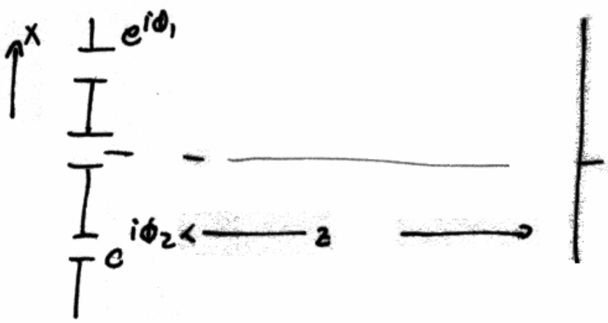
2·71/2·710

Problem Set 9 Solutions

We want a minima on axis. Let us consider that each off-axis slit has a certain phase delay associated with it as shown

(a)



The transparency can be written as

$$t(x) = e^{i\phi_1}\,\text{rect}\left(\frac{x-d}{a}\right) + \text{rect}\left(\frac{x}{a}\right) + e^{i\phi_2}\,\text{rect}\left(\frac{x+d}{a}\right)$$

The Fraunhofer diffraction pattern is the Fourier transform of $t(x)$ $\mathcal{F}\{t(x)\}$

$$T(f_x) = a\,e^{i\phi_1}e^{-i2\pi f_x d}\,\text{sinc}(af_x) + a\,\text{sinc}(af_x) + e^{i\phi_2}e^{+i2\pi f_x d}\,a\,\text{sinc}(af_x)$$

$$f_x = x'/\lambda z$$

$$\therefore\ T(x') = a\,\text{sinc}\left(\frac{ax'}{\lambda z}\right)\left[1 + e^{i\phi_1}e^{\frac{-i2\pi x' d}{\lambda z}} + e^{i\phi_2}e^{\frac{+i2\pi x' d}{\lambda z}}\right]$$

on axis minimum, $x'=0$

$$T(0) = a\left[1 + e^{i\phi_1} + e^{i\phi_2}\right] = 0$$

$\Rightarrow$ $\sin\phi_1 + \sin\phi_2 = 0$ $\Rightarrow$ $\phi_1 = 2m\pi - \phi_2$

$m = 0, 1, 2 \ldots$

$1 + \cos\phi_1 + \cos\phi_2 = 0$

Take $\phi_1 = -\phi_2$

$1 + 2\cos\phi_1 = 0$

$\Rightarrow$ $\cos\phi_1 = -1/2$

$\Rightarrow$ $\phi_1 = 2n\pi +$

$n = 0, 1, 2.$

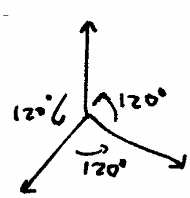
Take $n=0$, $\phi_1 = 120°$

$\phi_2 = -120°$

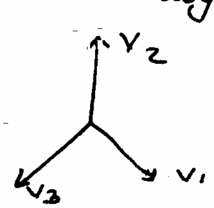
We see that we need to impose relative phase delays of 120° in between each slit to ensure destructive interference at $x'=0$

(b) Analogy $\rightarrow$

Mechanical analogy $\rightarrow$ Three coplanar, non colinear must be inclined 120° with respect to each other for equilibrium



