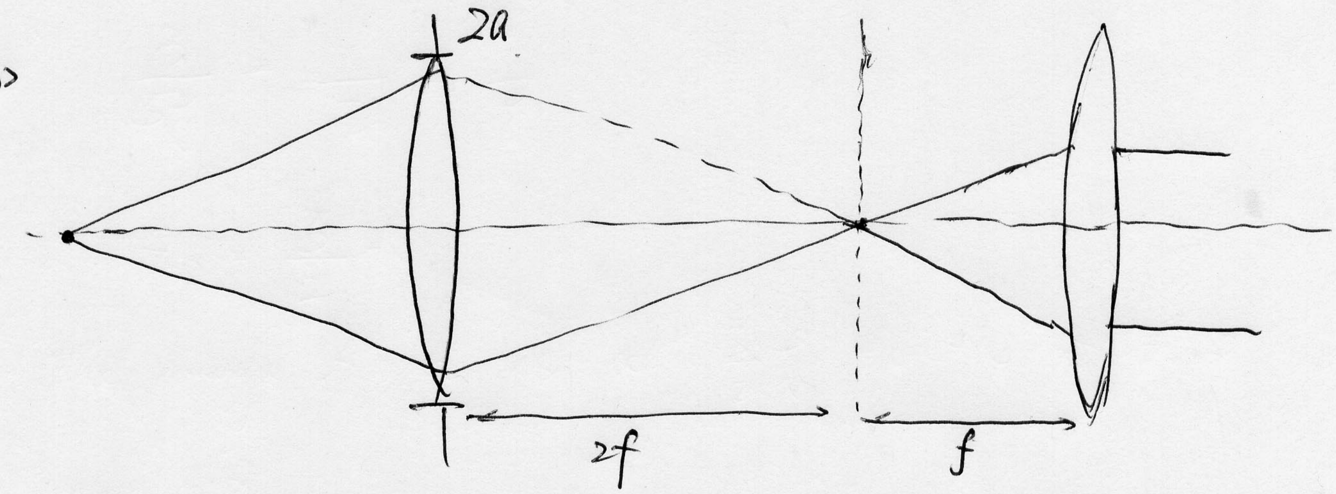
3.

a) Using Lens law cascadedly, the image will be at infinity.

b)



Inorder to obtain the field at 2f to the right of L1, we can think that the system of Lens L1 is illuminated by a point souce at 2f to the left of Lens L1, while the object (tranparency) is the aperture of 2a (the dimension of Lens L1) and the lens is infintely large.

From (5-57) in Goodman, the field at 2f to the right of L1 is

$$U_2(x) = \mathcal{F}\left\{ rect\left(\frac{x'}{2a}\right)\right\}\Big|_{u=\frac{x}{2\lambda f}}$$

