2.71 / 2.710 Optics. Problem Set #5 Solution

1. Frequencies & wavelengths

a). $$\lambda = \frac{c}{\nu} = \frac{3\times10^8\ m/s}{88.1\times10^6\cdot s^{-1}} = 3.405\ m$$

b). $\lambda = 20m.$

$$\nu = \frac{c}{\lambda} = \frac{3\times10^8 m/s}{20m} = 15\ MHz$$

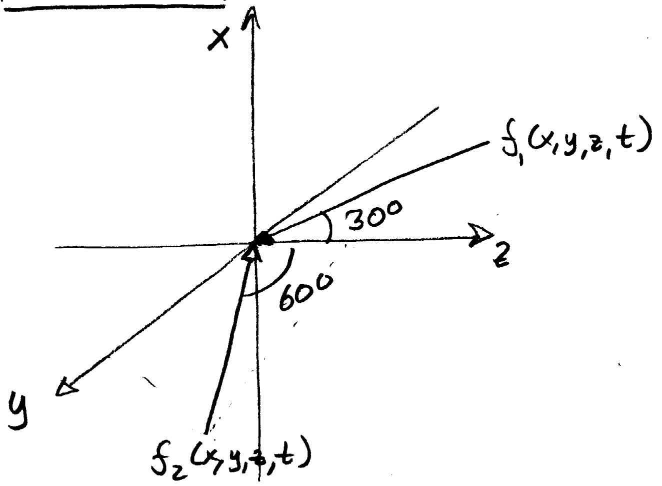
c) $$\lambda = \frac{c}{\nu} = \frac{340m/s}{440s^{-1}} = 0.7727\ m$$

d). $$\nu = \frac{c}{\lambda} = \frac{3\times10^8 m/s}{1\times10^{-9} m} = 3\times10^{17}\ Hz$$

e). $$\lambda = \frac{h}{mv} = \frac{6.626\times10^{-34}}{9.109\times10^{-31}\cdot10^5 m/s} = 7.27\ nm$$

f). $$\nu = \frac{\sqrt{gk\tanh(kd)}}{2\pi} = \frac{\sqrt{9.8\cdot\frac{2\pi}{0.1}\cdot\tanh(\frac{2\pi}{0.1}\cdot3)}}{2\pi} = 3.95\ Hz$$

Problem 4



$\lambda = 1\,\mu m$ $\quad c = 3\times10^8\ m\ sec^{-1}$

$k = \frac{2\pi}{\lambda} = 6.2832\times10^6\ m^{-1}$

Two representations for a plane wave

$f_r(x,y,z,t) = A\cos(\vec{k}\cdot\vec{r} - \omega t)$

$f_p(\vec{r},t) = Ae^{i(\vec{k}\cdot\vec{r} - \omega t)}$

(a) For $f_1(x,y,z,t)$:

$\vec{k}_1 = k_x\hat{i} + k_y\hat{j} + k_z\hat{z}$;

$$\begin{cases} \vec{k}_x = -k\sin 30 \\ \vec{k}_y = 0 \\ k_z = -k\cos 30 \end{cases}$$

$f_{r_1}(x,y,z,t) = A\cos\left(-\frac{2\pi}{\lambda}(\sin 30\ x + \cos 30\ z) - \omega t\right)$

$= A\cos\left(\pi\times10^6 x + 5.441\times10^6 z - \omega t\right)$

$f_p(\vec{r},t) = Ae^{i(\pi\times10^6 x + 5.441\times10^6 z - \omega t)}$

(b) For $f_2(x,y,z,t)$

$$\begin{cases} \vec{k}_x = 0 \\ \vec{k}_y = -k\sin 60 \\ \vec{k}_z = -k\cos 60 \end{cases}$$

$f_{r_2}(x,y,z,t) = A\cos\left(\frac{2\pi}{\lambda}(\sin 60\ y + \cos 60\ z) - \omega t\right)$

