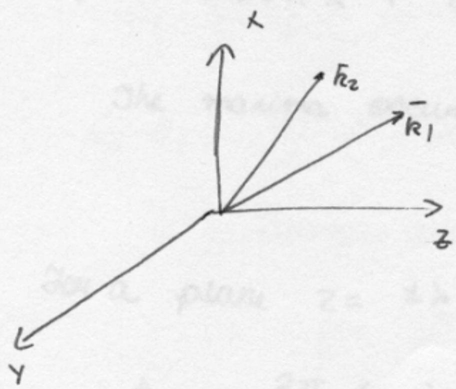
(2.71/2.710) Problem Set 7

Solutions

1.)



Lets call the waves (Ignore the $e^{-i\omega t}$ term)

$$E_1 = |E_1| e^{i \bar{k}_1 \cdot \bar{r}}$$

$$= |E_1| e^{i(k_1 \sin 30^\circ x + k_1 \cos 30^\circ z)}$$

$$E_2 = |E_2| e^{i \bar{k}_2 \cdot \bar{r}}$$

$$= |E_2| e^{i(k_2 \cos 45\, x + k_2 \sin 45 \sin 30\, y + k_2 \sin 45 \cos 30\, z)}$$

Consider any two waves $E_A = A e^{i\phi_A}$

$E_B = B e^{i\phi_B}$

The interference of these waves is given by

$$I = |E_A + E_B|^2 = A^2 + B^2 + 2AB \cos(\phi_A - \phi_B)$$

1) (a) xy plane $\Rightarrow z = 0$

Assume $|E_1| = |E_2| = 1$, $\lambda_1 = \lambda_2 \neq 1$.

$$\therefore \quad I = 2\left(1 + \cos(\phi_1 - \phi_2)\right)$$

$$\phi_1 = \frac{2\pi}{\lambda}\left(\sin 30^\circ x + \cancel{\cos 30 z}\right)$$

$$\phi_2 = \frac{2\pi}{\lambda}\left(\cos 45\, x + \sin 45 \sin 30\, y\right)$$

$$\phi_1 - \phi_2 = \frac{2\pi}{\lambda}\left\{ x(\sin 30 - \cos 45) - y \sin 45 \sin 30 \right\}$$

$\therefore$ The interference fringes are lines whose equations is

$$\phi = x\,(\sin 30 - \cancel{\cos}\,45) - y \sin 45 \sin 30$$

$$\therefore \phi = 0.2071x + 0.3536y$$

The maxima occur when $0.2071x + 0.3530y = m\lambda$
$(m = -1, 0, 1 \cdots)$

(b) For a plane $z = \lambda$,

$$\phi_1 = \frac{2\pi}{\lambda}\left(\sin 30^\circ x + \cos 30 \cdot \lambda\right)$$

$$\phi_2 = \frac{2\pi}{\lambda}\left(\cos 45 x + \sin 45 \sin 30 y + \sin 45 \cos 30 \lambda\right)$$

$$\phi_2 - \phi_1 = \frac{2\pi}{\lambda}\left\{x(\cos 45 - \sin 30) + y \sin 45 \sin 30 + \lambda(\sin 45 \cos 30 - \cos 30)\right\}$$

$$\therefore \phi_2 - \phi_1 = \frac{2\pi}{\lambda}\left\{0.2071x + 0.3530y - 0.2536\lambda\right\}$$

$\therefore$ The interference fringes are still lines that ~~to~~ are given by the same slope

