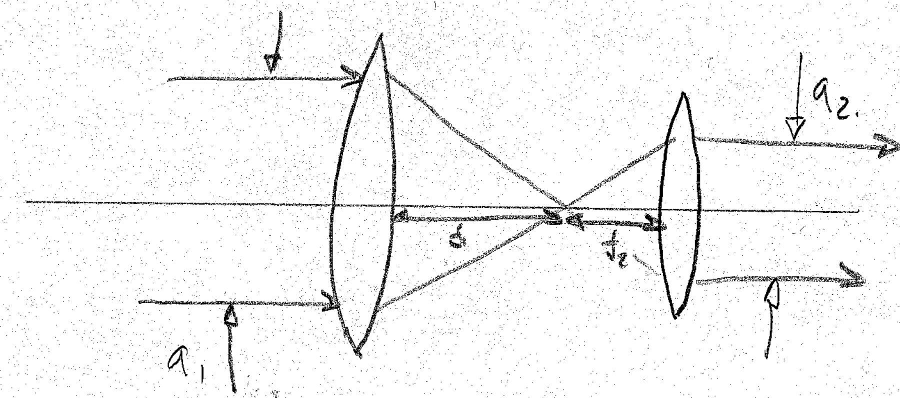
Problem ~~3~~ $\boxed{2}$

L_1 will bring a collimated beam to focus @ f_1. Similarly, if we place L_2, f_2 from this focus, light will emerge collimated.

By inspection:



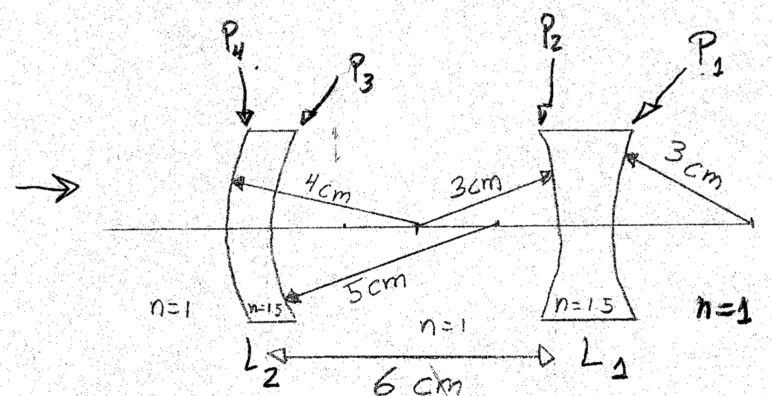
$$\text{Separation} = d = f_1 + f_2$$

$$\text{Width of outgoing Ray} = \frac{f_2}{f_1} a_1$$

~~Problem Set 2~~

- SOLUTION -

Problem ~~1~~ 3



Power of surface = $\dfrac{n_{\text{to the right}} - n_{\text{to the left}}}{\text{Radius of curvature}}$

$$P_1 = \frac{1-1.5}{3} = -1/6$$

$$P_2 = \frac{1.5-1}{-3} = -1/6$$

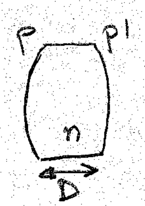
$$P_3 = \frac{1-1.5}{5} = -1/10$$

$$P_4 = \frac{1.5-1}{4} = 1/8$$

System Matrix

$$\begin{pmatrix} \alpha_{OUT} \\ x_{OUT} \end{pmatrix} = \begin{pmatrix} \text{Matrix for} \\ L_1 \end{pmatrix} \begin{pmatrix} \text{Propagation} \\ \text{of } 6\text{cm} \end{pmatrix} \begin{pmatrix} \text{Matrix for} \\ L_2 \end{pmatrix} \begin{pmatrix} \alpha_{in} \\ x_{in} \end{pmatrix}$$

Matrix for a thick lens =
$$\begin{pmatrix} 1 - \frac{P'D}{n} & -\left[P + P' - \frac{PP'D}{n}\right] \\ \frac{D}{n} & 1 - \frac{PD}{n} \end{pmatrix}$$



→ See Annex 1 for the derivation of this formula.

$$\begin{pmatrix} \text{Matrix for} \\ L_1 \end{pmatrix} = \begin{pmatrix} 1 - \frac{P_1 \cdot 1\text{cm}}{1.5} & -\left[P_2 + P_1 - \frac{P_2 P_1 \, 1\text{cm}}{1.5}\right] \\ \frac{1\text{cm}}{1.5} & 1 - \frac{P_2 \cdot 1\text{cm}}{1.5} \end{pmatrix} = \begin{pmatrix} 1.1\hat{1} & 0.352 \\ 0.666 & 1.1\hat{1} \end{pmatrix}$$

$$\begin{pmatrix} \text{Propagation} \\ \text{of } 6\text{cm} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 6 & 1 \end{pmatrix}$$