$$\left(\frac{z_1}{z_2(z_1-d)} = \frac{2f}{2f(2f-0)} = \frac{1}{2f} \right)$$

So $U_2(x) \propto \mathrm{sinc}\left(\frac{a x}{\lambda f}\right)$

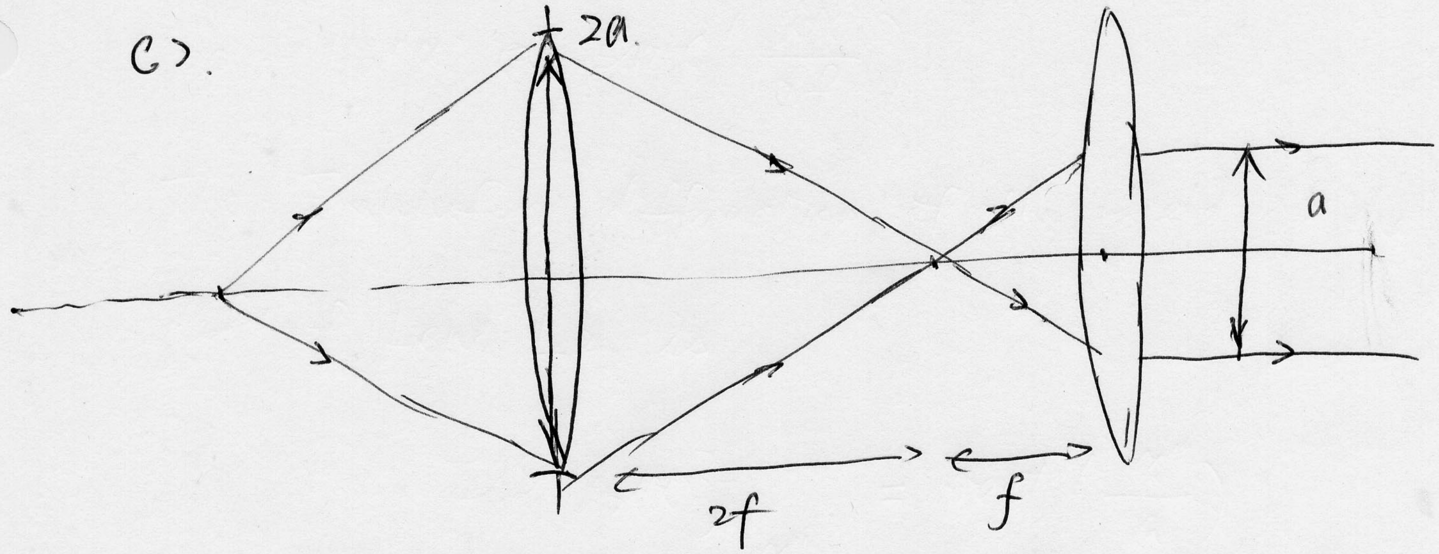
The Frauhofer diffraction of the field to the left of Lens L2 is

$$\mathcal{F}\{U_2(x)\}\Big|_{\frac{x''}{\lambda f}} = \mathrm{rect}\left(\frac{x''}{a}\right)$$

We can think that we have a transparency with function of $U_2(x)$ at f to the left of Lens L2 and use a plane wave to illuminate it, We can get it's Frauhofer diffraction at f to the right of Lens L2.

So what we get is a truncated plane wave with width of d.

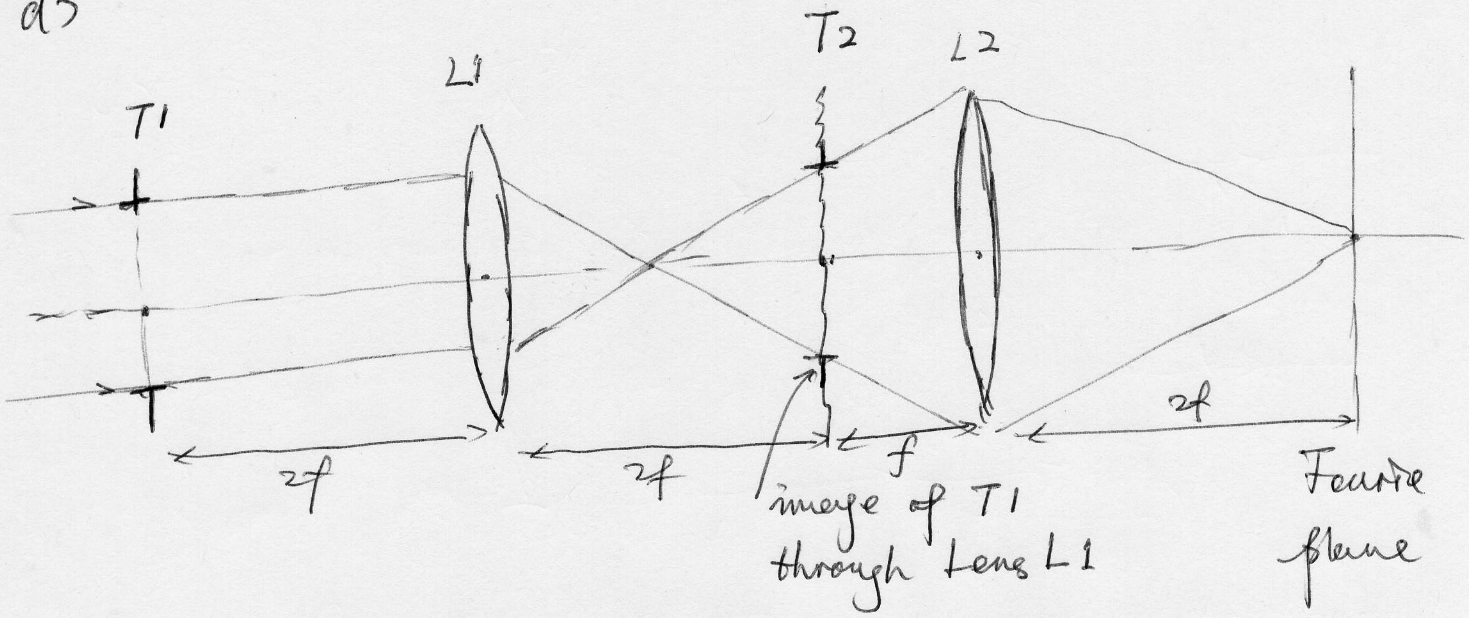
(c).