$$\phi = 0.2071x + 0.3536y - 0.2536\lambda$$

However the maxima have shifted and occur when

$$0.2071x + 0.3536y = (m + 0.2536)\lambda$$
$(m = \ldots -1, 0, 1 \cdots)$

(c) On y-z, $x=0$

$$\phi_1 = \frac{2\pi}{\lambda}(\cos 30\, z)$$

$$\phi_2 = \frac{2\pi}{\lambda}(\sin 45 \sin 30 y + \sin 45 \cos 30\, z)$$

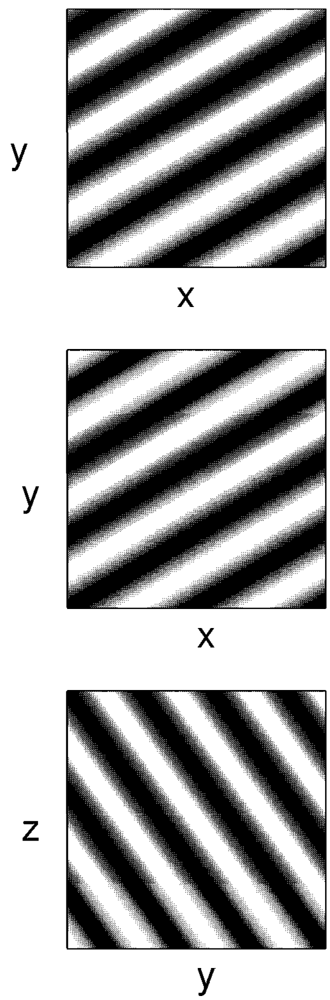
$$\phi_2 - \phi_1 = \frac{2\pi}{\lambda}(0.3536\, y - 0.2536\, z)$$

$\therefore$ Fringes are lines ~~are~~ that are determined by

$$\phi = 0.3536y - 0.2536\, z$$

The maxima occur @ $0.3536y - 0.2536\, z = m\lambda$

$(m = \cdots -1, 0, 1 \cdots)$


y
x
y
x
z
y

(2) Interference of on axis plane & spherical wave

$$E_p = |E_p| e^{\left(i\frac{2\pi}{\lambda}z\right)}$$

$$E_{sp} = |E_{sp}|\frac{1}{r} \quad e^{\left(\frac{i2\pi}{\lambda}z\right)} \quad e^{\frac{i\pi(x^2+y^2)}{\lambda z}}$$

↳ (Note : This is a paraxial approximation ~~to~~ for a spherical wave)

For $z = 1000\lambda$

assume $|E_p| = \frac{|E_{sp}|}{(1000\lambda)} = 1$

∴ we have ϕ_1 (similar to problem 1)

$$\phi_1 = \frac{2\pi}{\lambda}z$$

$$\phi_2 = \frac{2\pi}{\lambda}z + \frac{\pi(x^2+y^2)}{\lambda z}$$

$$\phi_2 - \phi_1 = \frac{\pi(x^2+y^2)}{\lambda z}$$

$$\phi = \frac{2\pi}{\lambda} \cdot \frac{(x^2+y^2)}{2z}$$

∴ The locus of the fringes is now given by

$\frac{x^2+y^2}{2z} = c$ i.e it ~~is a circle~~ consists of concentric rings.

The location of maxima are at

$$\frac{x^2+y^2}{2\times1000\lambda} = m\lambda$$

$$\therefore \quad x^2+y^2 = 2000\lambda^2 m$$

∴ The maxima are circles with radii $r = \lambda\sqrt{2000m}$
$(m = 0, 1, \cdots)$

for $d = 2000\lambda$, $|E_p| = 1$ But $|E_{sp}| = \frac{1}{2}$

$$I = 1 + \frac{1}{4} + 2 \times 1 \times \frac{1}{2} \cos(\phi_2 - \phi_1) = \frac{5}{4} + \cos(\phi_2 - \phi_1)$$
$$= \frac{5}{4}\left(1 + \frac{4}{5}\cos(\phi_2 - \phi_1)\right)$$

ie the contrast is reduced!

