2.710 Problem Set 8

– SOLUTIONS –

1 PARSEVAL'S THEOREM

Show:
$$\int_{-\infty}^{+\infty} |f(x)|^2 dx = \int_{-\infty}^{+\infty} |F(u)|^2 du$$

$$\int_{-\infty}^{+\infty} |f(x)|^2 dx = \int_{-\infty}^{+\infty} f(x) \cdot f^*(x)\, dx =$$

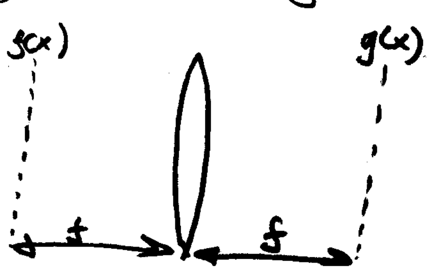
$$= \int_{-\infty}^{+\infty} f(x) \left[\int_{-\infty}^{+\infty} F^*(u)\, e^{-i 2\pi u x} du \right] dx$$

Reversing the order of integration we get:

$$\int_{-\infty}^{+\infty} |f(x)|^2 dx = \int_{-\infty}^{+\infty} F^*(u) \left[\int_{-\infty}^{+\infty} f(x)\, e^{-i 2\pi x u} dx \right] du.$$

$$= \int_{-\infty}^{+\infty} F^*(u)\, F(u)\, du = \int_{-\infty}^{+\infty} |F(u)|^2 du$$ Q.E.D.

(b) Take the following case:



We know that the relation between the input field and the output field is:

$$g(x) = C' \cdot \mathcal{F}\{f(x)\}$$ C': Complex Constant.

The intensity is then: $$|g(x)|^2 = |\mathcal{F}\{f(x)\}|^2$$

The total energy in the input beam is:

$$\int_{-\infty}^{+\infty} |f(x)|^2 dx.$$

And the total energy in the output beam is:

$$\int_{-\infty}^{+\infty} |\underbrace{\mathfrak{F}\{f(x)\}}_{F(u)}|^2 du$$

Hence,

$$\int_{-\infty}^{+\infty} |f(x)|^2 dx = \int_{-\infty}^{+\infty} |F(u)|^2 du.$$

$\Rightarrow$ This relation implies that energy is conserved in the case of a lossless optical system (like the ones we've modelled)

(a) Is the Fourier Transform Linear?

Recall linearity implies:

$$f_1(x) \longrightarrow \boxed{\mathcal{S}} \longrightarrow F(u)$$

$$g(x) \longrightarrow \boxed{\mathcal{S}} \longrightarrow G(u)$$

then $\alpha f(x) + \beta g(x) \longrightarrow \boxed{\mathcal{S}} \longrightarrow \alpha \tilde{F}(u) + \beta G(u)$

Apply to Fourier Transform:

$$\mathcal{F}\{\alpha f(x) + \beta g(x)\} = \int_{-\infty}^{+\infty} (\alpha f(x) + \beta g(x))\, e^{-j2\pi u x}\, dx$$

$$= \int_{-\infty}^{+\infty} \alpha f(x) e^{-j2\pi u x}\, dx + \int_{-\infty}^{+\infty} \beta\, g(x)\, e^{-j2\pi u x}\, dx = \underline{\underline{\alpha F(u) + \beta G(u)}}.$$

LINEAR SYSTEM.

(b) Given the fact that the Fourier Transform is <u>not</u> shift invariant, we can <u>not</u> define a transfer function.

3. $g(x', y') = f(x, y) * h(x, y)$.

$\Rightarrow G(u, v) = F(u, v) \cdot H(u, v)$.

$F(u, v) = \mathcal{F}\{f(x, y)\} = \left[\frac{1}{2}\delta(u) + \frac{1}{4}\delta(u - \frac{1}{\Lambda}) + \frac{1}{4}\delta(u + \frac{1}{\Lambda})\right]\delta(v)$.