From Geometrical Optics, we know that Lens L1 define the aperture of the system. We can get the width of the output plane wave easily from the plot above:

$$\frac{f}{2f} \cdot 2a = a$$

d>



First, without considering T1 and T2, we can find the Fourier plane (the image of the illumination source which is a plane wave for this case) at 2f to the right of L2.

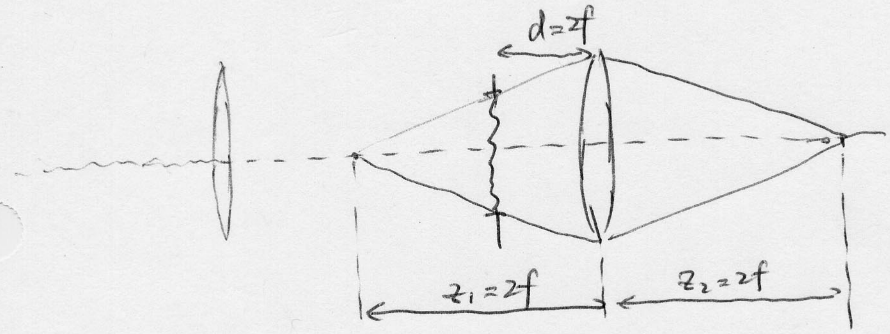
~~then we can predict we shall see Fourier transform of the [illegible]~~

Then the image of the aperture T1 through Lens L1 is exactly at the same place as the transparency T2. The two objects can be combined by multiplying them together.

New we can predict that at distance $2f$ to the right of Lens L2, we will see the Fourier transform of the product of T1 and T2.

If we assumed T2 with the function of $f(x)$, at the distance $2f$ to the right of L2, the field is

$$U(x') = \mathcal{F}\left\{ \mathrm{rect}\left(\frac{x}{w}\right) \cdot f(x) \right\}\Bigg|_{u=\frac{x'}{\lambda f}}$$