The fringes are still concentric rings given this time by

$$\frac{x^2+y^2}{2 \times 2000\lambda} = C$$

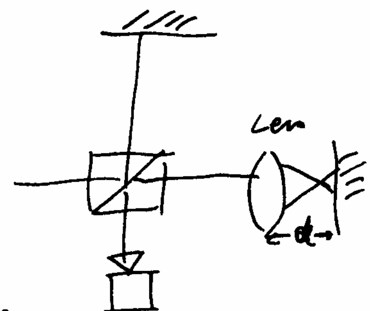
The maxima occur at

$$x^2+y^2 = m\lambda \times 4000\lambda$$

The maxima are circles with radii

$$r = 2000\lambda\sqrt{m} \quad (m = 0, 1, 2 \dots)$$

(c) We observe rings of ~~ever~~ INCREASING radii and DECRESING contrast. This is consistent with a diverging spherical wave

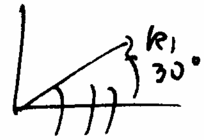


) Consider a Michelson with a lens.

Only if $d = f$ will we get 2 plane waves interfering.

In all other cases ie when $d \neq f$. we will see rings like the ones we have discussed. This is on account of the spherical wave that arises from the lens arm of the Michelson ~~as it is~~ on account of the lens's focussing action.

(3.) Off axis plane wave $\longrightarrow$



$$E_p = |E_p|\, e^{i\frac{2\pi}{\lambda}(z\cos 30 + x\sin 30)}$$

$$E_{sp} = \frac{|E_{sp}|}{\alpha}\, e^{i\frac{2\pi}{\lambda}z} \quad e^{i\pi\frac{x^2+y^2}{\lambda z}}$$

Assume $\quad \dfrac{E_{sp}}{1000\lambda} =$

$$\phi_1 = \frac{2\pi}{\lambda}(z\cos 30 + x\sin 30)$$

$$\phi_2 = \frac{2\pi}{\lambda}z + \pi\frac{x^2+y^2}{\lambda z}$$

$$\phi_2 - \phi_1 = \frac{2\pi}{\lambda}z(1-\cos 30) + \frac{2\pi(x^2+y^2)}{2\lambda z} + \frac{2\pi}{\lambda}(x\sin 30)$$

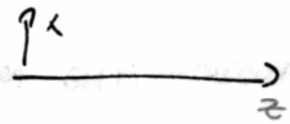
$$\frac{2\pi}{\lambda}\left\{ z(1-\cos 30) + \frac{1}{2z}(x^2+y^2) + x\sin 30 \right\}$$

$$\phi = z(1-\cos 30) + \frac{1}{2z}(x^2+y^2) + x\sin 30 \quad \text{— (1)}$$ is the locus of the fringes.

Notice that the fringes are no longer concentric circular rings. Additionally, the longitudinal distance z now plays a part in determining the fringe pattern as well (Contrast this with the earlier case where z only determined the radii of the rings and not their shapes!)

For $d = 2000\lambda$, the fringe pattern is still given by eqn. (1). However, as noted earlier, there is a contrast reduction in the fringes