Electrical analogy

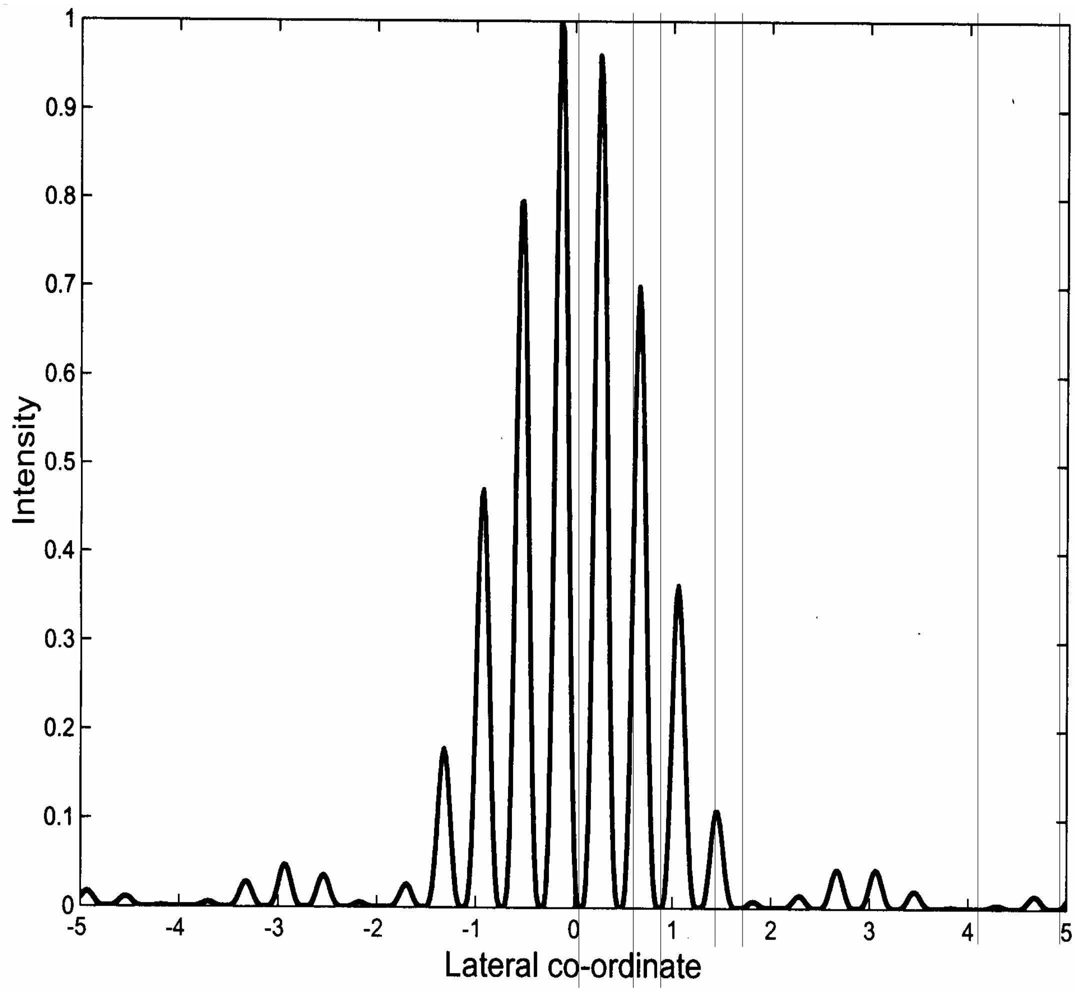


Voltages 120° out of phase are used to transmit power

(e) Assume $\phi \quad \phi_2 \quad 120^\circ \quad \frac{2\pi}{3}$

$$T(\ ') \quad a \, \mathrm{sinc}\left(\frac{ax}{\lambda z}\right)\left[\ 1 + \ 2\cos\left(\frac{2\pi x' d}{\lambda z} + \frac{2\pi}{3}\right)\right]$$

Plot attached



Intensity
Lateral co-ordinate
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-5
-4
-3
-2
-1
0
1
2
3
4
5

5. a) $I(x) = \frac{1}{2}\left[1 + \frac{1}{2}\cos\left(2\pi \frac{x}{40\mu m}\right) + \frac{1}{2}\cos\left(2\pi \frac{3x}{40\mu m}\right)\right]$

