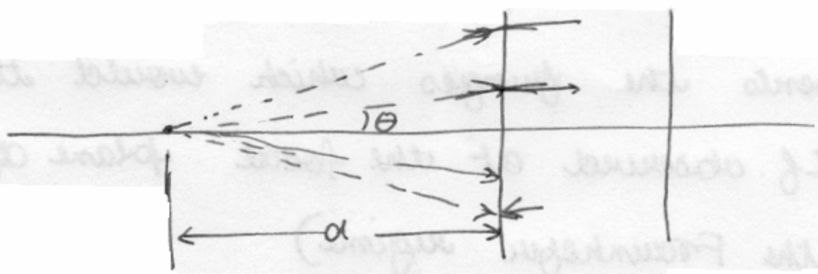
2.71/2.710

Problem Set #7

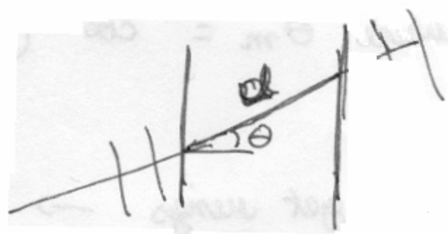
SOLUTIONS

1.)



Read out of Fabry Perot by a spherical wave

Fabry perot condition



$d = \frac{m\lambda}{2n}$ Now $d = \frac{L}{\cos\theta}$

$\frac{L}{\cos\theta} = \frac{m\lambda}{2n}$ $L = \frac{10\lambda}{n}$

$\Rightarrow \frac{1}{\cos\theta} = \frac{m}{20}$

Find values of θ for $m = 20, 21 \ldots \infty$

$\theta = 0, 17.753°, 24.62°, 29.591°,$ etc

(Remember that these θs are in medium n → We shall just stay in this medium since n is not given.

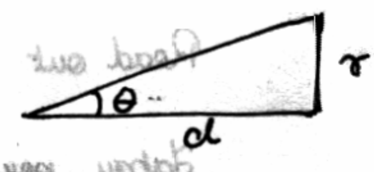
We can see that the rays that are accepted (100% transmission) are rays that have

$$\cos\theta = \frac{20}{m} \qquad (m \geq 20)$$

This represents the fringes which would the Fabry-Perot rings (If observed at the focal plane of a lens or in the Fraunhofer regime)

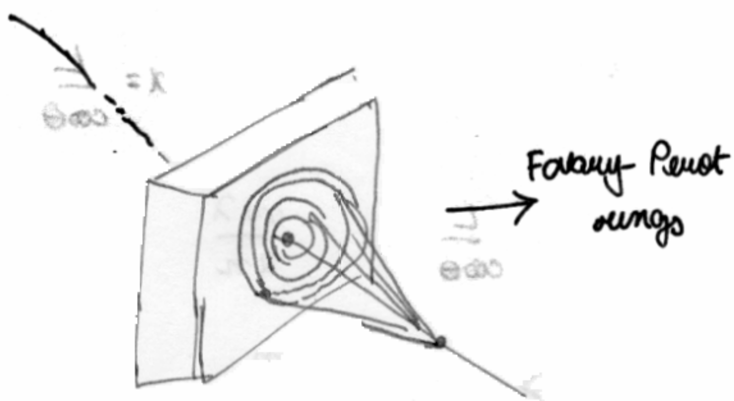
the radius of the ring would be

$$r_m \approx \tan(\theta_m)\, d$$



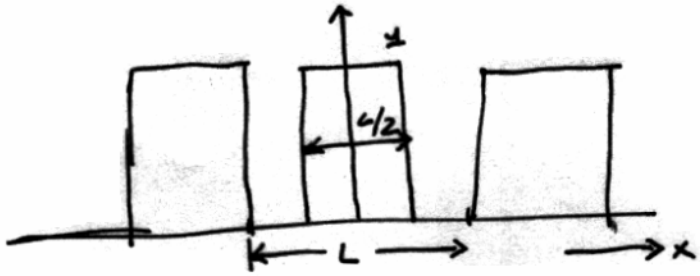
$$\text{where } \theta_m = \cos^{-1}\left(\frac{20}{m}\right) \quad \left\{ \text{for } m \geq 20 \right\}$$

To see ~~the~~ how you get rings →



(2)