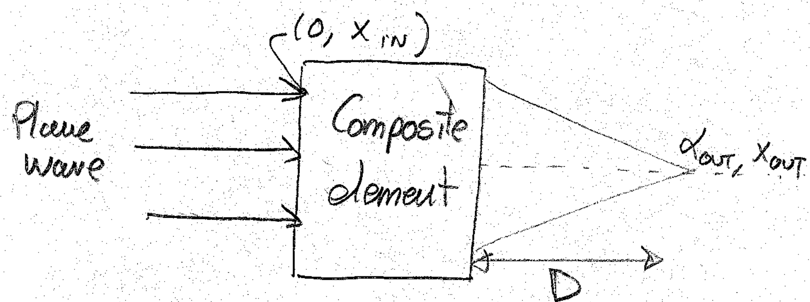
$$\begin{pmatrix} \text{Matrix for} \\ L_2 \end{pmatrix} = \begin{pmatrix} 1 - \frac{P_3 \, 1\text{cm}}{1.5} & -\left[P_4 + P_3 - \frac{P_4 P_3 \, 1\text{cm}}{1.5}\right] \\ \frac{1\text{cm}}{1.5} & 1 - \frac{P_4 \, 1\text{cm}}{1.5} \end{pmatrix} = \begin{pmatrix} 1.067 & -0.0\hat{3} \\ 0.67 & 0.917 \end{pmatrix}$$

$$\begin{pmatrix}\alpha_{OUT}\\ X_{OUT}\end{pmatrix} = \begin{pmatrix}1.\hat{11} & 0.352\\ 0.666 & 1.\hat{11}\end{pmatrix}\begin{pmatrix}1 & 0\\ 6 & 1\end{pmatrix}\begin{pmatrix}1.067 & -0.0\hat{3}\\ 0.67 & 0.917\end{pmatrix}\begin{pmatrix}\alpha_{IN}\\ X_{IN}\end{pmatrix}$$

$$= \underbrace{\begin{pmatrix}3.67 & 0.215\\ 8.56 & 0.77\end{pmatrix}}_{M}\begin{pmatrix}\alpha_{IN}\\ x_{IN}\end{pmatrix}$$

$$\text{Power} = -M_{12} = \underline{\underline{-0.215}}$$

(b) Plane wave incident from the left:



$$\begin{pmatrix}\alpha_{OUT}\\ X_{OUT}\end{pmatrix} = \begin{pmatrix}1 & 0\\ D & 1\end{pmatrix}\begin{pmatrix} M \end{pmatrix}\begin{pmatrix}0\\ X_{IN}\end{pmatrix}$$

$$= \begin{pmatrix}1 & 0\\ D & 1\end{pmatrix}\begin{pmatrix}3.67 & 0.215\\ 8.56 & 0.77\end{pmatrix}\begin{pmatrix}0\\ X_{IN}\end{pmatrix}$$

$$= \begin{pmatrix}1 & 0\\ D & 1\end{pmatrix}\begin{pmatrix}0.215\, X_{IN}\\ 0.77\, X_{IN}\end{pmatrix} = \begin{pmatrix}0.215\, X_{IN}\\ (0.215\cdot D + 0.77)\, X_{IN}\end{pmatrix}$$

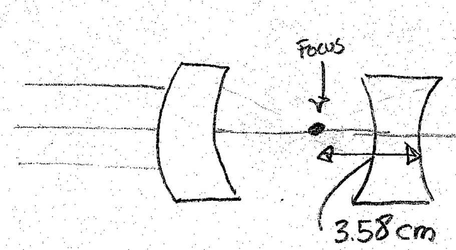
focal length $= f = \dfrac{1}{-0.215}$ **= - 4.65 cm (This is the Effective Focal Length (EFL)**
This distance is referenced to the principal planes of the system)

Focus Condition: $X_{out} = 0$

$$X_{OUT} = (0.215\,D + 0.77) = 0$$

$$\Rightarrow 0.215\,D + 0.77 = 0 \Rightarrow \boxed{D = -3.58 \text{ cm}}$$ **: This is the Back Focal Length (BFL)**

This distance is referenced to the last surface of the optical system, as depicted in the diagram



c) Object @ infinity

The image will be virtual.

ANNEX 1

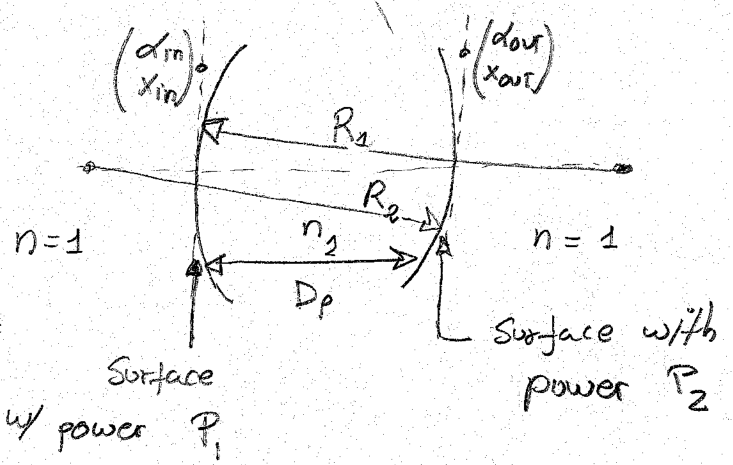
Derivation of Matrix for a thick Lens

Recall from notes:

- Refraction by spherical surface: $\begin{pmatrix} n'\alpha_1' \\ x_1' \end{pmatrix} = \begin{pmatrix} 1 & -\text{Power} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} n\alpha_1 \\ x_1 \end{pmatrix}$
- Translation through unif. medium: $\begin{pmatrix} n\alpha_1 \\ x_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{D_{01}}{n} & 1 \end{pmatrix} \begin{pmatrix} n\alpha_0 \\ x_0 \end{pmatrix}$

→ Distance to propagate.