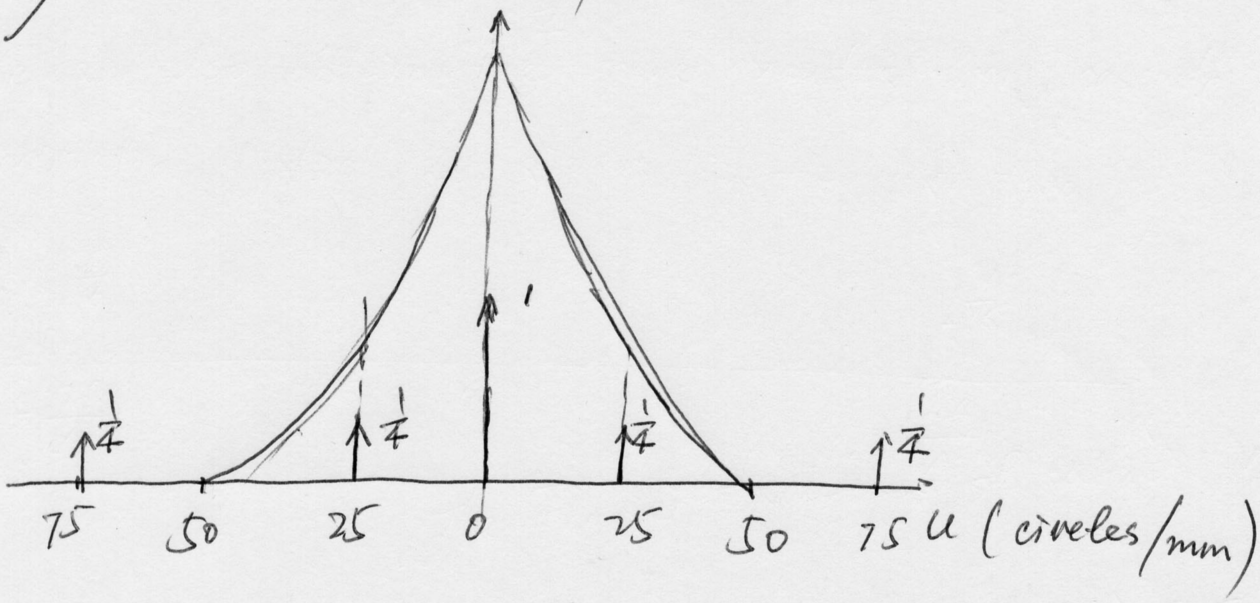
$I_{max} = 1 \qquad I_{min} = 0$

$$\text{contrast} = \frac{I_{max} - I_{min}}{I_{max} + I_{min}} = 1$$

b) The Fourier transform of $I(x)$ is

$$\frac{1}{2}\Big[\delta(u) + \frac{1}{4}\delta\left(u + \frac{1}{40\mu m}\right) + \frac{1}{4}\delta\left(u - \frac{1}{40\mu m}\right)$$
$$+ \frac{1}{4}\delta\left(u + \frac{3}{40\mu m}\right) + \frac{1}{4}\delta\left(u - \frac{1}{40\mu m}\right)\Big]$$

$\frac{1}{40\mu m} = 25$ circles/mm



The Fourier transform of output is

$$\frac{1}{2}\left[\delta(u) + \frac{1}{8}\delta\left(u + \frac{1}{40\mu m}\right) + \frac{1}{8}\delta\left(u - \frac{1}{40\mu m}\right)\right]$$

$$= \frac{1}{2}\left(1 + \frac{1}{4}\cos\left(2\pi\frac{x}{40\mu m}\right)\right)$$