(a)



$$f(x) \quad a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{\frac{L}{2}}\right) + b_n \sin\left(\frac{n\pi x}{\frac{L}{2}}\right)$$

b_n 0 for (the function is even

$$a_0 \quad \frac{1}{L}\int_{-L/2}^{L/2} f(x)\,dx$$

$$\frac{1}{L}\int_{-L/4}^{L/4} dx \qquad \frac{1}{L} \quad \frac{L}{2} \quad \frac{1}{2}$$

$$\frac{2}{L}\int_{-L/2}^{L/2} f(x)\cos\left(\frac{2n\pi x}{L}\right)dx \qquad \frac{2}{L}\int_{L/4}^{L/4} \cos\left(\frac{2n\pi x}{L}\right)dx$$

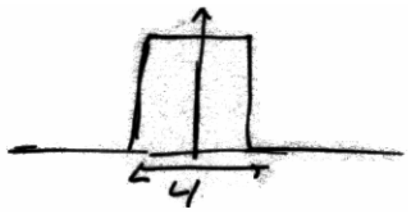
$$\frac{}{L} \quad \frac{\sin\left(\frac{2n\pi}{L} \times \frac{L}{4}\right) + \sin}{\frac{n\pi}{L}} \quad 2$$

$$= \quad \frac{2}{\cancel{L}} \times \frac{L}{n\pi} \times \cancel{2} \quad \sin\left(\frac{2n\pi}{4}\right)$$

$$\frac{\sin\left(\frac{n\pi}{2}\right)}{\left(\frac{n\pi}{2}\right)}$$

$$f(x) = \frac{1}{2} + \sum^{\infty} \frac{\sin\left(\frac{n\pi}{2}\right)}{\left(\frac{n\pi}{2}\right)} \cos\left(\frac{n\pi x}{L/2}\right)$$

This is the required Fourier series



$f(x)$ $\quad \text{rect}\left(\frac{x}{L/2}\right)$

$F(f_x)$ $\quad \frac{L}{2}\,\text{sinc}\left(\frac{L}{2} f_x\right)$ $\quad$ { From Fourier transform definition }

For a boxcar $\quad \mathcal{F}\{f_x\}$ $\quad \frac{L}{2}\,\text{sinc}\left(\frac{L}{2} f_x\right)$

Compare with $\quad a_n = \dfrac{\sin\left(\frac{n\pi}{2}\right)}{\left(\frac{n\pi}{2}\right)}$

Observe that

$$a_n = \frac{L}{2}\,\text{sinc}\left(\frac{L}{2} f_x\right) \quad \text{for } \cancel{f_x = \ldots \frac{4\pi}{L} \ldots} \text{ etc}$$

$$f_x = \frac{1}{L}, \frac{2}{L} \ldots \frac{m}{L}$$

$$\cancel{\frac{L}{2} \times \sin\left(\pi \times \frac{L}{2} \times \frac{2\pi}{L}\right.}$$

$$\frac{L}{2} \quad \frac{\sin\left(\frac{\cancel{L}}{2} \times \pi \times \frac{n}{\cancel{L}}\right)}{\pi \times \frac{\cancel{L}}{2} \times \frac{n}{\cancel{L}}}$$

$$\frac{L}{2} \quad \frac{\sin\left(\frac{n\pi}{2}\right)}{\left(\frac{n\pi}{2}\right)}$$

$\Rightarrow$ Analogy with the Fourier series !

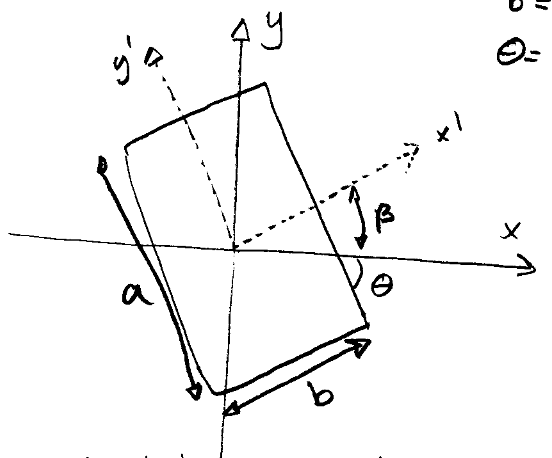
Problem Set 7

- Solutions -

Problem 1

Tilted Aperture

$a = 10\ \mu m$
$b = 5\ \mu m$
$\theta = 60°$

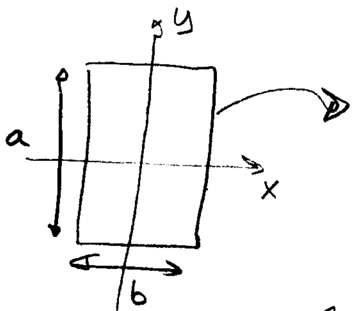


$\beta + \theta = 90°$

Rotation:

$$x' = x\cos\beta + y\sin\beta$$
$$y' = -x\sin\beta + y\cos\beta$$

First, let's find the Fourier transform of the unrotated aperture:



$$f(x,y) = \text{rect}\left(\frac{x}{b}\right)\text{rect}\left(\frac{y}{a}\right)$$

Since the function is separable in rectangular coordinates, the Fourier transform is very easy:

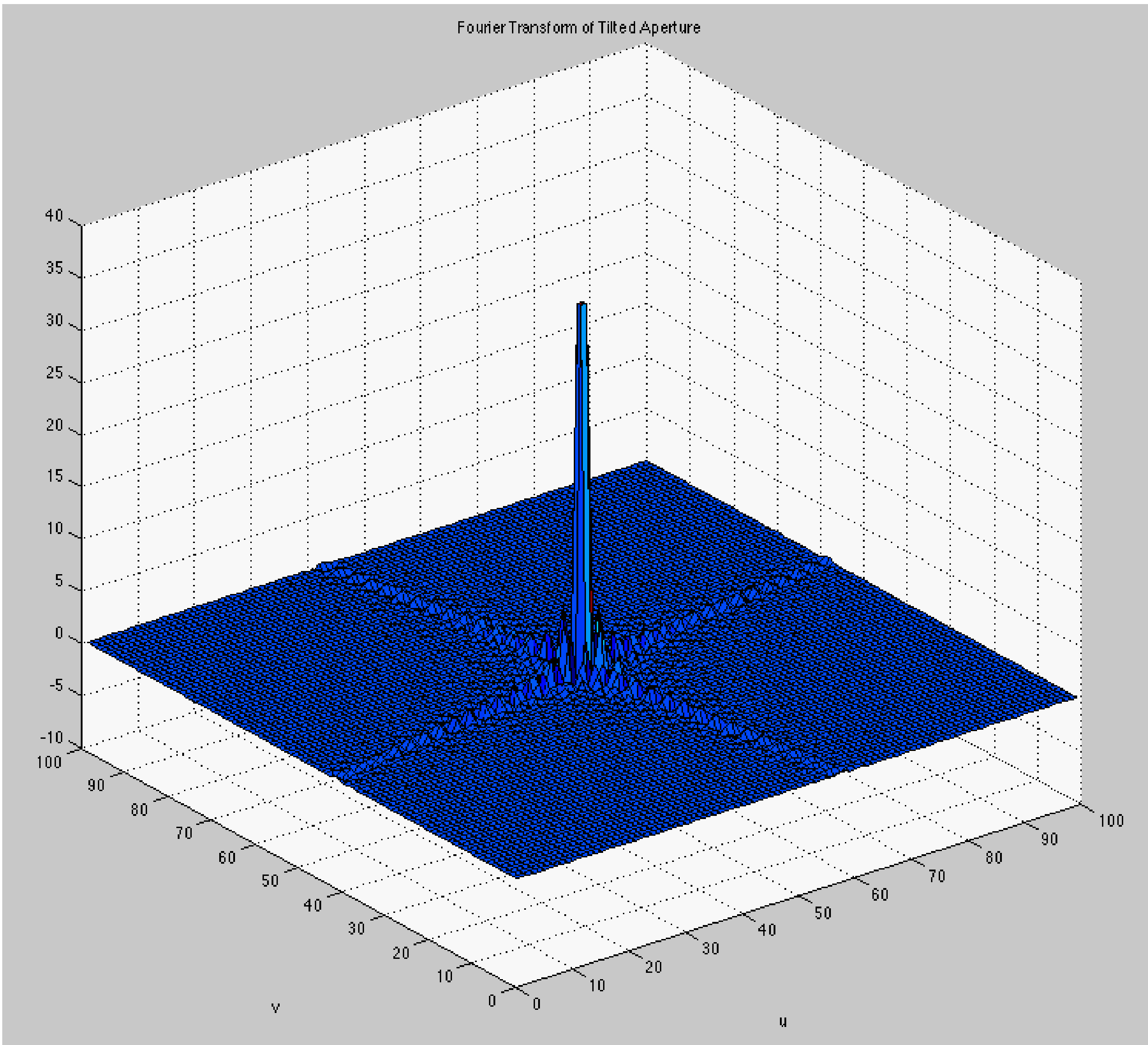
$$\mathcal{F}\{f(x,y)\} = \mathcal{F}(u,v) = ab\ \text{sinc}(bu)\,\text{sinc}(av)$$