(4) Counter propagating waves $\rightarrow$



$$E \quad 1 e^{i(kz \quad \omega t)}$$

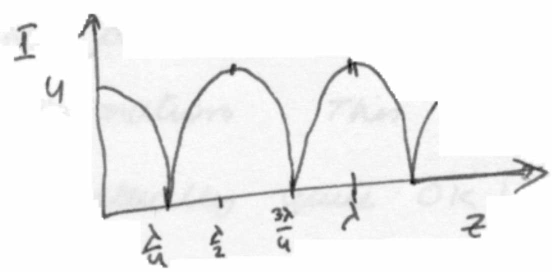
$$E \quad 1 e^{i(kz \quad \omega t)}$$

$$I \quad |E_+ + E_-|^2$$

$$= 2(\quad + \cos(\phi_+ \quad \phi_-))$$

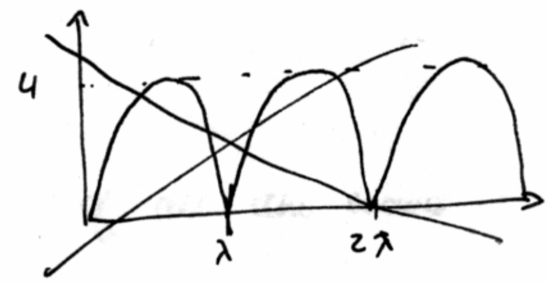
$$\phi_+ \quad kz - \omega t$$

$$\phi \quad -kz - \omega t$$



$$\phi_+ \quad \phi_- = 2kz$$

$$I \quad 2\{1 + \cos(2kz)\}$$



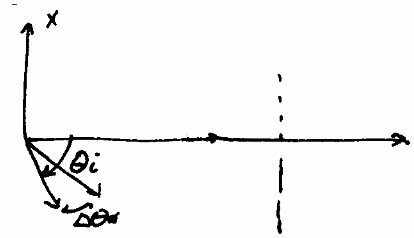
$$= 2 \times 2 \cos^2(kz)$$

$$4\cos^2(kz)$$

We see that the interference pattern results a wave that is truly stationary i.e it does not change with time even though it has a form similar to a travelling wave

This is why it is called standing wave

(5) Fan of 'N' plane waves propagating symmetrically



We need to assume that the plane waves are consistent with the paraxial approximation. This may not sound reasonable but it is actually quite OK!

For instance if $N = 30$, $\Delta\theta = 1°$, then $\theta_{fan} = 30° \approx \frac{\pi}{6}$ rad

and $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2} \approx \frac{\pi}{6}$!

The interference pattern is the sum of all the waves

The m^{th} wave is

$$E_m = \exp\left\{ik\left(\cos(\theta_i + m\Delta\theta)\, z + \sin(\theta_i + m\Delta\theta)\, x\right)\right\}$$

Since $\theta_i, \Delta\theta \to 0$,

$\cos(\theta_i + m\Delta\theta) \to 1$

$$\sin(\theta_i + m\Delta\theta) = \sin\theta_i \cos(m\Delta\theta) + \cos\theta_i \sin(m\Delta\theta)$$
$$\approx \theta_i + m\Delta\theta$$

$$\therefore E_m \approx \exp\left\{i\frac{2\pi}{\lambda}\left(z + (\theta_i + m\Delta\theta)x\right)\right\}$$

Summing up,

$$E_{field} = \sum_{m=0}^{N-1} E_m$$

$$E = e^{\left(i\frac{2\pi}{\lambda}(z+\theta_i x)\right)} \sum_{m=0}^{N-1} \exp\left\{ i\frac{2\pi}{\lambda} m \,\Delta\theta\, x \right\}$$

↳ Recognize that this is a geometric series

$$\therefore E = e^{\left(i\frac{2\pi}{\lambda}(z+\theta_i x)\right)} \frac{1 - \exp\left(i\frac{2\pi}{\lambda} N\Delta\theta x\right)}{1 - \exp\left(i\frac{2\pi}{\lambda}\Delta\theta x\right)}$$

Lets say $\frac{2\pi}{\lambda} N\Delta\theta x = \phi_1$ $\qquad$ $\frac{2\pi}{\lambda}\Delta\theta x = \phi_2$

$$\therefore E_1 = e^{\left(i\frac{2\pi}{\lambda}(z+\theta_i x)\right)} \frac{1-e^{i\phi_1}}{1-e^{i\phi_2}}$$

$$= e^{\left(i\frac{2\pi}{\lambda}(z+\theta_i x)\right)} \frac{e^{i\phi_1/2}}{e^{i\phi_2/2}} \times \frac{e^{-i\phi_1/2} - e^{i\phi_1/2}}{e^{-i\phi_2/2} - e^{i\phi_2/2}}$$

$\to -\left(\sin\frac{\phi_1}{2}\right) 2i$

$\to -(\sin \phi_2$

$$|E_1|^2 = \frac{\sin^2(\phi_1/2)}{\sin^2(\phi_2/2)}$$

$$= \frac{\sin^2\left(\frac{N\pi}{\lambda}\Delta\theta x\right)}{\sin^2\left(\frac{\pi}{\lambda}\Delta\theta x\right)}$$

The interference pattern thus looks like a series of spikes (These are also referred to as orders of diffraction)