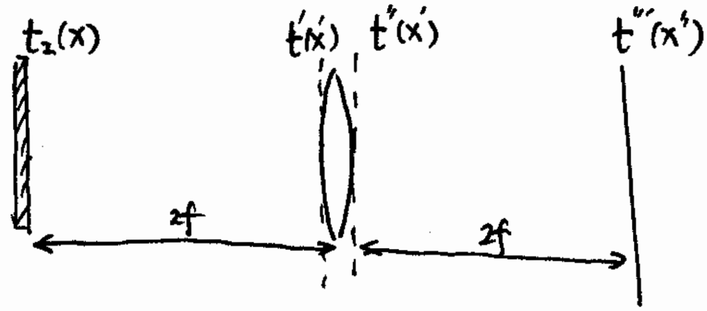
$$\text{contrast} = \frac{1}{4}$$

c) The coherent transfer function looks like triangle function

$$\Lambda\left(\frac{u}{25 mm^{-1}}\right)$$

The cut-off frequency is half of incoherent imaging system. 25 circles/mm

Problem 3.



From page 113, "Goodman", we know the impulse response of this system can be written as:

$$\tilde{h}(x'') = \int_{-\infty}^{+\infty} P(2\lambda f \tilde{x}) \exp[-j2\pi x'' \tilde{x}]\, d\tilde{x}$$

where $\tilde{x} = \dfrac{x'}{2\lambda f}$

so the Amplitude Transfer Function of the system is.

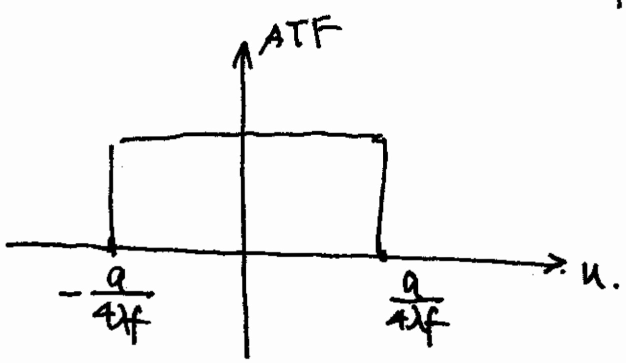
$$H(u) = \mathcal{F}\{\tilde{h}(x'')\}$$
$$= \frac{1}{2\lambda f} P(-2\lambda f u).$$

we assume the diameter of the lens is a.

so $P(x')$ take the form of.

$$P(x') = \mathrm{rect}\left(\frac{x'}{a}\right).$$

Then $H(u) = \dfrac{1}{2\lambda f} \mathrm{rect}\left(\dfrac{2\lambda f u}{a}\right).$



Then sinusoidal freq. of the input is -

$$u_i = \frac{1}{\Lambda}.$$

so. if $\frac{1}{\Lambda} \leq \frac{a}{4\lambda f}$.

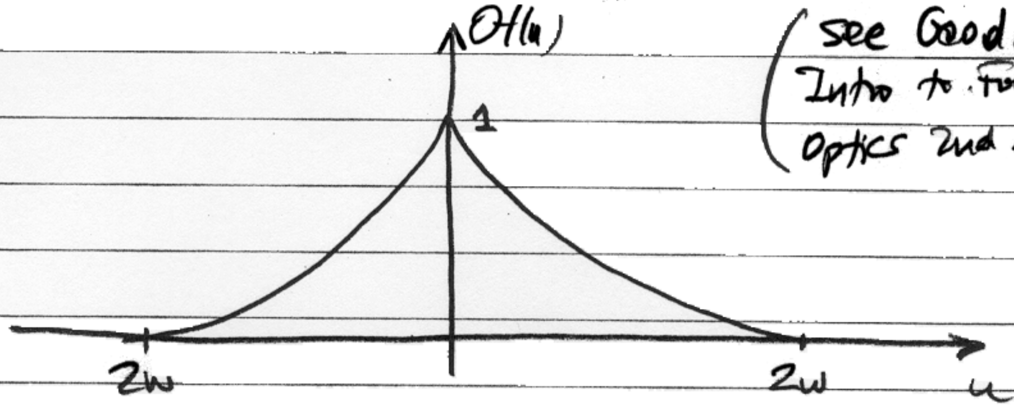
$$\Rightarrow \boxed{\Lambda \geq \frac{4\lambda f}{a}}$$

the image can be formed by coherent illumination.