$H(u, v) = x_0 \,\mathrm{rect}(x_0 u)\cdot\delta(v) \Leftarrow \begin{cases} -\frac{1}{2x_0} \le u \le \frac{1}{2x_0} & = 1 \\ |u| > \frac{1}{2x_0} & = 0. \end{cases}$

so. when $-\frac{1}{2x_0} \le \frac{1}{\Lambda} \le \frac{1}{2x_0} \Rightarrow -\frac{\Lambda}{2} \le x_0 \le \frac{\Lambda}{2}$

$G(u, v) = \frac{x_0}{2}\left[\delta(u) + \frac{1}{2}\delta(u - \frac{1}{\Lambda}) + \frac{1}{2}\delta(u + \frac{1}{\Lambda})\right]\delta(v)$.

when. $|x_0| > \frac{\Lambda}{2}$

$G(u, v) = \frac{x_0}{2}\delta(u)\,\delta(v)$.

$$g(x', y') = \mathcal{F}^{-1}\{g(x', y')\}. = \begin{cases} \frac{x_0}{2}\left[1 + \cos(2\pi\frac{x'}{\Lambda})\right] & -\frac{\Lambda}{2} \le x_0 \le \frac{\Lambda}{2} \\ \frac{x_0}{2} & |x_0| > \frac{\Lambda}{2} \end{cases}$$

4.a). The field ~~after~~ before aperture is.

$G(\frac{x'}{\lambda f})$. where $G(u) = \mathcal{F}\{g(x)\}$.

$g(x) = \frac{1}{2}\left[1 + \cos(2\pi\frac{x}{\Lambda})\right]$.

$\Rightarrow G(u) = \frac{1}{2}\left[\delta(u) + \frac{1}{2}\delta(u - \frac{1}{\Lambda}) + \frac{1}{2}\delta(u + \frac{1}{\Lambda})\right]$.

field $g(x') = \frac{1}{2}\left[\delta(x') + \frac{1}{2}\delta(\frac{x'}{\lambda f} - \frac{1}{\Lambda}) + \frac{1}{2}\delta(\frac{x'}{\lambda f} + \frac{1}{\Lambda})\right]$

$= \frac{1}{2}\left[\delta(x') + \frac{1}{2}\delta(x' - \frac{\lambda f}{\Lambda}) + \frac{1}{2}\delta(x' + \frac{\lambda f}{\Lambda})\right]$.

After the aperture.

① If $\frac{a}{2} \ge \frac{\lambda f}{\Lambda} \Rightarrow a \ge \frac{2\lambda f}{\Lambda}$

$$g_{out}(x') = \frac{1}{2}\left[\delta(x') + \frac{1}{2}\delta(x' - \frac{\lambda f}{\Lambda}) + \frac{1}{2}\delta(x' + \frac{\lambda f}{\Lambda})\right].$$

② If $\frac{a}{2} < \frac{\lambda f}{\Lambda} \Rightarrow a < \frac{2\lambda f}{\Lambda}$.

$$g_{out}(x') = \frac{1}{2}\delta(x').$$

4.b). Here ~~the~~ after the aperture, the optical system calculated. $G(u,v)$ in the coordinates $x' = u \cdot \lambda f$ and $y' = v \cdot \lambda f$.

The aperture placed at the Fourier plane is equivalent to the "sinc" function in input plane (Fourier pairs).

This relationship can be find as.

$H(u,v) = x_0 \operatorname{rect}(x_0 u)$ in P. 3.