- Let's apply the above equations to a thick lens:



$$\begin{pmatrix} \overset{1}{n''}\alpha_{OUT} \\ x_{OUT} \end{pmatrix} = \begin{pmatrix} \text{refraction by} \\ \text{power } P_2 \end{pmatrix} \begin{pmatrix} \text{propagation} \\ \text{by distance } D_\ell \end{pmatrix} \begin{pmatrix} \text{refraction by} \\ \text{power } P_1 \end{pmatrix} \begin{pmatrix} \overset{1}{n''}\alpha_{in} \\ x_{in} \end{pmatrix}$$

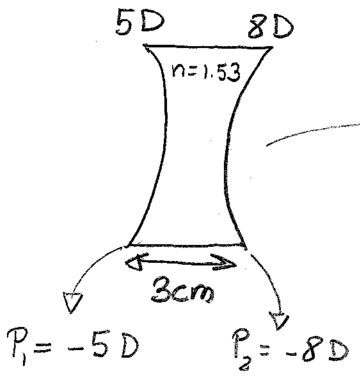
$$= \begin{pmatrix} 1 & -P_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{D_\ell}{n} & 1 \end{pmatrix} \begin{pmatrix} 1 & -P_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha_{in} \\ x_{in} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -P_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -P_1 \\ \frac{D_\ell}{n} & -\frac{D_\ell}{n}P_1 + 1 \end{pmatrix} \begin{pmatrix} \alpha_{in} \\ x_{in} \end{pmatrix}$$

$$\boxed{\begin{pmatrix} \alpha_{OUT} \\ x_{OUT} \end{pmatrix} = \begin{pmatrix} 1 - \frac{P_2 D_\ell}{n} & -\left[P_1 + P_2 - \frac{P_1 P_2 D_\ell}{n}\right] \\ \frac{D_\ell}{n} & 1 - \frac{P_1 D_\ell}{n} \end{pmatrix} \begin{pmatrix} \alpha_{in} \\ x_{in} \end{pmatrix}}$$

Problem ~~3~~ 4

(a) Focal and Principal Planes



$$\text{Matrix} = \begin{pmatrix} 1 - \frac{P_2 \cdot 3cm}{n} & -\left[P_1 + P_2 - \frac{P_1 P_2 \, 3cm}{n}\right] \\ \frac{3cm}{n} & 1 - \frac{P_1 \, 3cm}{n} \end{pmatrix}$$

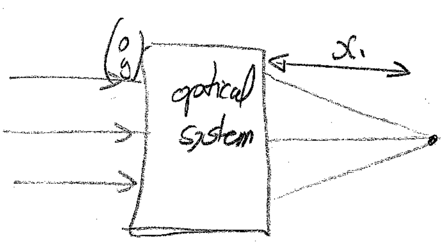
$$= \begin{pmatrix} 1 + \frac{8(0.03)}{1.53} & -\left[-5-8 - \frac{5 \cdot 8 (0.03)}{1.53}\right] \\ \frac{0.03}{1.53} & 1 + \frac{5(0.03)}{1.53} \end{pmatrix}$$

$$= \begin{pmatrix} 1.16 & 13.78 \\ 0.0196 & 1.098 \end{pmatrix}$$

Ppal Planes:

First let's calculate the BFL and the FFL:

BFL:

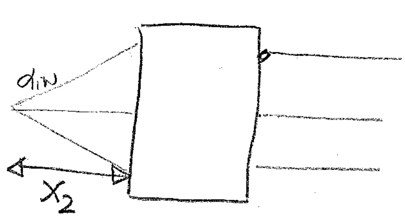


$$\begin{pmatrix} 1 & 0 \\ x_1 & 1 \end{pmatrix}\begin{pmatrix} 1.16 & 13.78 \\ 0.0196 & 1.098 \end{pmatrix}\begin{pmatrix} 0 \\ y \end{pmatrix} = \begin{pmatrix} \alpha_{OUT} \\ x_{OUT} \end{pmatrix}$$

$$= \begin{pmatrix} 1.16 & 13.78 \\ 1.16x + 0.0196 & 13.78x + 1.098 \end{pmatrix}\begin{pmatrix} 0 \\ y \end{pmatrix} = \begin{pmatrix} \alpha_{OUT} \\ x_{OUT} \end{pmatrix}$$

Focus Condition: $x_{OUT} = 0$; $\rightarrow$ $13.78x_1 + 1.098 = 0$ $\rightarrow$ $\boxed{x_1 = -7.97 \text{ cm}}$ BFL

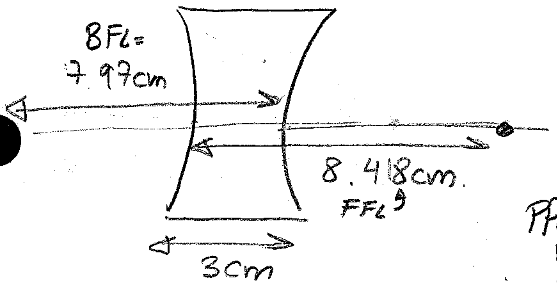
FFL:



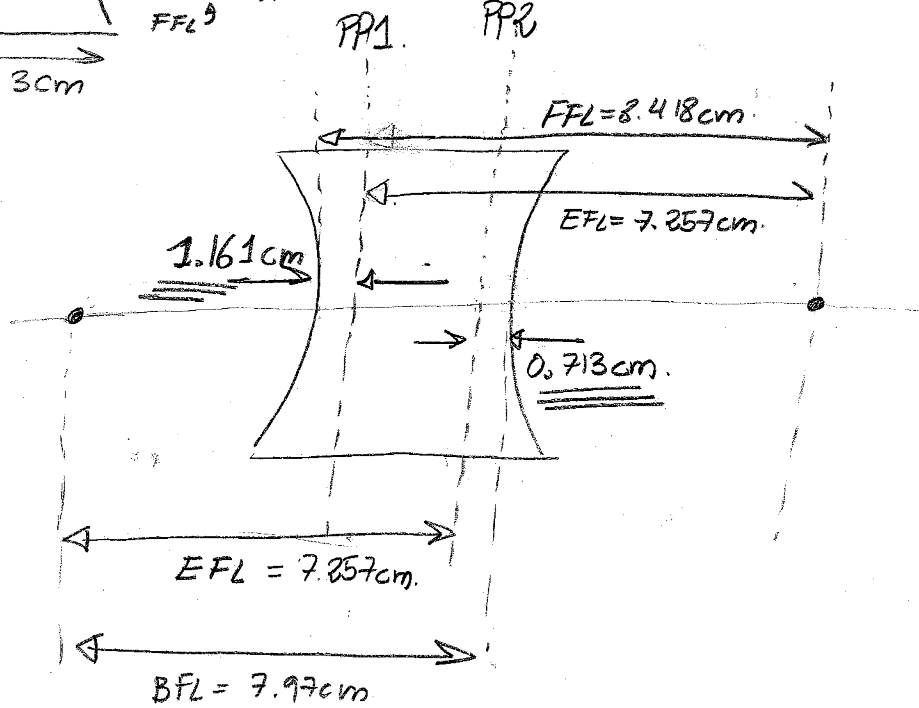
$$\begin{pmatrix} 1.16 & 13.78 \\ 0.0196 & 1.098 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ x_2 & 1 \end{pmatrix}\begin{pmatrix} \alpha_{IN} \\ x_{IN} \end{pmatrix} = \begin{pmatrix} \alpha_{OUT} \\ x_{OUT} \end{pmatrix}$$

$$\begin{pmatrix} 1.16 + 13.78x_2 & 13.78 \\ 1.098x_2 + 0.0196 & 1.098 \end{pmatrix}\begin{pmatrix} \alpha_{IN} \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha_{OUT} \\ x_{OUT} \end{pmatrix}$$

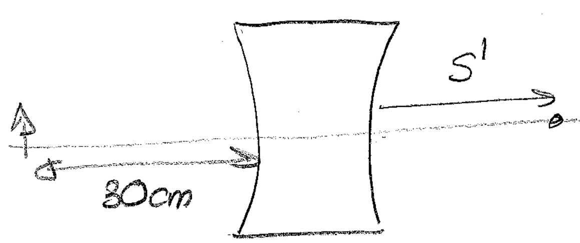
$\rightarrow$ $\alpha_{OUT} = 0$ $\rightarrow$ $1.16 + 13.78x_2 = 0$ $\rightarrow$ $\boxed{x_2 = -8.418 \text{ cm}}$



Effective Focal Length (EFL) $= \dfrac{1}{\text{Power}} = \underline{-7.257\text{ cm}}$



(b) Object placed 30 cm to the left of 1st lens vertex. Where will the image be formed?



$$\begin{pmatrix}\alpha_{OUT}\\ X_{OUT}\end{pmatrix} = \begin{pmatrix}1 & 0\\ s' & 1\end{pmatrix}\begin{pmatrix}1.16 & 13.78\\ 0.0196 & 1.098\end{pmatrix}\begin{pmatrix}1 & 0\\ 0.3 & 1\end{pmatrix}\begin{pmatrix}\alpha_{IN}\\ X_{IN}\end{pmatrix}$$

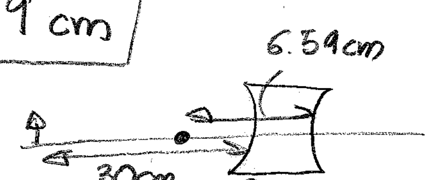
$$= \begin{pmatrix}1 & 0\\ s' & 1\end{pmatrix}\begin{pmatrix}5.294 & 13.78\\ 0.349 & 1.098\end{pmatrix}\begin{pmatrix}\alpha_{IN}\\ X_{IN}\end{pmatrix}$$

Imaging Condition: $\dfrac{\partial X_{OUT}}{\partial \alpha_{IN}} = 0$

$$\begin{pmatrix}\alpha_{OUT}\\ X_{OUT}\end{pmatrix} = \begin{pmatrix}5.294 & 13.78\\ 5.294s' + 0.349 & 13.78s' + 1.098\end{pmatrix}\begin{pmatrix}\alpha_{IN}\\ X_{IN}\end{pmatrix}$$

