2.710: Solutions to Home work 1

Problem 1: Optics Buzzwords

You will get full credit for this problem if your comments about the topic indicate a certain depth of understanding about what you have written and that you have done some research.

Problem 2: The lifeguard

This problem essentially asks you to derive Snell's law using the analogy of the lifeguard. The lifeguard must get to the swimmer as quickly as possible in order to prevent him/her from drowning. The lifeguard has different speeds running on the sand and swimming in the water. We seek to find the optimal path (i.e. the direction and distance that the lifeguard must travel on sand and water) that minimizes the time required to reach the swimmer.

Similarly, Fermat's principle tells us that a photon seeks to minimize the time of travel between two points. For the simple case of light travelling from one medium to another, the problem is exactly the same as the lifeguard problem. We are interested in the conditions that describe the path that minimizes the time of travel and not surprisingly, we find the condition to be Snell's law.

We have to plan the lifeguard's path that reaches the swimmer in minimum time (Figure 1).

Running speed on sand = c.

Swimming speed = c/n.

$$\text{Velocity(v)} = \frac{\text{distance(x)}}{\text{time(t)}}.$$

$$\Rightarrow t = \frac{x}{v}.$$

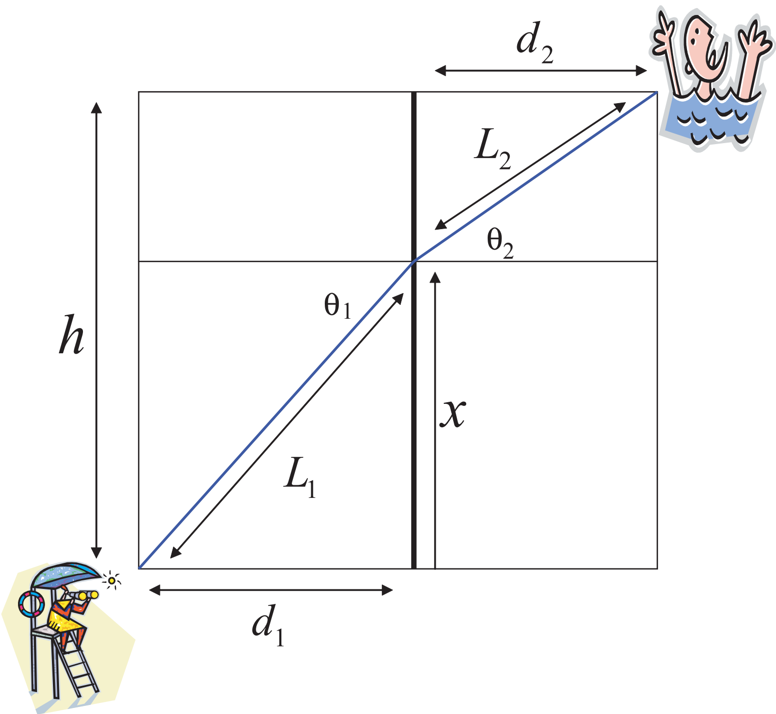

Figure 1: The lifeguard problem.

Look at Figure 1 and realize that

$$L_1 = \sqrt{x^2 + d_1^2} \qquad \text{and}$$

$$L_2 = \sqrt{(h-x)^2 + d_2^2}$$

The total time of flight t is then

$$t = \frac{1}{c}(L_1 + nL_2)$$

We seek to find the condition that would minimize the time of flight by differentiating the above equation with respect to x and setting the derivative to zero.

$$\frac{\partial t}{\partial x} = \frac{1}{c}(\frac{\partial L_1}{\partial x} + n\frac{\partial L_2}{\partial x}) = 0$$

$$\Rightarrow \frac{1}{c}\left(\frac{x}{\sqrt{x^2 + d_1^2}} - \frac{n(h-x)}{\sqrt{(h-x)^2 + d_2^2}}\right) = 0$$

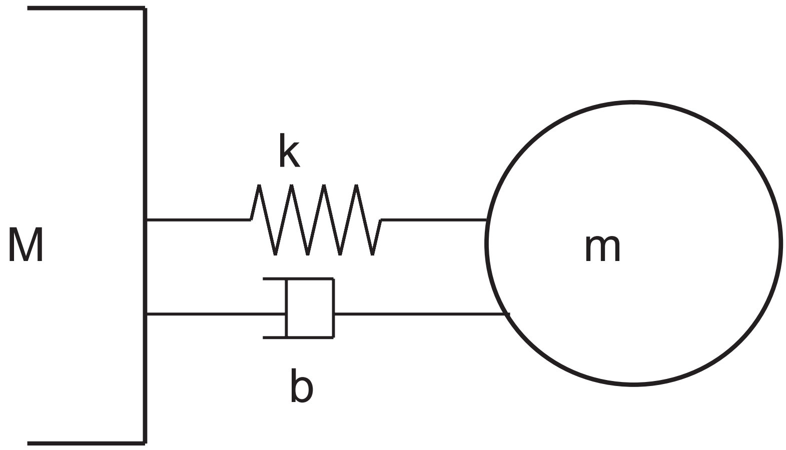

Figure 2: 1 DOF model of Nitrogen atom.

$$\Rightarrow \sin\theta_1 = n\sin\theta_2.$$

which is Snell's law!