Here, the aperture is. $\operatorname{rect}(\frac{x'}{a})$. and $x' = u \cdot \lambda f$.

$$\Rightarrow \operatorname{rect}\left(\frac{u \lambda f}{a}\right). \Rightarrow \boxed{x_0 = \frac{\lambda f}{a}}.$$

5. The second lens do another Fourier transform, which is equivalent to Inverse Fourier tranform except that the sign is flipped,

So we have at x'' plane is.

$$g_{out}(x'') = \begin{cases} \frac{1}{2}\left[1 + \cos(2\pi \frac{x''}{\Lambda})\right] & a \geq \frac{2\lambda f}{\Lambda} \\ \frac{1}{2} & a < \frac{2\lambda f}{\Lambda} \end{cases}$$

cosine is a even function the flip of sign can not be seen here

Problem 6 solution

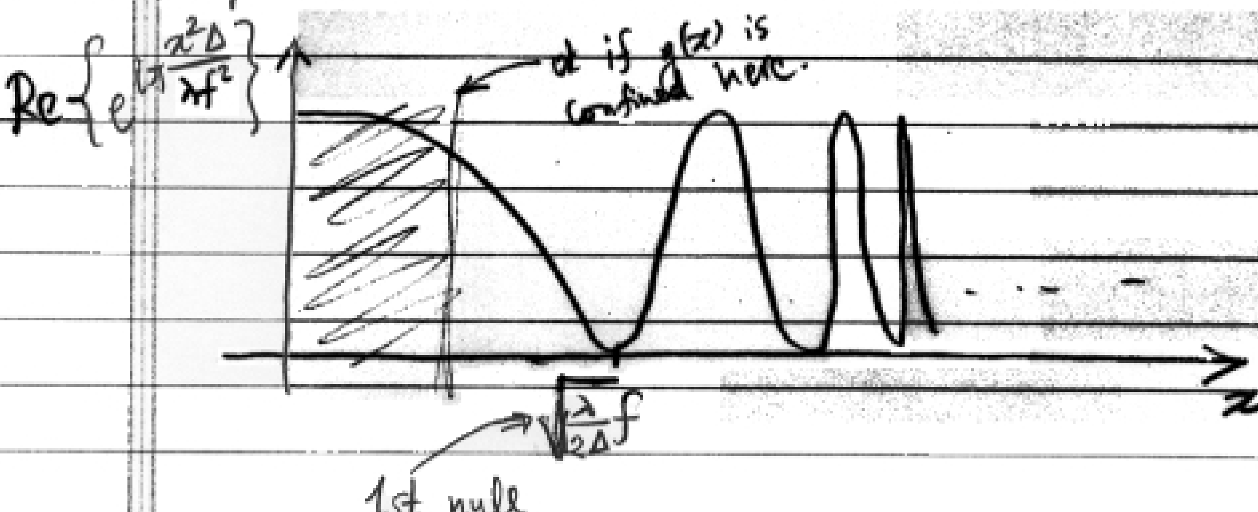
Field at displaced plane:

$$\int \underbrace{g(x)}_{\text{input pattern } |x|<\frac{D}{2}} \underbrace{e^{-i\pi \frac{x^2}{\lambda f}}}_{\text{lens}} \underbrace{e^{i\pi \frac{(x'-x)^2}{\lambda(f-\Delta)}} dx}_{\text{Fresnel kernel}}$$

$$= (x)\, e^{-i\pi \frac{x^2}{\lambda f}}\, e^{+i\pi \frac{x'^2+x^2-2xx'}{\lambda(f-\Delta)}}$$

$$= e^{+i\pi \frac{x'^2}{\lambda(f-\Delta)}} \int \underbrace{g(x)\, e^{-i2\pi \frac{xx'}{\lambda(f-\Delta)}}}_{\text{Fourier transform term}} \underbrace{e^{i\pi \frac{x^2}{\lambda}\left(\frac{1}{f-\Delta}-\frac{1}{f}\right)}}_{\text{defocus term}} dx$$

Defocus term $\quad e^{i\pi \frac{x^2}{\lambda}\left(\frac{1}{f-\Delta}-\frac{1}{f}\right)} \simeq e^{i\pi \frac{x^2\Delta}{\lambda f^2}}$