For incoherent illumination. we know the cut-off freq. is doubled. so the eretiaria is.

$$\boxed{\Lambda \geq \frac{2\lambda f}{a}.}$$

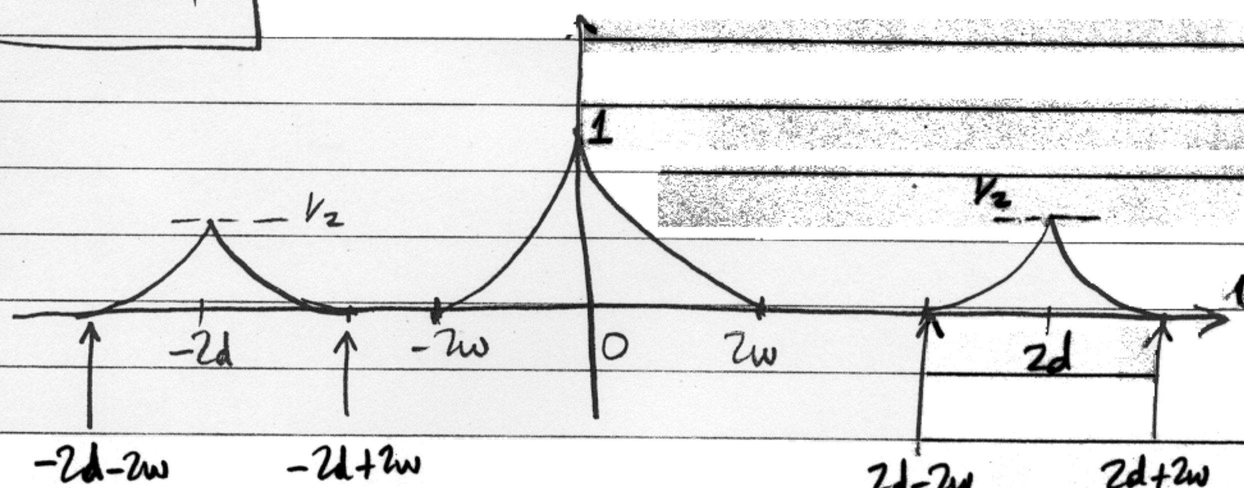
(4) Autocorrelation of a ~~single~~ circular aperture of radius w.



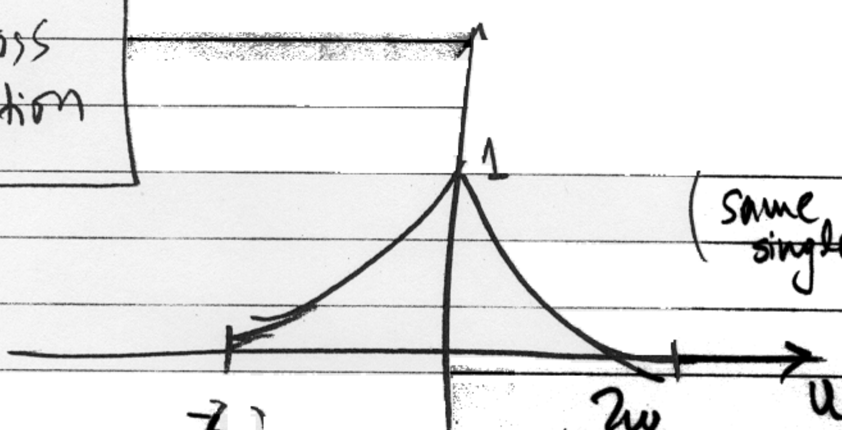
(see Goodman Intro to Fourier Optics 2nd ed p.1

For 2 apertures arranged across the y axis,

v-cross section



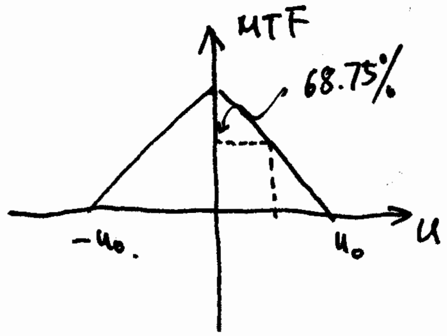
u- cross section



(same as single ci

Problem 5.