Problem 3: Why is the sky blue?

This problem seeks to help you recollect some of the linear systems theory that will be useful in this class. It is also instructive to understand the dynamics of why the sky is blue. To solve this problem, you need to formulate a simple 1 Degree of freedom (DOF) model for the Nitrogen atom. We will find the response of the Nitrogen atom subjected to a periodic excitation by a photon and then apply Rayleigh's scattering assumption to see why the sky is blue.

Part 3a

The model for a Nitrogen atom is shown in Figure 2. The mass of the electron is 'm', 'k' and 'b' model the stiffness and damping between the massive nucleus ($M \gg m$) and the light electron. The equation of motion for the electron subjected to a force 'F' is

$$m\ddot{x} + b\dot{x} + kx = F.$$

Taking the Laplace transform of this equation we get the transfer function ,

$$\frac{X}{F} = \frac{1}{ms^2 + bs + k}.$$

Applying Rayleigh's approximation,

$$|A(\omega)|^2 = \left| \frac{s^2 X}{F} \right|^2.$$

The extra s^2 in the numerator is on account of the fact that the scattering is proportional to the acceleration of the particle. The driving force is assumed to be sinusoidal. Recall that the Fourier transform of a linear system tells us about its response to sinusoidal inputs. Hence, we need to find the transfer function in the Fourier domain. This is obtained by replacing s by $i\omega$,

$$|A(\omega)|^2 = \left| \frac{-\omega^2}{(k - m\omega^2) + i\omega b} \right|^2.$$

$$\Rightarrow |A(\omega)|^2 = \frac{\omega^4}{(k - m\omega^2)^2 + (\omega b)^2}.$$

Part 3b

For a simple second order system, recall that

$$\text{Natural frequency,} \quad \omega_n = \sqrt{\frac{k}{m}}$$

$$\text{Damping ratio,} \quad \zeta = \frac{b}{2\sqrt{mk}}$$

$$\text{Resonance frequency,} \quad \omega_r = \omega_n \sqrt{1 - \zeta^2}$$

This is not a simple second order system on account of the $-\omega^2$ term in the numerator. You can find that the maximum of the transfer occurs for

$$\omega_r = \sqrt{\frac{2k^2}{2mk - b^2}}.$$

However, for $b = 0$,

$$\omega_r = \sqrt{\frac{k}{m}}.$$

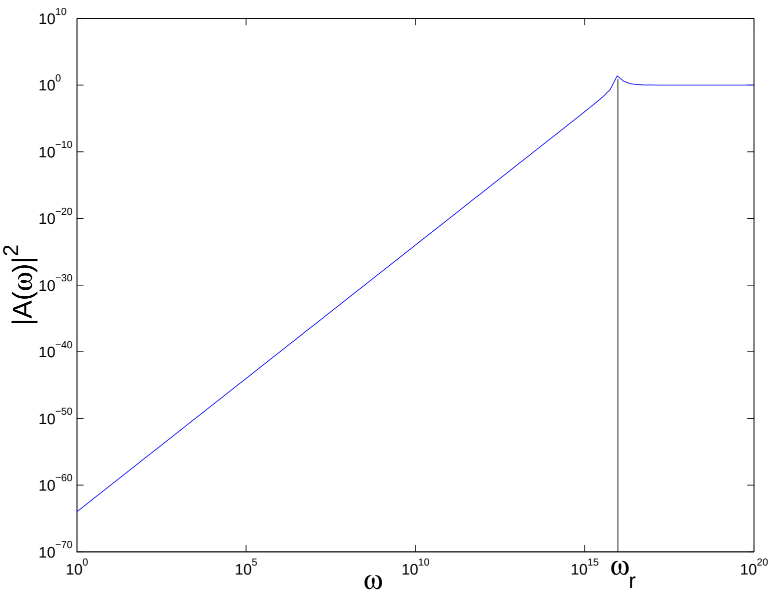

Figure 3: Transfer function $|A(\omega)|^2$.

So the resonance frequency is equal to the natural frequency if $\zeta \approx 0$. The transfer function is shown in Figure 3

Part 3c

Here, we assume that the damping ratio is negligible in the Nitrogen atom (This is justified because typically dissipation within atoms is usually very small). In this case,

$$\omega_r \approx \omega_n = \sqrt{\frac{140}{9.31 \times 10^{-31}}} = 1.24 \times 10^{16}\text{rad/sec}$$

In the visible spectrum, violet has the highest frequency $\omega_{\text{violet}} = 4.71 \times 10^{15}$ rad/sec. Thus, the resonance frequency of the Nitrogen atom is higher than all optical frequencies within the visible spectrum of light.

Part 3d

Assuming zero dissipation, the transfer function of the system is

$$|A(\omega)|^2 = \frac{\omega^4}{(k - m\omega^2)^2},$$

$$\Rightarrow |A(\omega)|^2 = c \times \frac{\omega^4}{(1 - (\omega/\omega_r)^2)^2}.$$

where c is a constant. If $\omega \ll \omega_r$, the denominator reduces to 1 and the resulting expression is

$$|A(\omega)|^2 = c \times \omega^4.$$

Thus we see that in the visible range, higher frequencies are scattered more strongly than the lower frequencies. As a result of greater scattering, the blue end of the spectrum is what is visible most when we look at the sky.