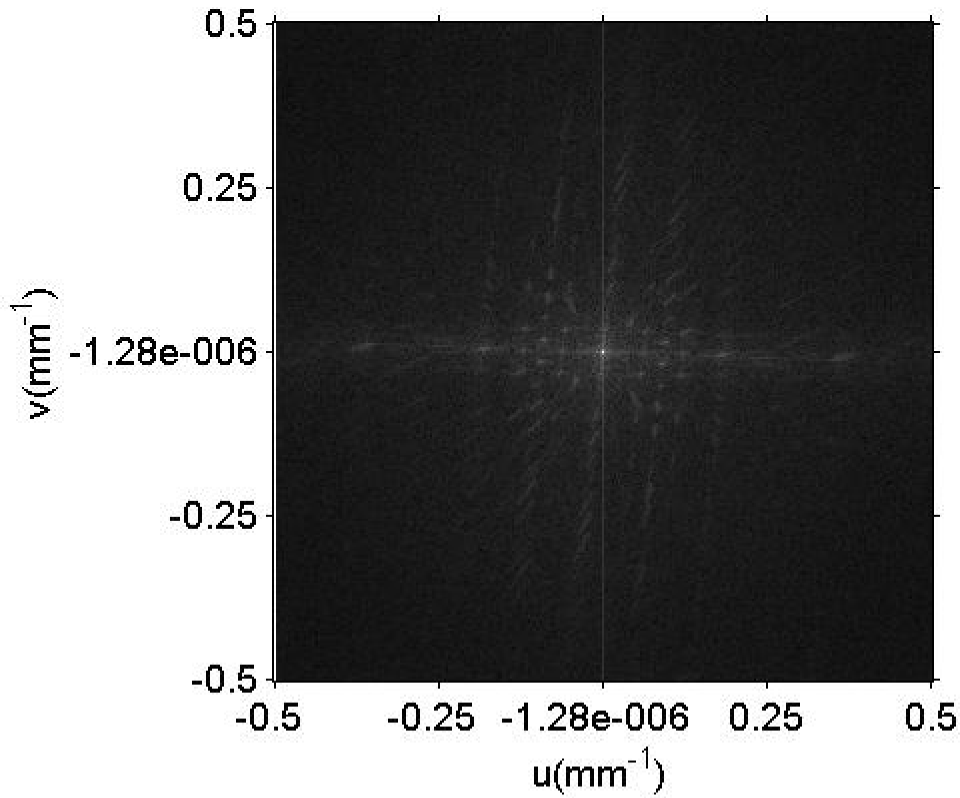
$$\frac{\Delta x}{\delta x} = \frac{2u_{\max}}{\delta u} \equiv N$$: 1D Space–Bandwidth Product (SBP)

aka number of pixels in the space domain

2D SBP ~ N^2

SBP: example





space domain

$\delta x = 1 \qquad \Delta x = 1024$

Fourier domain

(aka spatial frequency domain)

$u_{\max} = 0.5 \qquad \delta u = 1/1024 = 9.765625 \times 10^{-4}$