MASSACHUSETTS INSTITUTE OF TECHNOLOGY

2.71 Optics **Fall '05**

Problem Set #8 Posted Wednesday, Nov. 09, 2005 — Due Wednesday, Nov. 16, 2005

1. **Paserval's theorem.** Let $f(x)$ denote a square integrable and sufficiently smooth function and $F(u)$ its Fourier transform.

   1.a) Show that
   $$\int_{-\infty}^{\infty} |f(x)|^2 \, \mathrm{d}x = \int_{-\infty}^{\infty} |F(u)|^2 \, \mathrm{d}u$$

   1.b) Argue that this result expresses energy conservation in the context of an optical system.

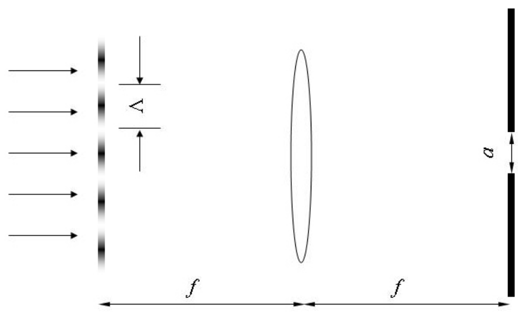
2. The Fourier transform may be regarded as a mapping of functions into their transforms and therefore satisfies the definition of a system.

   2.a) Is this system linear?

   2.b) Can you specify a transfer function for this system? If yes, what is it? If not, why not?

3. Compute the following convolution analytically in the *simplest* possible way:
   $$g\left(x', y'\right) = f\left(x, y\right) \star h\left(x, y\right),$$
   where
   $$f\left(x, y\right) = \frac{1}{2}\left\{1 + \cos\left(2\pi \frac{x}{\Lambda}\right)\right\},$$
   $$h\left(x, y\right) = \mathrm{sinc}\left(\frac{x}{x_0}\right).$$

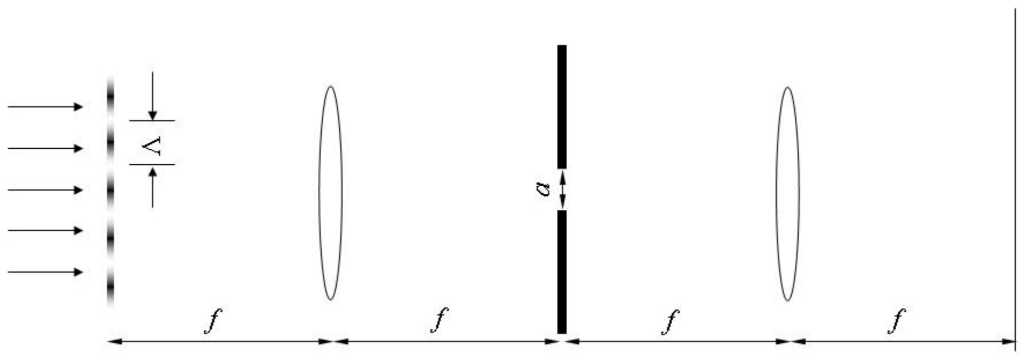
4. Consider the optical system drawn below, where a sinusoidal amplitude grating of perfect contrast and period Λ is placed at the front focal plane of a lens of focal length f. A rectangular aperture (slit) of size a is placed at the back focal plane of the lens.



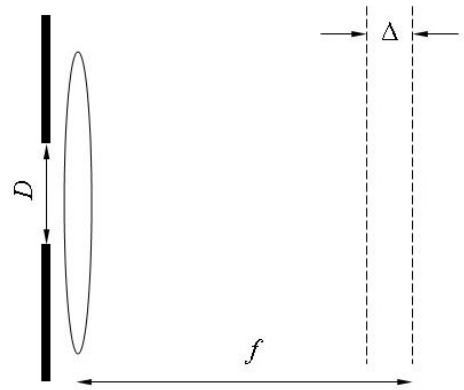
4.a) What is the field immediately past the aperture?

4.b) What is the relationship between the answer to this problem and the previous problem? Quantify.

5. Now consider the optical system drawn below, whose first part is identical to that of the previous problem, and an extra lens of focal length f has been added so that the aperture is located at this lens' front focal plane. What is the field observed at the back focal plane?



6. A unit-amplitude, normally incident monochromatic plane wave illuminates an object of maximum linear dimension D, situated immediately in front of a larger positive lens of focal length f, as shown below. Due to a positioning error, the intensity distribution is measured across a plane at a distance $d - \Delta$ behind the lens. How small must Δ be if the measured intensity distribution is to accurately represent the Fraunhofer diffraction pattern of the object?



7. **Image processing.** Download an image from your favorite image-intensive website (e.g., imdb.com), convert it to grayscale by adding the R, G, B color values at each pixel, and crop its central portion $g(x, y)$ so that it have square shape (e.g., 128×128.) Please do not use any images that might be considered offensive.

   7.a) Plot your image next to the amplitude $|G(u, v)|$ of its Fourier transform $G(u, v)$ (more details may be visible if instead you plot $\log_1 0|G(u, v)|$.)

   7.b) Select a 5×5 square region $\mathcal{S}$ around the origin of the Fourier transform domain and define the new function $G_1(u, v)$ such that

$$G_1(u, v) = \begin{cases} G(u, v) & \text{outside } \mathcal{S} \\ 0 & \text{inside } \mathcal{S} \end{cases}$$

   Plot the inverse Fourier transform $g_1(x, y)$ of $G_1(u, v)$.

   7.c) Define the new function $G_2(u, v)$ such that

$$G_2(u, v) = \begin{cases} G(u, v) & \text{inside } \mathcal{S} \\ 0 & \text{outside } \mathcal{S} \end{cases}$$

   Plot the inverse Fourier transform $g_2(x, y)$ of $G_2(u, v)$.

   7.d) Comment on the appearances of $g_1(x, y)$, $g_2(x, y)$ and how these appearances are affected by the size of the region $\mathcal{S}$.

   If you use MATLAB to solve this problem, you will find the following functions useful: (i) `fft2` computes the 2D Fourier transform of an image and returns it with some quadrants swapped, (ii) `fftshift` rearranges the quadrants of the Fourier transform in their proper order, (iii) `ifft2` computes the inverse 2D Fourier transform (iv) `imagesc; colormap gray` displays a real grayscale image, (v) `print -dps [filename]` prints a figure into a postscript file which you can then print at any Athena printer using the `lpr` command.