$\downarrow$

$5.294s' + 0.349 = 0 \rightarrow \boxed{s' = -6.59\text{ cm}}$



(c) Thin Lens Approximation

$$\text{Matrix for thin lens} = \begin{pmatrix} 1 & -(P_1+P_2) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -(-13) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 13 \\ 0 & 1 \end{pmatrix}$$

Image Formation:

$$\begin{pmatrix} \alpha_{OUT} \\ x_{OUT} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ s' & 1 \end{pmatrix} \begin{pmatrix} 1 & 13 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0.3 & 1 \end{pmatrix} \begin{pmatrix} \alpha_{IN} \\ x_{IN} \end{pmatrix}$$

Image Condition: $\frac{\partial x_{OUT}}{\partial \alpha_{IN}} = 0 \longrightarrow$

$$\begin{pmatrix} \alpha_{OUT} \\ x_{OUT} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ s' & 1 \end{pmatrix} \begin{pmatrix} 4.9 & 13 \\ 0.3 & 1 \end{pmatrix} \begin{pmatrix} \alpha_{IN} \\ x_{IN} \end{pmatrix}$$

$$= \begin{pmatrix} 4.9 & 13. \\ 4.9s'+0.3 & 13s'+1 \end{pmatrix} \begin{pmatrix} \alpha_{IN} \\ x_{IN} \end{pmatrix}$$

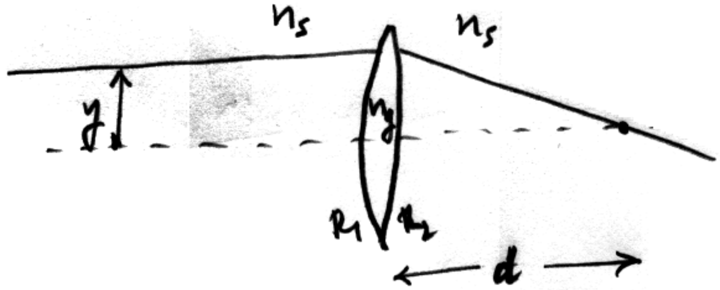
$$\downarrow$$

$$4.9s' + 0.3 = 0 \longrightarrow \boxed{s'_{\text{thin-lens}} = -6.122 \text{ cm.}}$$

$$\% \text{ Error} = \left(100 - \frac{6.122}{6.59} \times 100\right) = \underline{\underline{7.1\,\%}}$$

① A thin bi-convex lens of index 1.5 is known to have focal length of 50 cm in air when immersed in a transparent liquid medium, the focal length is measured to be 250 cm, What is the refractive index n of the liquid

Soln.



n_g = ndex of glass

n_s = Index of surrounding medium

d = focal length

power $P = (n_g - n_s)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$

→ find the focal length d

$$\begin{pmatrix} 1 & 0 \\ \frac{d}{n_s} & 1 \end{pmatrix}\begin{pmatrix} 1 & -P \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ y \end{pmatrix} = \begin{pmatrix} 1 & -P \\ \frac{d}{n_s} & 1-\frac{Pd}{n_s} \end{pmatrix}\begin{pmatrix} 0 \\ y \end{pmatrix} = \begin{pmatrix} -Py \\ \left(1-\frac{Pd}{n_s}\right)y \end{pmatrix}$$

$$\left(1 - \frac{Pd}{n_s}\right)y = 0 \ \text{(for all } y) \Rightarrow \boxed{d = \frac{n_s}{P}}$$

air: $\dfrac{1}{(1.5-1)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)} = 50$

liquid: $\dfrac{n}{(1.5-n)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)} = 250$

$$\xrightarrow{\text{divide}} \frac{\frac{n}{1.5-n}}{\frac{1}{1.5-1}} = \frac{250}{50} \Rightarrow n = 1.36$$

② You'd like to look through a lens at your pet kitten and see it standing right-side up but shrunk to 1/3 its normal height. If the absolute value of the focal l. is f, determine what kind of lens is needed (i.e. positive or negative) as well as the object & image distances in terms of f.