Then note that a rotation in one domain translates into a rotation by the same amount in the other domain.

$$\mathcal{F}\{f(x,y)_{\text{tilted}}\} = ab\ \text{sinc}\big(b(u\cos\beta + v\sin\beta)\big)\,\text{sinc}\big(a(-u\sin\beta + v\cos\beta)\big)$$

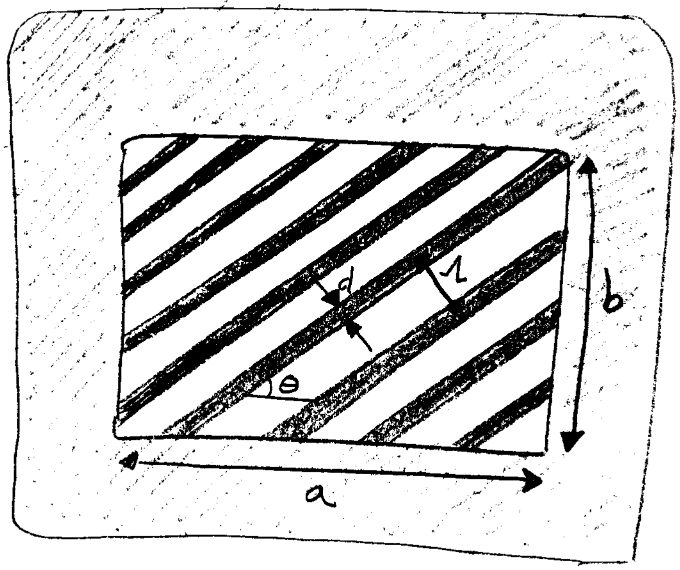
See figure on next page.

Problem 1



Fourier Transform of Tilted Aperture
40
35
30
25
20
15
10
5
0
-5
-10
100
90
80
70
60
50
40
30
20
10
0
v
0
10
20
30
40
50
60
70
80
90
100
u

Problem 2 Tilted Binary Grating

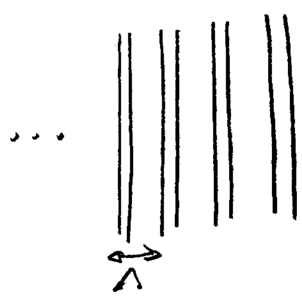


$\Lambda = 10\,\mu m$
$d = 2\,\mu m$
$\theta = 30^\circ$
$a = 5\,mm$
$b = 3\,mm$

Lets solve this problem by decomposing the structure into simple isolated components.

First, the grating:

1) Assume an infinite square grating (& not tilted)

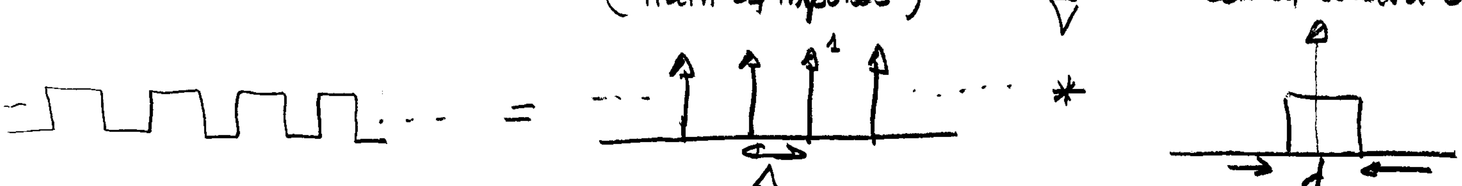


... ⇒ in Cross-section:

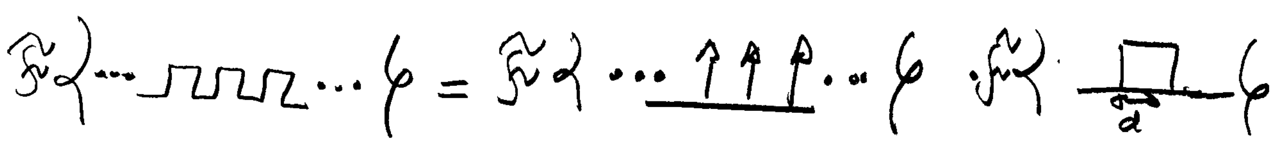
We can represent the grating as follows:

convolved

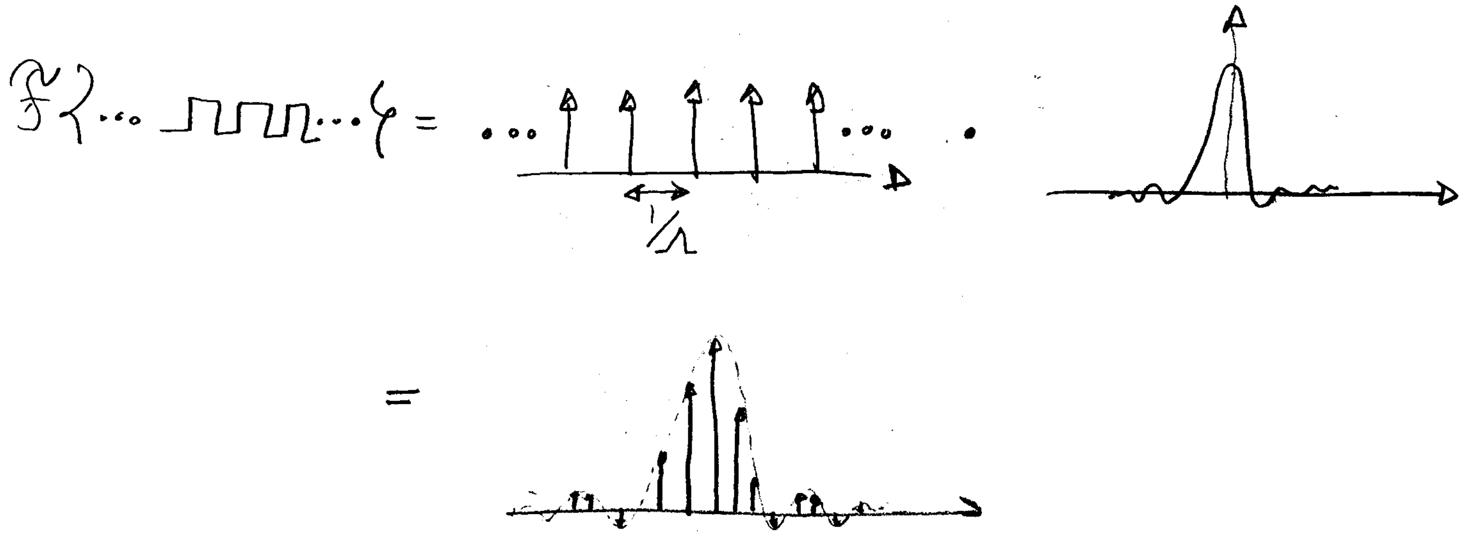
(Train of impulses) Box of width d



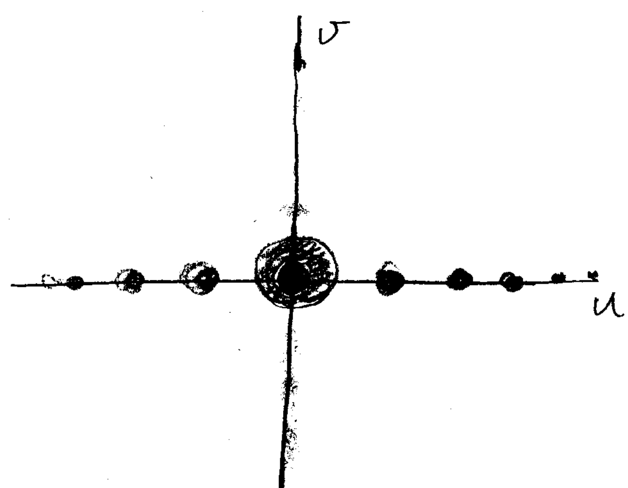
Then the Fourier Transfor is simple