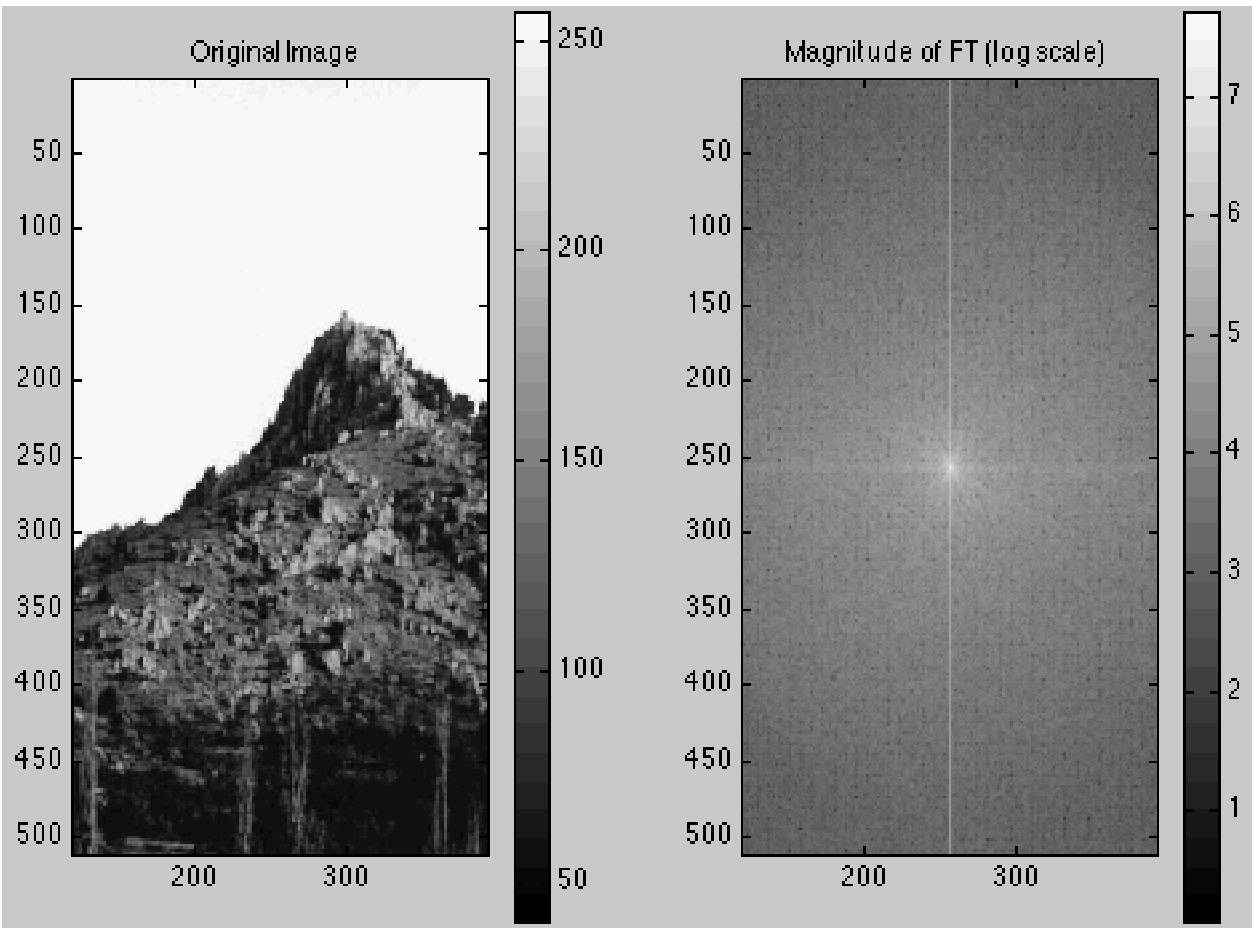


Defocus does not do much damage if $\frac{D}{2} \lesssim \frac{1}{2}\sqrt{\frac{\lambda}{2\Delta}}f$