The MTF for a diffraction-limited system is a triangle.



More rigorously, the MTF is given as page 143 "Goodman" if the aperture is circular. However, for simplicity, we use triangle as here.

Then we know

$$u_0 = \frac{25\,mm^{-1}}{1-0.6875} = 80\ mm^{-1}.$$

(a) The contrast at $50mm^{-1}$ is.

$$\left(1-\frac{50}{80}\right)\times 100\% = 37.5\%.$$

(b) If the system is illuminated by coherent light, the cut-off freq. is $40\ mm^{-1}$. So $50mm^{-1}$ fringe is not visible.

(c) If the illumination is off-axis, then the spatial freq. of the object can be shifted down by the illumination. so it is possible to see the freq. if the illumination is inclined.

Problem 6.

(a) The incoherent PSF is.

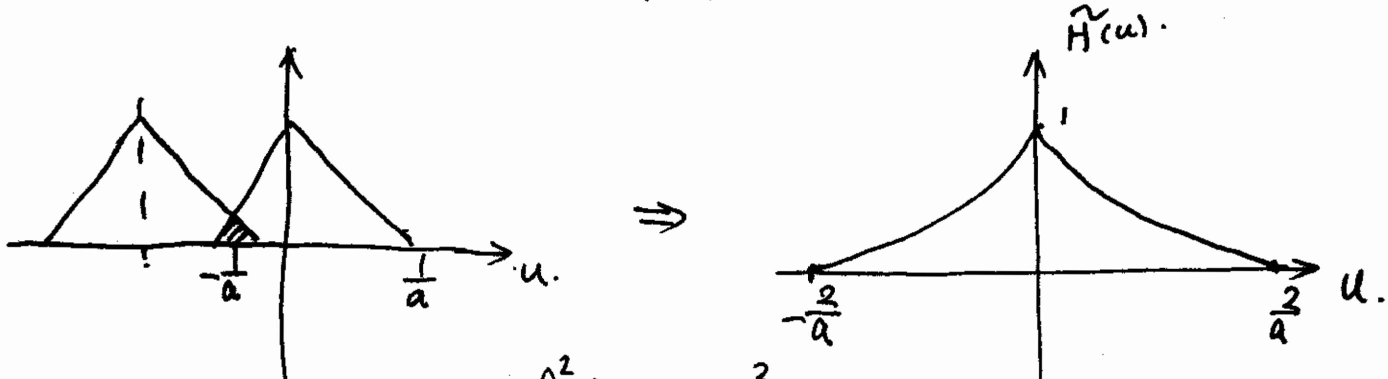
$$\tilde{h}(x) = |h(x)|^2 = \mathrm{sinc}^4\left(\frac{x}{b}\right).$$

(b). MTF is

$$\tilde{H}(u) = \mathcal{F}\{\tilde{h}(x)\} = [a\,\mathrm{rect}(au)] * [a\,\mathrm{rect}(au)] * [a\,\mathrm{rect}(au)] * [a\,\mathrm{rect}(au)].$$

If we normalize the MTF to "1". then.

$$\tilde{H}(u) = a^4 \cdot \Lambda(au) * \Lambda(au).$$



$$\tilde{H}(u) = \begin{cases} \frac{a^2}{4}\left(|u| - \frac{2}{a}\right)^2 & |u| \leq \frac{2}{a} \\ 0 & |u| > \frac{2}{a}. \end{cases}$$