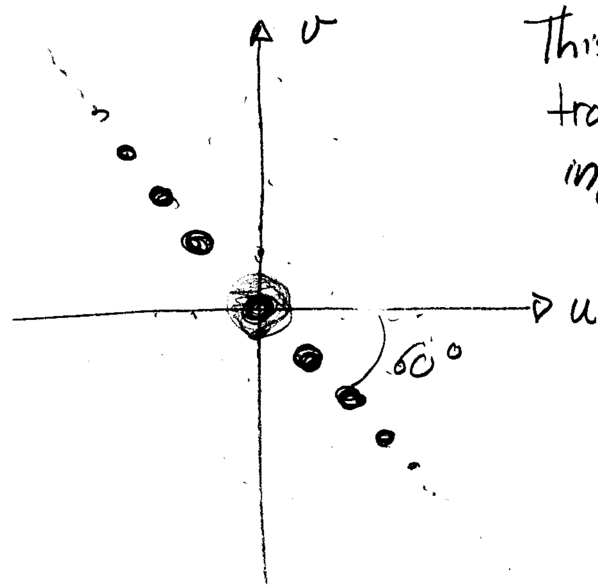
Since convolution in one domain is a multiplication in the other domain.



In 2D:



Now Rotate by $90-\theta = 60^\circ$

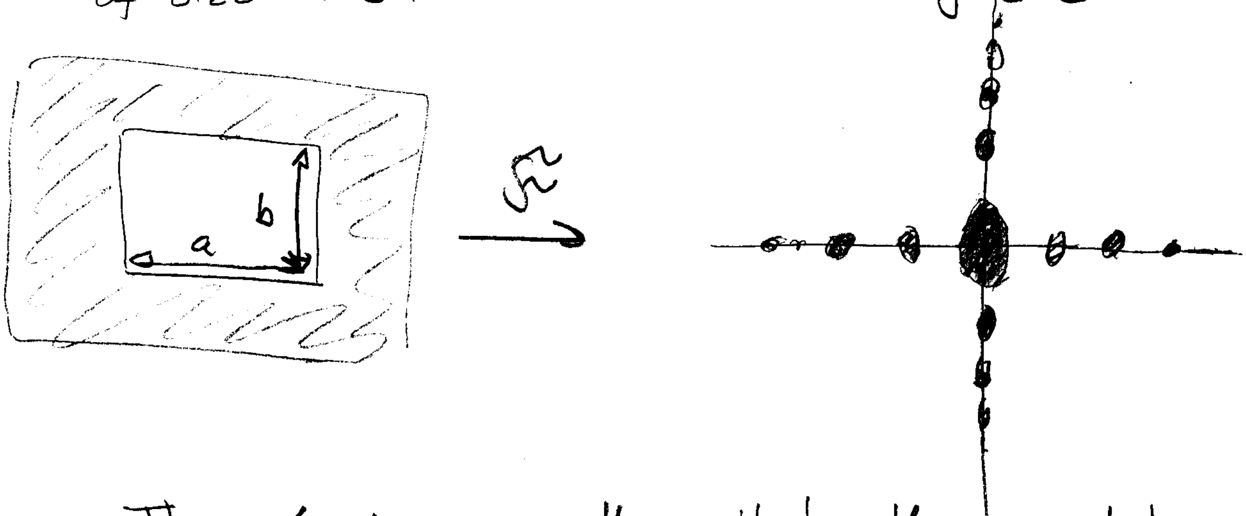


This is the Fourier transform of tilted infinite square grating.