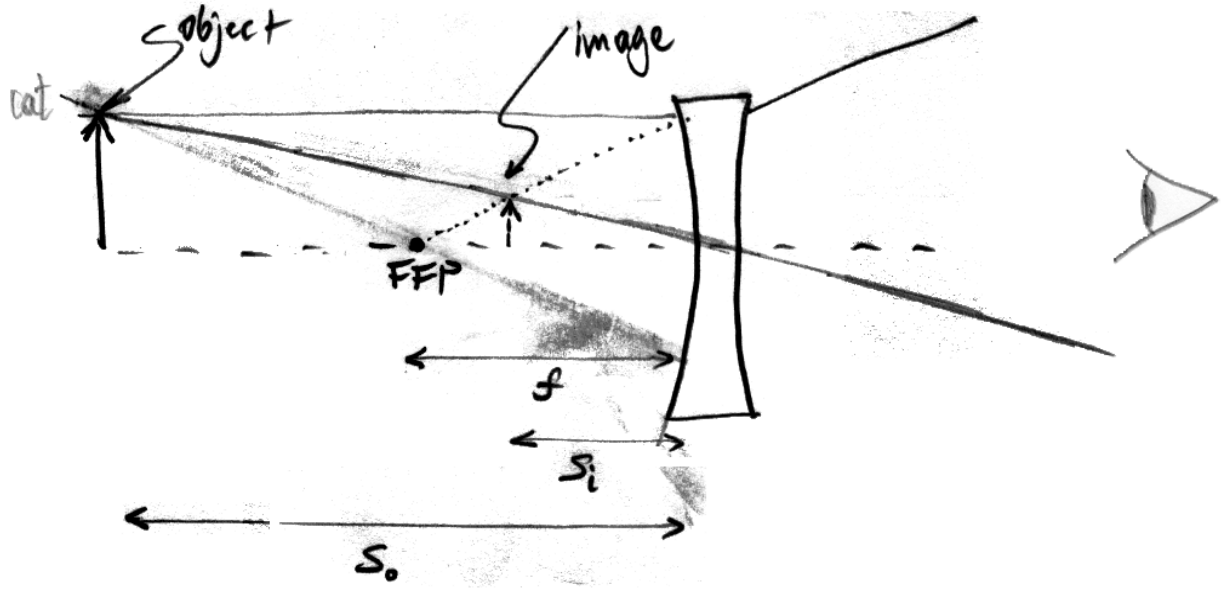
Soln.

image de-magnified & erect ⇒ need negative lens (see Hecht Table 5.3 p. 165)

$$M_{lateral} = -\frac{s_i}{s_o} = \frac{1}{3}$$

$$\frac{1}{s_i} + \frac{1}{s_o} = -\frac{1}{f} \Rightarrow \frac{1}{s_i} + \frac{1}{(-3s_i)} = -\frac{1}{f} \Rightarrow s_i = -\frac{2f}{3}$$

$\Rightarrow s_o =$



1. **(45%)** We intend to use a spherical ball lens of radius R and refractive index n as magnifier in an imaging system, as shown in Figure A. The refractive index satisfies the relationship $1 < n < 4/3$, and the medium surrounding the ball lens is air (refractive index = 1).

   1.a) Calculate the effective focal length (EFL) of the ball lens. Use the thick lens model with appropriate parameters.

   1.b) Locate the back focal length (BFL), the front focal length (FFL) and the principal planes of the ball lens.

   1.c) An object located at distance d to the left of the back surface of the ball lens, as shown in Figure A, where

$$d = R\frac{4-3n}{4(n-1)}.$$

   Show that the object is one half (EFL) behind the principal plane, and use this fact to find the location of the image plane.

   1.d) Is the image real or virtual? Is it erect or inverted? What is the magnification?

   1.e) Locate the aperture stop and calculate the numerical aperture (NA) of the optical system of Figure A.

   1.f) Sketch how a human observer using the optical system of Figure A as input to her eye would form the final image of the object on her retina.

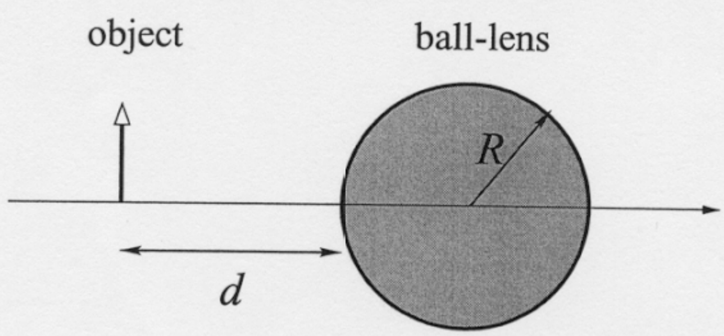
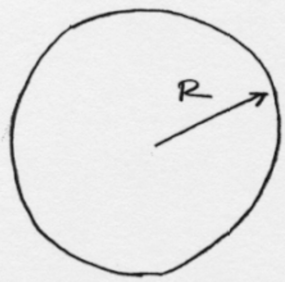
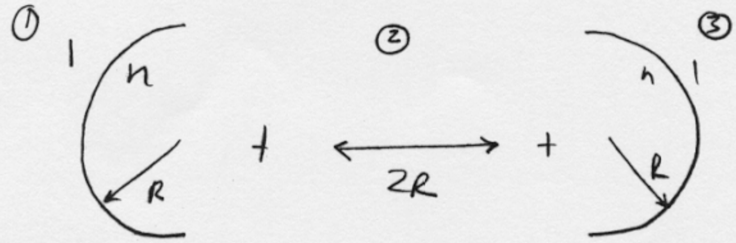


Figure A

Problem 1.