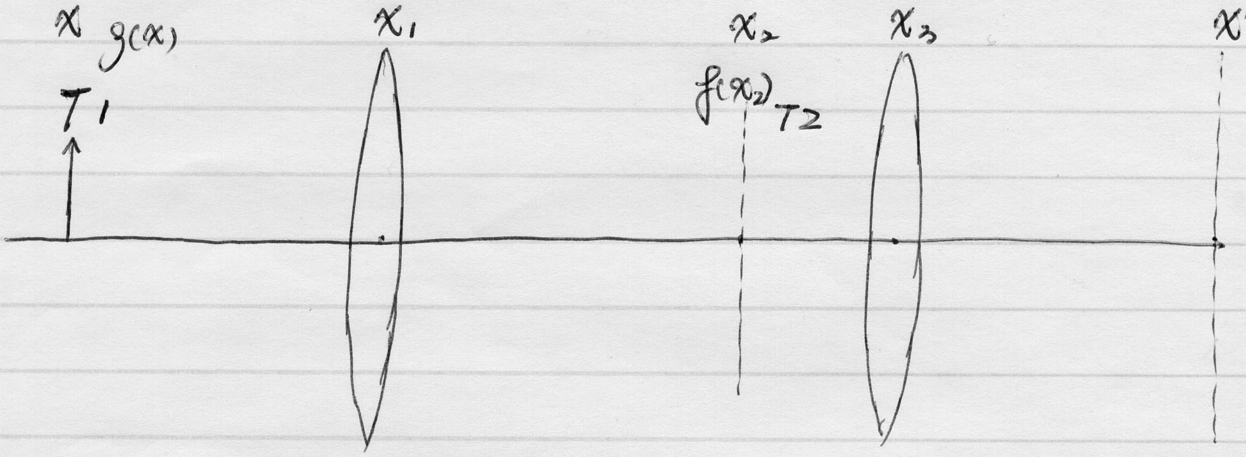


$$\frac{z_1}{z_2(z_1-d)} = \frac{1}{f}$$

We can also obtain the same results from cascade derivation which is given in the next two pages.

Object T1 : $g(x)$

Transparency T2 : $f(x_2)$



$$\iiiint g(x)\cdot \exp\left\{j\frac{\pi}{\lambda}\cdot\frac{(x_1-x)^2}{2f}\right\} dx \cdot \exp\left\{-j\frac{\pi}{\lambda}\cdot\frac{x_1^2}{f}\right\}$$

$$\exp\left\{j\frac{\pi}{\lambda}\cdot\frac{(x_2-x_1)^2}{2f}\right\}\cdot dx_1 \cdot f(x_2)\cdot \exp\left\{j\frac{\pi}{\lambda}\cdot\frac{(x_3-x_2)^2}{f}\right\} dx_2$$

$$\cdot \exp\left\{-j\frac{\pi}{\lambda}\cdot\frac{x_3^2}{f}\right\}\cdot \exp\left\{j\frac{\pi}{\lambda}\cdot\frac{(x'-x_3)^2}{z}\right\} dx_3$$

$$= \iiiint g(x)\cdot f(x_2)\exp\left\{j\frac{\pi}{\lambda}\left[\frac{\cancel{x_1^2}-2xx_1+x^2}{2f}-\frac{\cancel{x_1^2}}{f}+\frac{x_2^2-2x_2x_1+\cancel{x_1^2}}{2f}\right.\right.$$

$$\left.\left.+\frac{\cancel{x_3^2}+x_2^2-2x_3x_2}{f}-\frac{\cancel{x_3^2}}{f}+\frac{x'^2+x_3^2-2x'x_3}{z}\right]\right\} dx\, dx_1\, dx_2\, dx_3$$

$$= \iiiint g(x)\, f(x_2)\ \exp\left\{-j\frac{\pi}{\lambda}\frac{(x+x_2)}{f}x_1\right\}\exp\left\{j\frac{\pi}{\lambda}\left[\frac{x^2}{2f}+\frac{x_2^2}{2f}+\frac{x_2^2}{f}\right.\right.$$

$$\left.\left.-\frac{2x_3x_2}{f}+\frac{x'^2+x_3^2-2x'x_3}{z}\right]\right\} dx_1\ dx\, dx_2\, dx_3$$

$$= \iiint g(x)\, f(x_2) \cdot \delta(x+x_2) \cdot \exp\left\{ j\frac{\pi}{\lambda}\left[\frac{x^2}{2f} + \frac{3}{2f}x_2^2 - \frac{2x_3x_2}{f} + \right.\right.$$