$= A\cos\left(5.44\times10^6 y + \pi\times10^6 z - \omega t\right)$

$f_p(\vec{r},t) = Ae^{i(5.44\times10^6 y + \pi\times10^6 z - \omega t)}$

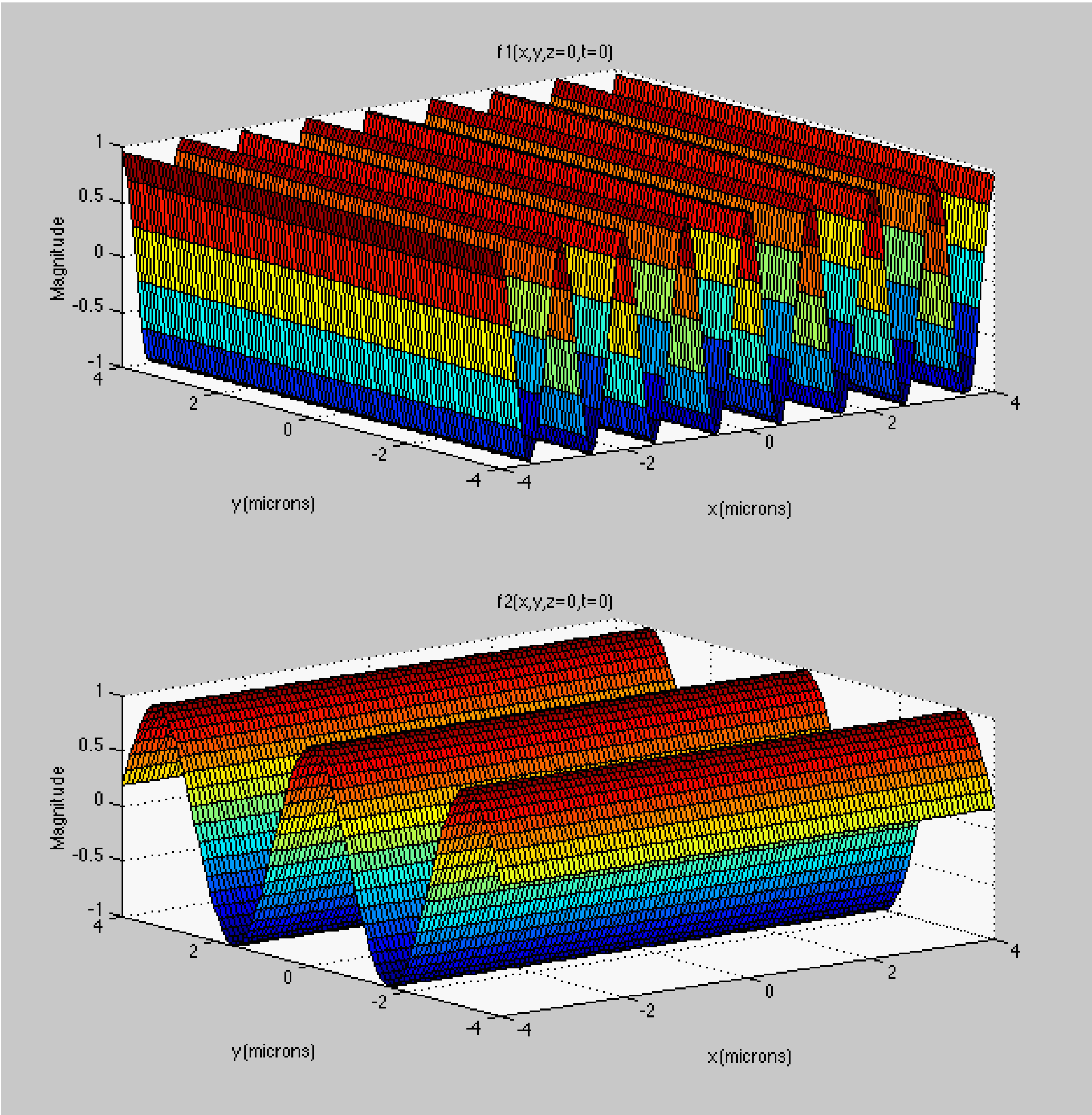
(c) See MATLAB Plots

(d) Superposition @ $z=0$

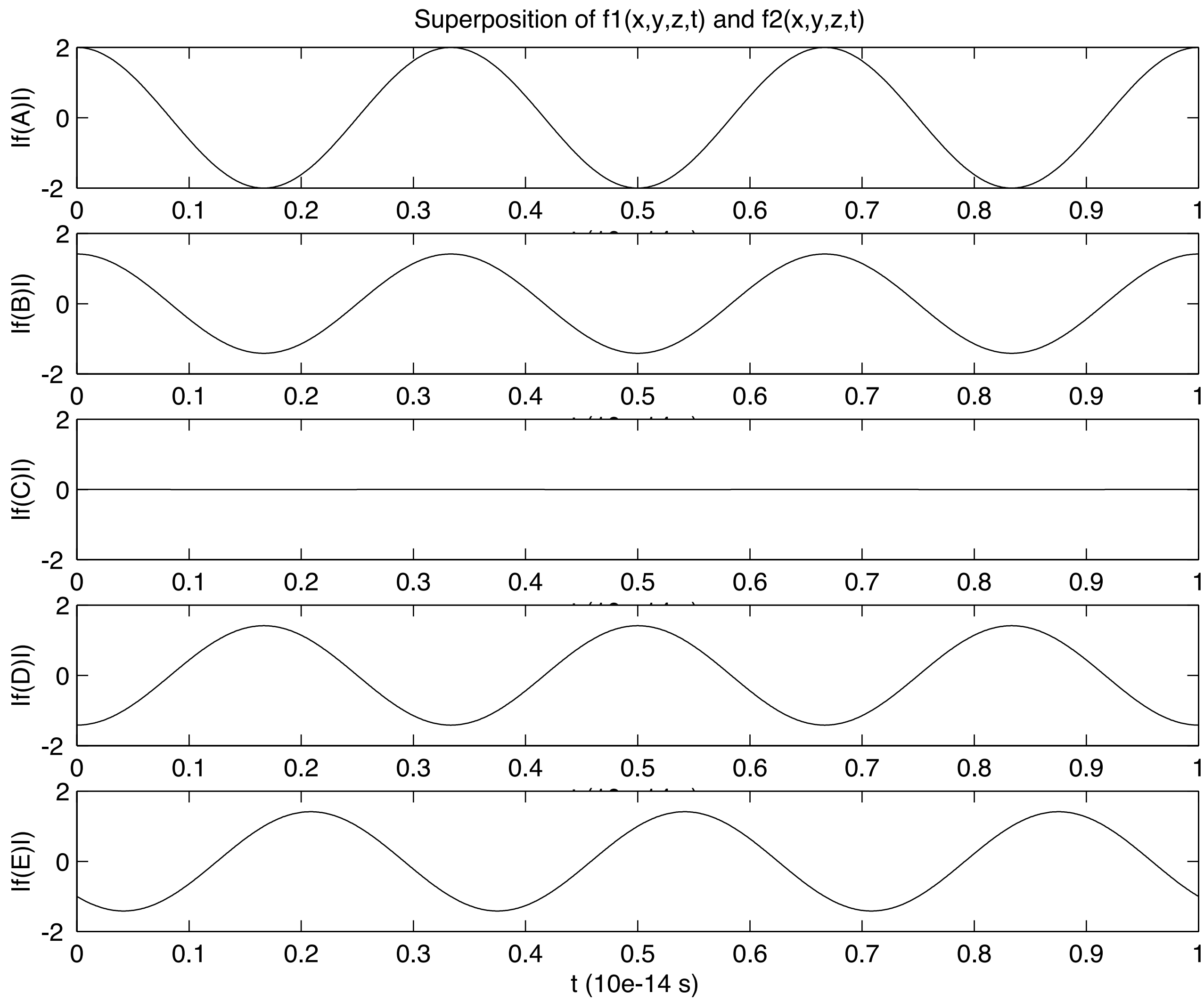
$f_{super}(\vec{r},t) = f_1 + f_2 = \mathrm{Re}\left\{A\left(e^{i(\pi\times10^6 x)} + e^{i(5.44\times10^6 y)}\right)e^{-i\omega t}\right\}$

See MATLAB Plots

(c)


f1(x,y,z=0,t=0)
Magnitude
1
0.5
0
-0.5
-1
4
2
0
-2
-4
y(microns)
-4
-2
0
2
4
x(microns)
f2(x,y,z=0,t=0)
Magnitude
1
0.5
0
-0.5
-1
4
2
0
-2
-4
y(microns)
-4
-2
0
2
4
x(microns)

(d)



We can see that the superposition of f_1 and f_2 adds constructively or deconstructively depending on location. For instance, at point A, we observe perfect constructive interference (maximum magnitude equals 2), while at point C we get the opposite effect (both waves exactly cancel each other).

3. Superposition of Waves.

3.a) $E_1(z,t) = E_0 \cos(k_1 z - \omega_1 t) \qquad \nu_1 = 100\,Hz$

$E_2(z,t) = E_0 \cos(k_2 z - \omega_2 t) \qquad \nu_2 = 101\,Hz$

$$E_1(z,t) + E_2(z,t) = E_0\left[\cos(k_1 z - \omega_1 t) + \cos(k_2 z - \omega_2 t)\right]$$

As trianglation:

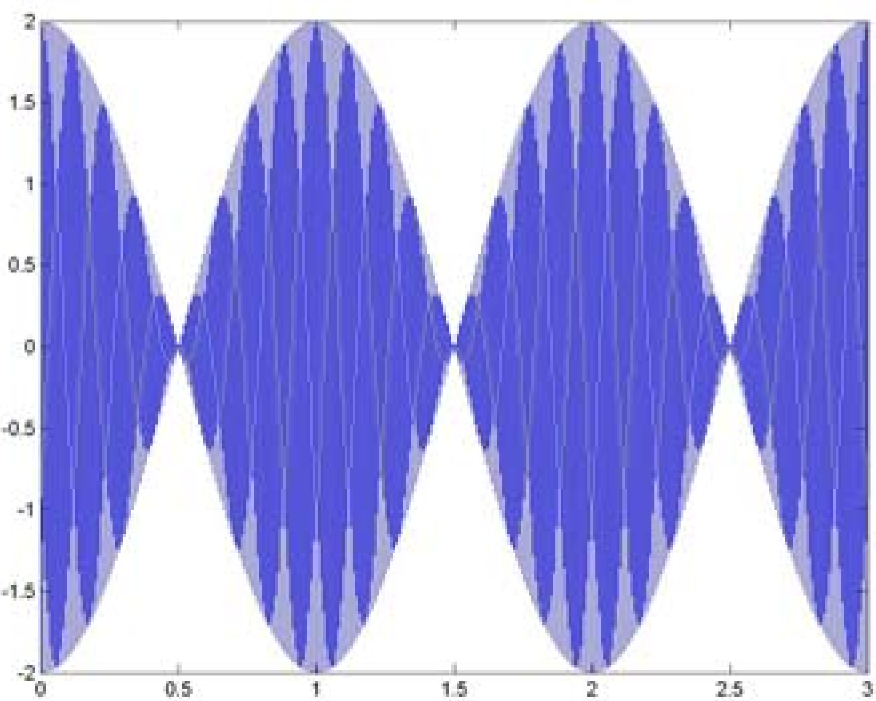
$$\boxed{\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}}$$

$$= 2E_0\left[\cos\left(\frac{(k_1+k_2)z - (\omega_1+\omega_2)t}{2}\right)\cos\left(\frac{(k_1-k_2)z - (\omega_1-\omega_2)t}{2}\right)\right]$$

$$= \underbrace{2\cdot\cos\left[\left(\frac{k_1-k_2}{2}\right)z - \left(\frac{\omega_1-\omega_2}{2}\right)t\right]}_{\text{low freq., slow varying envelop.}} E_0\cdot\underbrace{\cos\left[\left(\frac{k_1+k_2}{2}\right)z - \left(\frac{\omega_1+\omega_2}{2}\right)t\right]}_{\text{high freq., fast varying}}$$

$$\omega_{high} = \frac{\omega_1+\omega_2}{2} = 100.5\,Hz.$$

$$\omega_{low} = \frac{\omega_2-\omega_1}{2} = 0.5\,Hz.$$



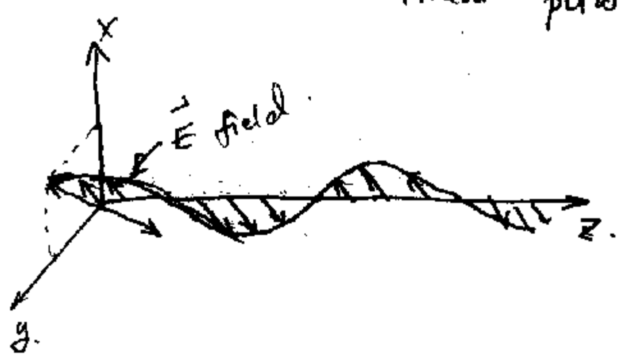
4. Superposition of polarized waves.

4a) $E_1(z,t) = \hat{x} E_0 \cos(kz - \omega t)$

$E_2(z,t) = \hat{y} E_0 \cos(kz - \omega t)$.

$$E(z,t) = E_1(z,t) + E_2(z,t)$$

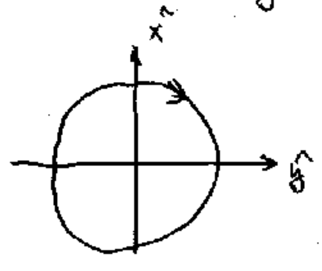
$$= \underbrace{(\hat{x} + \hat{y})}_{\text{linear polarization, } 45^\circ \text{ degree.}} E_0 \cos(kz - \omega t).$$



4.b). E_2 delayed by $\pi/2$.

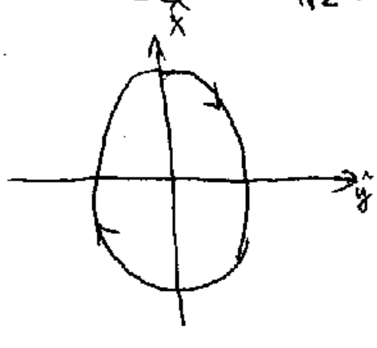
$$E_2(z,t) = \hat{y} E_0 \cos(kz - \omega t + \tfrac{\pi}{2}).$$

$$= \hat{y} E_0 \sin(kz - \omega t).$$



Right hand circular polarization.

4c). $E_2(z,t) = \frac{1}{\sqrt{2}} \hat{y} E_0 \sin(kz - \omega t)$.



elliptical polarization.