(1.a)





positive spherical surface,

propagation by 2R

negative spherical surface

Matrix method:

$$\begin{pmatrix} 1 & -\frac{1-n}{-R} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \frac{2R}{n} & 1 \end{pmatrix}\begin{pmatrix} 1 & -\frac{n-1}{R} \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1+\frac{1-n}{R}\cdot\frac{2R}{n} & \frac{1-n}{R} \\ \frac{2R}{n} & 1 \end{pmatrix}\begin{pmatrix} 1 & -\frac{n-1}{R} \\ 0 & 1 \end{pmatrix}$$

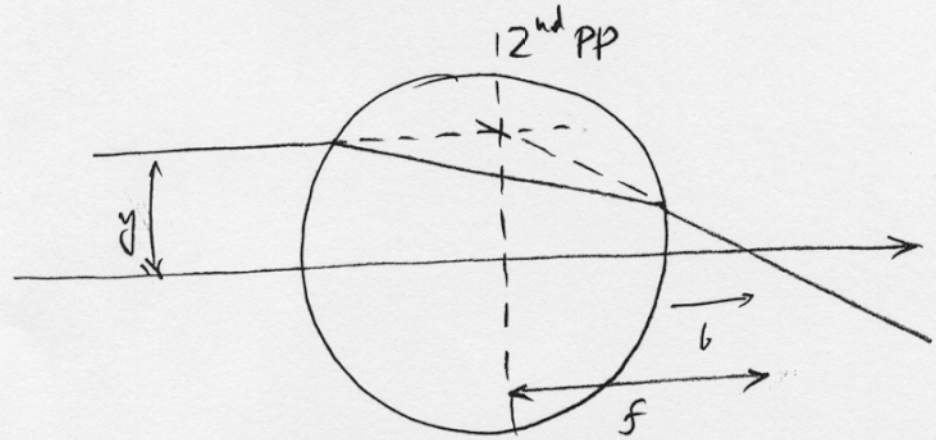
$$= \begin{pmatrix} \frac{2}{n}-1 & \frac{1-n}{R} \\ \frac{2R}{n} & 1 \end{pmatrix}\begin{pmatrix} 1 & -\frac{n-1}{R} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{2}{n}-1 & -\frac{2}{n}\frac{n-1}{R}+\frac{n-1}{R}+\frac{1-n}{R} \\ \frac{2R}{n} & -\frac{2R}{n}\cdot\frac{n-1}{R}+1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2}{n}-1 & -\frac{2(n-1)}{nR} \\ \frac{2R}{n} & \frac{2}{n}-1 \end{pmatrix}$$

$$\longrightarrow \text{EFL} = \frac{nR}{2(n-1)} \equiv f$$

(note $f>0$ always)

(1.6)



$$\begin{pmatrix}1&0\\ b&1\end{pmatrix}\begin{pmatrix}\frac{2}{n}-1 & -\frac{2(n-1)}{nR}\\ \frac{2R}{n} & \frac{2}{n}-1\end{pmatrix}\begin{pmatrix}0\\ y\end{pmatrix}=$$

$$=\begin{pmatrix}\frac{2}{n}-1 & -\frac{2(n-1)}{nR}\\ \left(\frac{2}{n}-1\right)b+\frac{2R}{n} & -\frac{2(n-1)b}{nR}+\left(\frac{2}{n}-1\right)\end{pmatrix}\begin{pmatrix}0\\ y\end{pmatrix} = \begin{pmatrix} -\frac{2(n-1)y}{nR}\\ \left[\underbrace{-\frac{2(n-1)b}{nR}+\left(\frac{2}{n}-1\right)}\right]y\end{pmatrix}$$

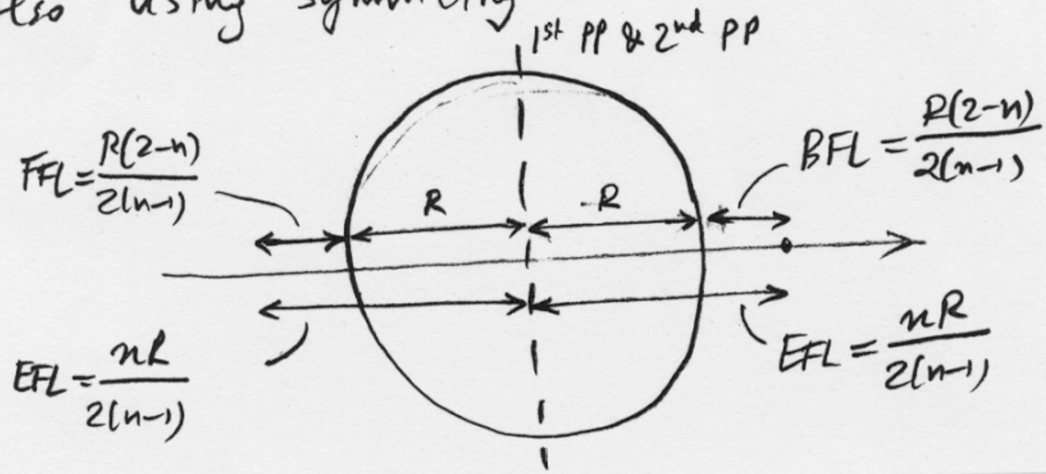
$$= 0 \Rightarrow b = \frac{nR\cdot\frac{2-n}{n}}{2(n-1)} = \frac{R(2-n)}{2(n-1)} = BFL$$

(note $b<0$ i.e. 2nd FP inside ball if $\underline{\underline{n>2}}$)