$$\left.\left. \frac{x'^2 + x_3^2 - 2x'x_3}{z} \right]\right\} dx\, dx_2\, dx_3$$

$$= \iint f(x_2)\, g(-x_2)\, \exp\left\{ j\frac{\pi}{\lambda}\left[\frac{2x_2^2}{f} - \frac{2x_3x_2}{f} + \frac{x'^2 + x_3^2 - 2x'x_3}{z} \right]\right\} dx_2\, dx_3$$

$$= \iint f(x_2)\, g(-x_2) \cdot \exp\left\{ j\frac{\pi}{\lambda}\, \frac{2x_2^2}{f} \right\} \exp\left\{ j\frac{\pi}{\lambda} \cdot \frac{x'^2}{z} \right\}$$

$$\exp\left\{ j\frac{\pi}{\lambda} \cdot \left[\frac{x_3^2}{z} - \left(\frac{2x_2}{f} + \frac{2x'}{z} \right) x_3 \right]\right\} dx_3\, dx_2$$

$$= \int f(x_2)\, g(-x_2) \cdot \exp\left\{ j\frac{\pi}{\lambda} \cdot \frac{2x_2^2}{f} \right\} \exp\left\{ j\frac{\pi}{\lambda}\, \frac{x'^2}{z} \right\}$$

$$\exp\left\{ -j\frac{\pi}{\lambda} \cdot z \cdot \left(\frac{x_2}{f} + \frac{x'}{z} \right)^2 \right\} dx_2$$

$$= \int f(x_2)\, g(-x_2)\, \exp\left\{ j\frac{\pi}{\lambda} \left(\frac{2}{f} - \frac{z}{f^2} \right) x_2^2 \right\} \exp\left\{ -j\frac{2\pi}{\lambda} \cdot \frac{x'}{f} \cdot x_2 \right\} dx_2$$

So if $z = 2f$.

$$= \int f(x_2)\, g(-x_2)\, \exp\left\{ -j\frac{2\pi}{\lambda} \cdot \frac{x'}{f} \cdot x_2 \right\} dx_2$$

c) The transparency can be expressed as

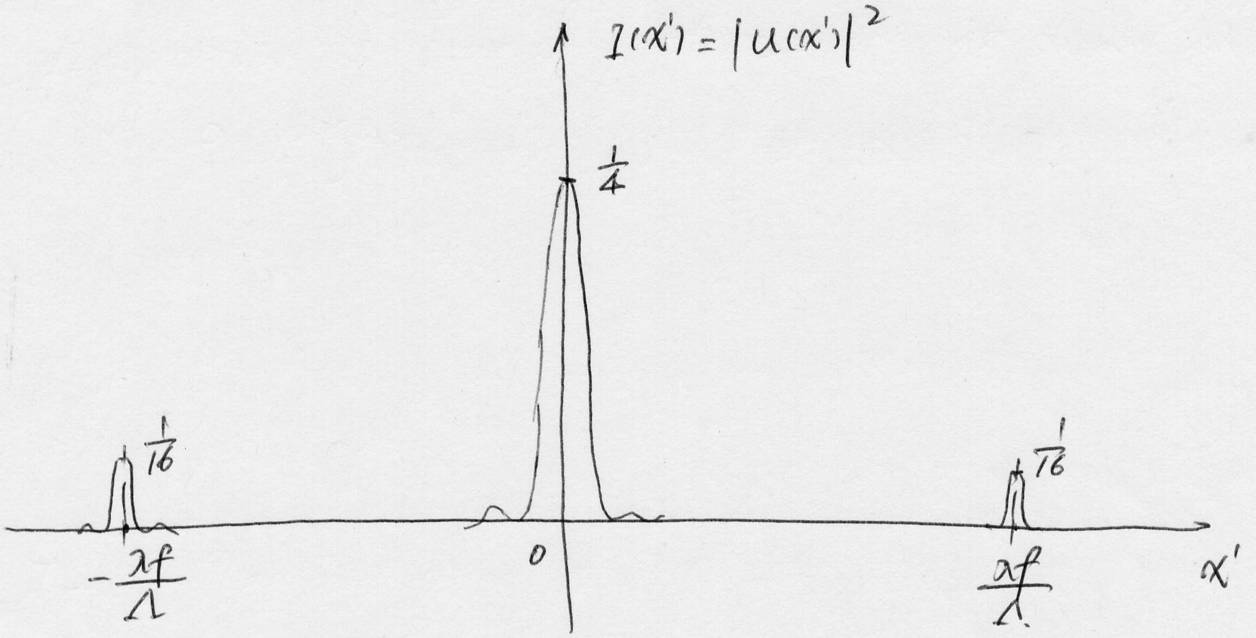
$$f(x) = \frac{1}{2}\left(1 + \cos\left(2\pi \frac{x}{\Lambda}\right)\right)$$