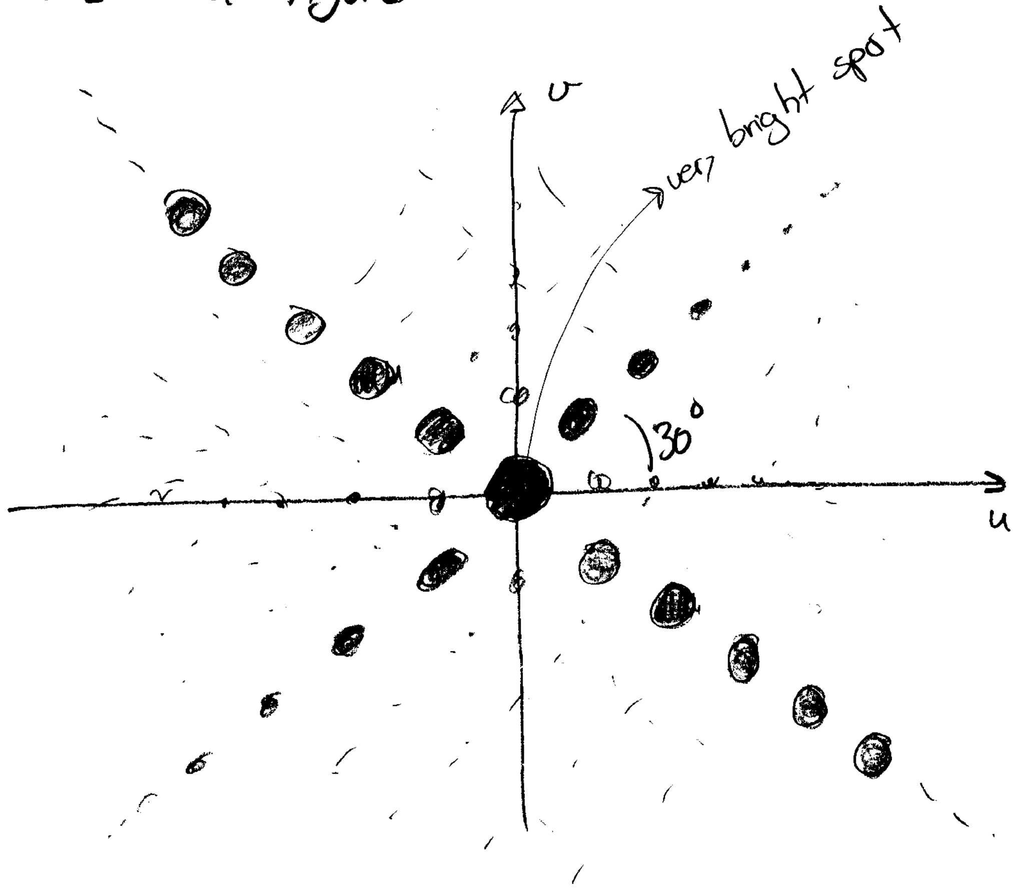
Figure 1

All that remains to be done is to convolve in frequency domain the previous result w/ the Fourier transform of an aperture of size a×b.

Figure 2



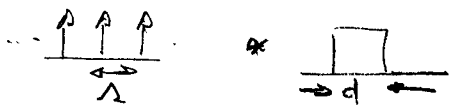
Then, the final result will be the convolution of Figure 1 and Figure 2.



Fourier Transform of tilted binary grating.

Analytically:

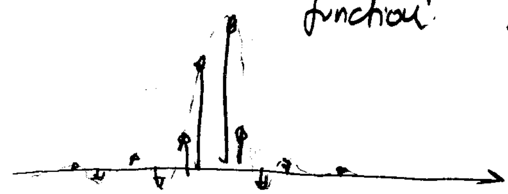
First the grating (square & infinite):



$$\sum_{n=-\infty}^{+\infty} \delta(y - n\Lambda) \xrightarrow{\mathcal{F}}$$

The Fourier Transform of the square wave will be of the form:

$$\rightarrow \sum_{n=-\infty}^{+\infty} \underbrace{\frac{2\sin\left(n\frac{2\pi}{\Lambda}\frac{d}{2}\right)}{n}}_{\text{sinc envelope function}} \underbrace{\delta\left(u - n\frac{2\pi}{\Lambda}, v\right)}_{\text{Impulses spaced by } \frac{2\pi}{\Lambda}}$$



Now, rotate this result cw 60°.

Rotation transformation: $u' = u\cos 60^\circ + v\sin 60$
$v' = -v\sin 60 + v\cos 60$

Then:

$$\underbrace{\sum_{n=-\infty}^{+\infty} \frac{2\sin\left(n\frac{2\pi}{\Lambda}\frac{d}{2}\right)}{n}\,\delta\left(\frac{1}{2}u + \frac{\sqrt{3}}{2}v - n\frac{2\pi}{\Lambda},\ -\frac{\sqrt{3}}{2}u + \frac{1}{2}v\right)}_{\text{Fourier Transform of infinite square grating rotated.}}$$

All that remains to be done is to convolve the above result with the Fourier transform of the square aperture.

$$\text{Solution} = \left[\sum_{n=-\infty}^{+\infty} \frac{2\sin\left(n\frac{2\pi}{\Lambda}\frac{d}{2}\right)}{n}\,\delta\left(\frac{1}{2}u + \frac{\sqrt{3}}{2}v - n\frac{2\pi}{\Lambda},\ -\frac{\sqrt{3}}{2}u + \frac{1}{2}v\right)\right] * ab\,\mathrm{sinc}(au)\,\mathrm{sinc}(bv)$$