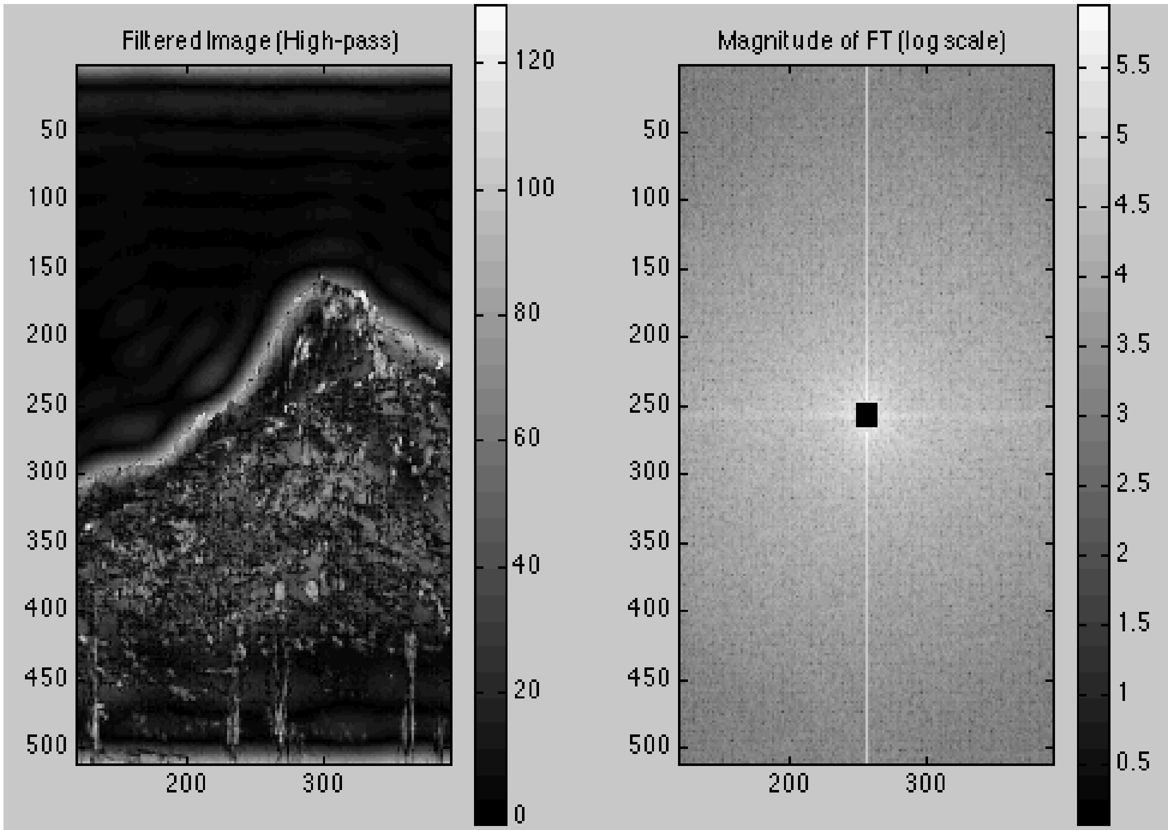
i.e. $D \lesssim \sqrt{2\Delta}\ J$.