The aperture is $\mathrm{rect}\left(\frac{x}{w}\right)$

$$U(x') = \mathcal{F}\left\{ f(x) \cdot \mathrm{rect}\left(\frac{x}{w}\right)\right\}\Big|_{u=\frac{x'}{\lambda f}}$$

$$= \frac{1}{2}\,\mathrm{sinc}\left(\frac{w x'}{\lambda f}\right) + \frac{1}{4}\,\mathrm{sinc}\left[w\left(\frac{x'}{\lambda f} - \frac{1}{\Lambda}\right)\right]$$

$$+ \frac{1}{4}\,\mathrm{sinc}\left[w\left(\frac{x'}{\lambda f} + \frac{1}{\Lambda}\right)\right]$$



Problem #2 Solution.

We know for a general transparency.

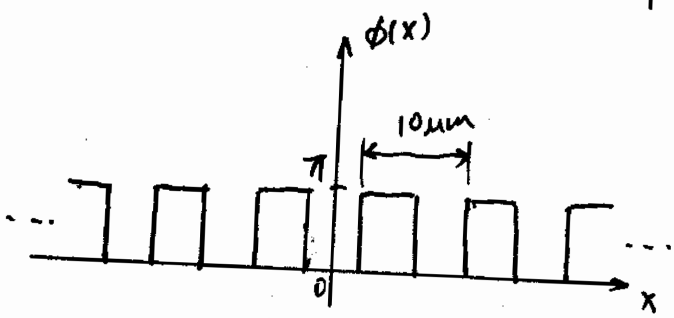
$$t(x) = \underbrace{A(x)}_{\text{"attenuation"}}\, e^{\,i\overbrace{\phi(x)}^{\text{"phase delay"}}}$$

Here for the glass plate object. $A(x) = 1$.

The phase delay between the peaks and grooves of the glass is.

$$2\pi \cdot \frac{(n-1)\cdot 1\,\mu m}{\lambda} = 2\pi \cdot \frac{0.5\cdot 1}{1} = \pi .$$

If we ~~let~~ use the peaks as the reference. the $\phi(x)$ is plotted as.



Note: here we change the phase $-\pi$ to π, because they are equivalent.

so the transparancy function is the same as the one shown in class

(Course Note: WK 13-b. page 13).

$$t(x) = \sum_{\substack{n=-\infty \\ n\neq 0}}^{+\infty} \text{sinc}\left(\frac{n}{2}\right) \exp\left(i\, 2\pi \frac{nx}{\Lambda}\right) \qquad \text{here } \Lambda = 10\,\mu m .$$

And we know physically it means after the transparancy, it represents multiple plane waves diffracts at the directions of $\sin\theta_n = \frac{n\lambda}{\Lambda}$

2.a).

As $t(x) = e^{j\phi(x)}$.