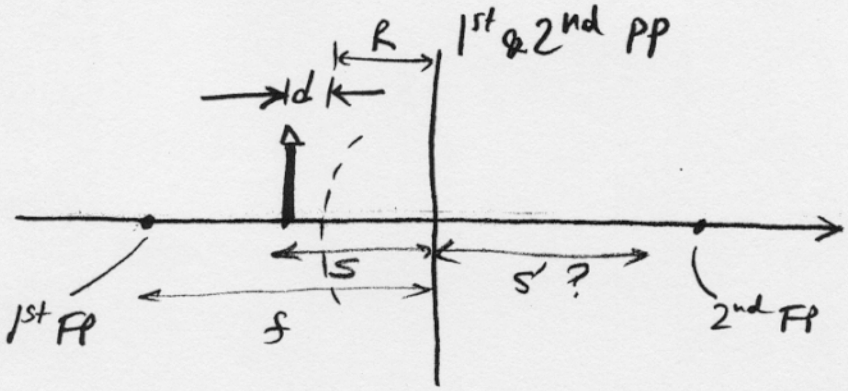
Location of 2nd PP wrt back surface of the ball

$$b-f = \frac{R(2-n)}{2(n-1)} - \frac{nR}{2(n-1)} = \frac{R}{2(n-1)}(2-n-n) = \frac{2R(1-n)}{2(n-1)} = -R$$

Also using symmetry



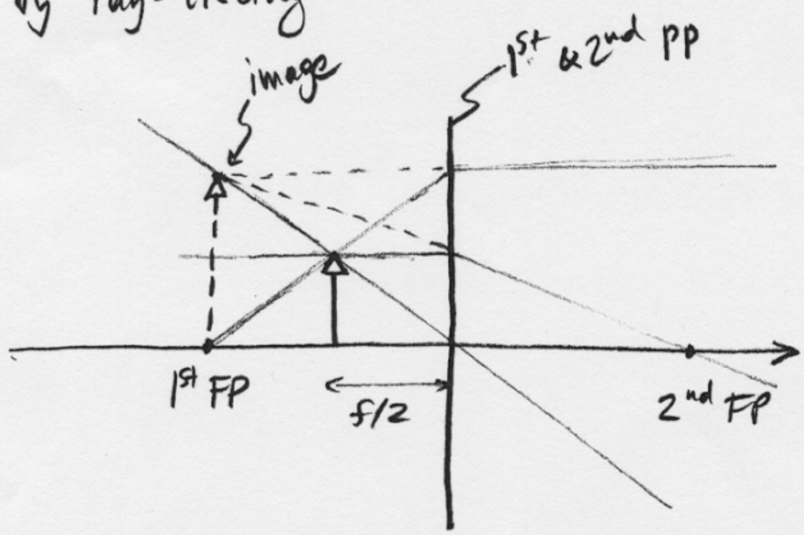
(1.c)



$$s = R\,\frac{4-3n}{4(n-1)} + R = R\,\frac{4-3n+4n-4}{4(n-1)} = \frac{Rn}{4(n-1)} = \frac{f}{2}$$

$$\frac{1}{s} + \frac{1}{s'} = \frac{1}{f} \Rightarrow \frac{2}{f} + \frac{1}{s'} = \frac{1}{f} \Rightarrow \frac{1}{s'} = -\frac{1}{f} \Rightarrow s' = -f$$

OR by ray-tracing



(1.d) virtual (need to extend rays backwards to intersect them)

erect

magnified (laterally), $m_x = -\frac{s'}{s} = +2$.

7. Image stabilizer / Anti-shake.

(a) System matrix from A to A'

$$M = \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{1}{f} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ d-s & 1 \end{pmatrix}$$

$$\frac{1}{f} = \frac{n'-n}{R_1} + \frac{n-n'}{R_2}$$

$$= \frac{0.5}{50} + \frac{0.5}{0.5}$$

$$= \frac{1}{50}$$

$\Rightarrow f = 5$ cm.

$d = 36$ cm.

$$M = \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{1}{5} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 36-s & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -\frac{1}{5} \\ s & 1-\frac{s}{5} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 36-s & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1-\frac{1}{5}(36-s) & -\frac{1}{5} \\ \boxed{s+(1-\frac{s}{5})(36-s)} & 1-\frac{s}{5} \end{pmatrix}$$

Imaging condition: $\qquad = 0.$

$$\Rightarrow s + (1-\frac{s}{5})(36-s) = 0.$$

$$s_1 = 30\text{cm}, \quad s_2 = 6\text{cm}$$

As $s < d-s$.

$$\Rightarrow \boxed{s=6}\text{cm}.$$

(b). When camera is tilted by 1°, A is off optical axis by $(d-s)\cdot\frac{1^\circ}{180^\circ}\cdot\pi$

$= 300mm \cdot \frac{\pi}{180}$

$\doteq 5.236mm.$

$M_{LT} = -\frac{s}{d-s} = -\frac{6}{30}$

so $\overline{AA''} = |M_{LT}\cdot 5.236| = 1.0472$ mm

CCD / CMOS chip must move $\boxed{1.0472 \text{ mm.}}$

(C). Assume lens moves laterally by δ, so optical axis moves by δ. then object A is off axis by

$(5.236-\delta).$

Since the image must be at the same position of A'. and now A' is off axis by δ.

$\Rightarrow \delta = |M_{LT}\cdot(5.236-\delta)|$

$\Rightarrow \delta = \frac{1}{5}(5.236-\delta).$

$$\boxed{\delta = \frac{1}{6}\times 5.236 = 0.8727 \text{ mm}}$$