Problem 5

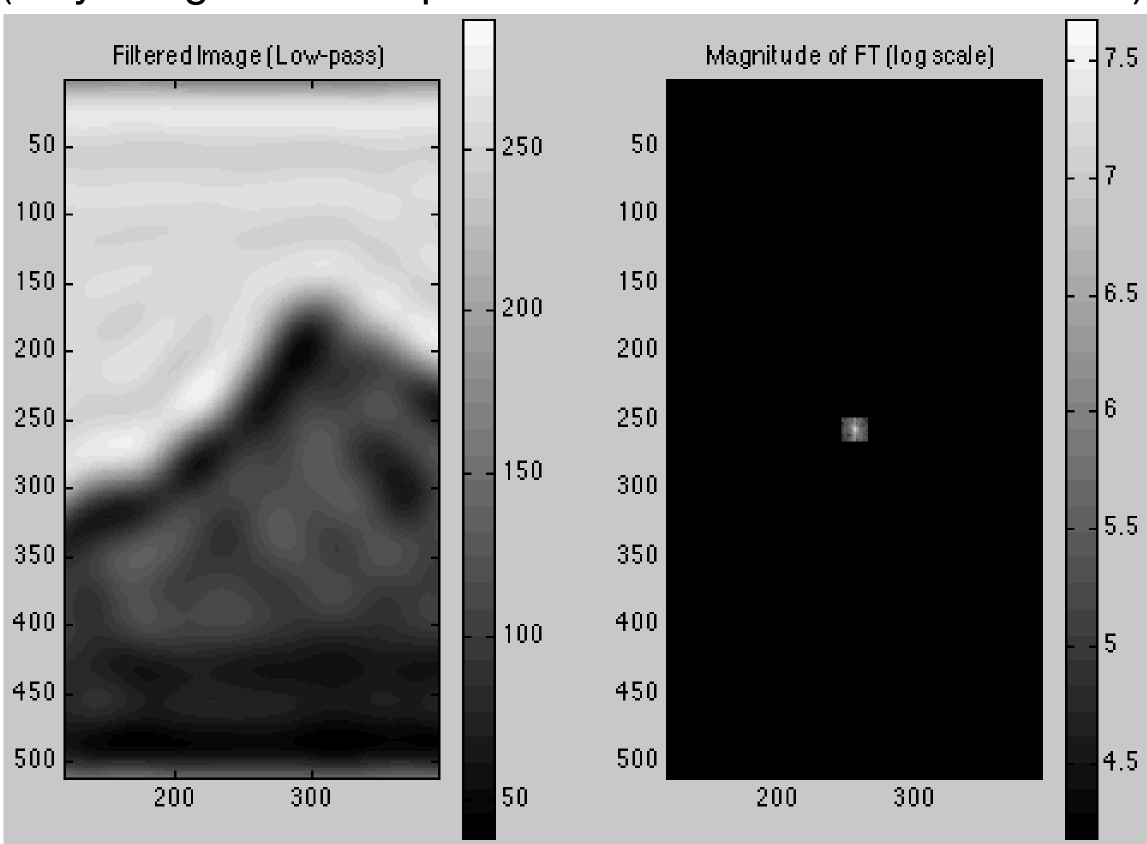
Plot of the Original Image and its Fourier Transform



High-pass Filter: Edges get enhanced



Low-pass Filter: Soften the Edges
(only the general shape of the mountain can be detected)



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% Code for Problem 3, HW7, 2.710 (Optics)
% Solutions Fall 2001, Dario Gil

clear all;

x=8;%Pixels blocked = 2*(x+1)

A=imread('Tejera','jpg'); %Read the image to be processed


figure(1)

subplot(1,2,1), imagesc(A)
axis equal; colorbar;
title('Original Image')

subplot(1,2,2), imagesc(log10(abs(fftshift(fft2(A)))))
axis equal; colorbar;
title('Magnitude of FT (log scale)')
colormap('gray')

% Part (b)
FTA=fftshift(fft2(A));
B=FTA;
i=length(A)/2-x:1:length(A)/2+(x+1);
j=i;
B(i,j)=0;

figure(2)
subplot(1,2,1), imagesc(abs(ifft2(B)))
axis equal; colorbar;

title('Filtered Image (High-pass)')
subplot(1,2,2), imagesc(log10(abs(B)))
axis equal; colorbar;
title('Magnitude of FT (log scale)')
colormap('gray')

%part(c)

C=zeros(size(A));
C(i,j)=1;
C=C.*FTA;

figure(3)
subplot(1,2,1), imagesc(abs(ifft2(C)))
axis equal; colorbar;

title('Filtered Image (Low-pass)')
subplot(1,2,2), imagesc(log10(abs(C)))
axis equal; colorbar;
title('Magnitude of FT (log scale)')
colormap('gray')

zoom on
```