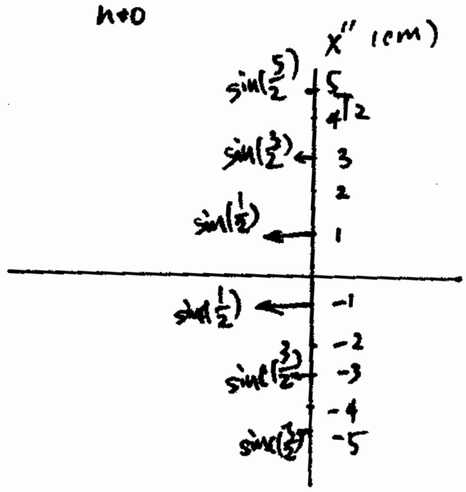
Intensity after T_1 is

$|t(x)|^2 = 1$.

2.b)

T_2 is at the Fourier Transform plane of T_1, so the field is

$$t'(x'') = \sum_{\substack{n=-\infty \\ n\neq 0}}^{+\infty} \text{sinc}\left(\frac{n}{2}\right) \delta\left(x'' - \frac{n}{\Lambda}\lambda f_1\right)$$



$$\frac{\lambda f_1}{\Lambda} = \frac{1 \cdot 100 \times 1000}{10} = 10^4\ \mu m$$
$$= 10\ mm.$$
$$= 1\ cm$$

T_2 has an open aperture of 7 cm, so only the ± 1, ± 3 orders can pass through T_2, and the ± 1 order get an additional

$$\frac{(n-1)\cdot 0.5}{1} \times 2\pi = \frac{\pi}{2} \quad \text{phase delay.}$$

so the field after T_2 is.

$$t''(x'') = j\,\text{sinc}\left(\tfrac{1}{2}\right)\left[\delta(x'' - 1cm) + \delta(x'' + 1cm)\right]$$
$$+ \text{sinc}\left(\tfrac{3}{2}\right)\left[\delta(x'' + 3cm) + \delta(x'' - 3cm)\right]$$

so at the observation plane, the field is the Fourier Transform of $t''(x'')$

$$t'''(x'') = j\,\mathrm{sinc}\left(\tfrac{1}{2}\right)\left[e^{j2\pi\frac{x'}{\lambda f_2}\cdot 1cm} + e^{-j2\pi\frac{x'}{\lambda f_2}\cdot 1cm}\right]$$
$$+ \mathrm{sinc}\left(\tfrac{3}{2}\right)\left[e^{j2\pi\frac{x'}{\lambda f_2}3cm} + e^{-j2\pi\frac{x'}{\lambda f_2}3cm}\right]$$

$$= j\,\mathrm{sinc}\left(\tfrac{1}{2}\right)\cdot 2\cdot\cos\left[2\pi\frac{x'}{10\mu m}\right] + \mathrm{sinc}\left(\tfrac{3}{2}\right)\cdot 2\cos\left[2\pi\frac{3x'}{10\mu m}\right]$$

Intensity.

$$|t''(x'')|^2 = 4\left\{\mathrm{sinc}^2\left(\tfrac{1}{2}\right)\cos^2\left[2\pi\frac{x'}{10\mu m}\right] + \mathrm{sinc}^2\left(\tfrac{3}{2}\right)\cos^2\left[2\pi\frac{3x'}{10\mu m}\right]\right\}.$$

2.d) This technique is called phase contrast microscopy, the transparency T_2 is named as "Zernike phase mask" it helps to image a pure phase object (T_1, in practice, a lot of bio-specimen are good examples of phase object) to an amplitude distribution at the image plane. (see plot next page)

2.e) If $b\to\infty$, all the angular spectrum are accepted. so.

$$\cancel{|t''(x'')|^2 =}$$

$$t'''(x'') = 2j\,\mathrm{sinc}\left(\tfrac{1}{2}\right)\cos\left(2\pi\frac{x'}{\Lambda}\right) + 2\sum_{\substack{n=3,\\ n=\text{odd}}}^{n=+\infty}\mathrm{sinc}\left(\tfrac{n}{2}\right)\cos\left(2\pi\frac{nx'}{\Lambda}\right).$$

Intensity

$$|t''(x'')|^2 = 4\left\{\left[\sum_{\substack{n=3\\ n=\text{odd}}}^{n=+\infty}\mathrm{sinc}\left(\tfrac{n}{2}\right)\cos\left(2\pi\frac{nx'}{\Lambda}\right)\right]^2 + \mathrm{sinc}^2\left(\tfrac{1}{2}\right)\cos^2\left(2\pi\frac{x'}{\Lambda}\right)\right\}.$$

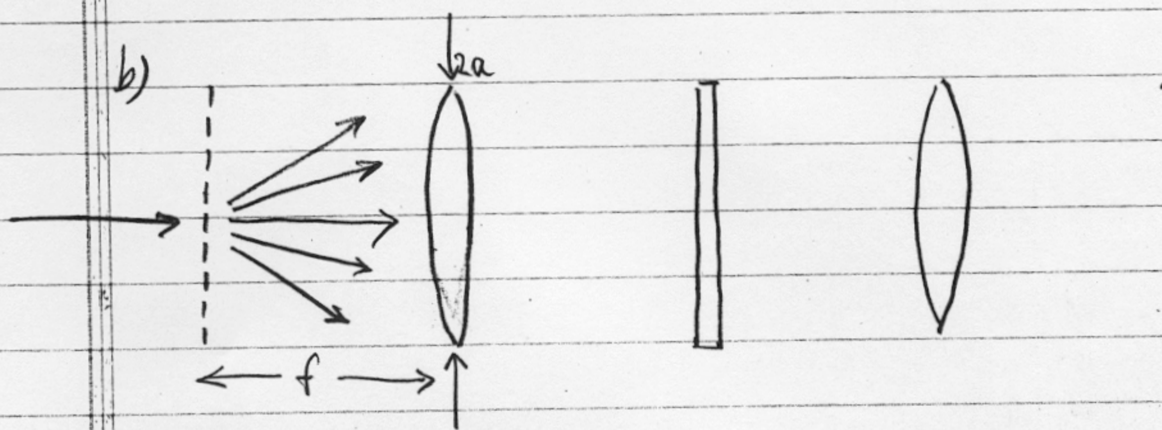
The cross terms from the first square term degrades the contrast of the intensity distribution.

So the large aperture is not desirable in this case.

2f) If $a = 0.5cm$, then the ± 1 orders are not phase shifted. and $b \to \infty$. the field is not modifieed at all by T_2. so the output filed will be $t(x)$ and not intensity contrast can be observed.

a) Nonlinearity will be significant only at the peaks of the Airy disks. The system has a low F/# ($\Leftrightarrow$ high NA) $\Rightarrow$ the Airy disks are very tight and the assumption is justified.

b)



Diffraction angle $\theta = \frac{\lambda}{X} = \frac{1}{4}$

System aperture $= \frac{a}{f} = \frac{1}{2}$ (F/1)

$\Rightarrow$ system admits orders $0, \pm 1, \pm 2$.

0-th order intensity: $\left(\frac{1}{2}\right)^2 I_0 = \frac{1}{4} I_0 \Rightarrow$ transparency transmits $0.1 I_0$

$\pm$1st order intensity: $\left(\frac{1}{2} \frac{\sin\frac{\pi}{2}}{\frac{\pi}{2}}\right)^2 I_0 = 0.101\ I_0 \Rightarrow$

$\Rightarrow$ transparency transmits $0.1 I_0$

$\pm$2nd order intensity: 0

$\Rightarrow$ output $I(x') = 0.1 I_0 \left[1 + 2\cos\left(\frac{2\pi x}{X}\right)\right]$