3. In this question we will be looking at how an instrumentation amplifier works, by breaking it up into stages. Use the ideal op amp approximation: The voltage at both input terminals should be identical, with no current flowing between them.

Solution

a) For this unit shown above, what is the voltage at $V_{\text{out}}$ in terms of $V_a$ and $V_b$? (Hint: Start by finding the voltage at the + terminal of the op amp).

Solution

Step 1: Recognize that there is a voltage divider at the (+) terminal. Then:

$$V_+ = V_b \left( \frac{R_3}{R_3 + R_2} \right)$$

Step 2: Ideal op amp: $V_- = V_+$, so the current through $R_2$ is $I_2 = \frac{1}{R_2} \left( V_a - V_b \left( \frac{R_3}{R_3 + R_2} \right) \right)$

Step 3: Ideal op amp: All current through $R_2$ continues through $R_3$. Then

$$V_{\text{out}} = V_- - I_2 R_3$$

$$V_{\text{out}} = V_b \left( \frac{R_3}{R_3 + R_2} \right) - \frac{R_3}{R_2} \left( V_a - V_b \left( \frac{R_3}{R_3 + R_2} \right) \right)$$

Step 4: Algebra. Notice that, algebraically, $\left( \frac{R_3}{R_3 + R_2} \right) \left( 1 + \frac{R_3}{R_2} \right) = \frac{R_3}{R_2}$. This will simplify things immensely.

$$V_{\text{out}} = \frac{R_3}{R_2} (V_b - V_a)$$
If they don’t notice that \( \left( \frac{R_3}{R_3 + R_2} \right) \left( 1 + \frac{R_3}{R_2} \right) = \frac{R_3}{R_2} \), the expression will be much uglier, and looks like this:

\[
V_{\text{out}} = V_b \left( \frac{R_3}{R_3 + R_2} \right) \left( 1 + \frac{R_3}{R_2} \right) - \frac{R_3}{R_2} V_a
\]

b) For the unit shown above, what is the voltage at \( V_\alpha \) in terms of \( V_1 \) and \( V_c \)? What kind of amplifier does this resemble if \( V_c = 0 \), and what does \( V_\alpha/V_1 \) become in that case?

Solution

Step 1: The voltage at \( V_\alpha \) is given by noticing the voltage divider at the inverting input:

\[
V_\alpha = (V_\alpha - V_c) \frac{R_{\text{gain}}/2}{(R_1 + R_{\text{gain}}/2)} + V_c
\]

Step 2: Then because \( V_\alpha = V_1 \) for an ideal op amp,

\[
V_1 = (V_\alpha - V_c) \frac{R_{\text{gain}}/2}{(R_1 + R_{\text{gain}}/2)} + V_c
\]

Rearranging to solve for \( V_\alpha \):

\[
V_\alpha = V_c + (V_1 - V_c) \frac{R_1 + R_{\text{gain}}/2}{R_{\text{gain}}/2}
\]

If \( V_c = 0 \), then this is a non-inverting amplifier, and \( V_\alpha/V_1 \) becomes

\[
\frac{V_\alpha}{V_1} = \frac{R_1 + R_{\text{gain}}/2}{R_{\text{gain}}/2}
\]
c) Two such amplifiers are connected together, as shown above. What is the voltage $V_c$ in terms of $V_1$ and $V_2$? Knowing this, and using the results from part b, what are $V_a$ and $V_b$ in terms of $V_1$ and $V_2$?

Solution

Step 1. Notice that the inverting (-) terminals of each op amp connect to the resistor on either side of $V_c$. Because the op amps are ideal, the voltage at their inverting terminal is equal to the voltage at the (+) terminal. So on one side it is $V_1$, and on the other, $V_2$.

Step 2. Then, the rest of the circuit can be ignored; $V_c$ is found just by a voltage divider between $V_1$ and $V_2$. It turns out to be the average voltage.

$$V_c = \frac{V_1 + V_2}{2}$$

Step 3. Knowing $V_c$, the top and bottom halves of the circuit can be treated in isolation, and using the results from part b,

$$V_a = V_c + (V_1 - V_c) \frac{R_1 + R_{\text{gain}}/2}{R_{\text{gain}}/2} = \frac{V_1 + V_2}{2} + \frac{V_1 - V_2}{2} \frac{R_1 + R_{\text{gain}}/2}{R_{\text{gain}}/2}$$

$$V_b = V_c + (V_2 - V_c) \frac{R_1 + R_{\text{gain}}/2}{R_{\text{gain}}/2} = \frac{V_1 + V_2}{2} + \frac{V_2 - V_1}{2} \frac{R_1 + R_{\text{gain}}/2}{R_{\text{gain}}/2}$$

After some algebra:

$$V_a = \frac{V_1 + V_2}{2} + \frac{(V_1 - V_2)}{2} \frac{2R_1 + R_{\text{gain}}}{R_{\text{gain}}}$$

$$V_b = \frac{V_1 + V_2}{2} + \frac{(V_2 - V_1)}{2} \frac{2R_1 + R_{\text{gain}}}{R_{\text{gain}}}$$
d) Now the circuit from part a is attached to the front, to create the full “instrumentation amplifier” circuit above. What is $V_{out}$ in terms of $V_1$ and $V_2$? How is it affected by changing $R_{gain}$?

Solution

Step 1. Using the results from part a,

$$V_{out} = \frac{R_3}{R_2} (V_b - V_a)$$

And part c,

$$V_a = \frac{V_1 + V_2}{2} + \left(\frac{V_1 - V_2}{2}\right) \frac{2R_1 + R_{gain}}{R_{gain}}$$

$$V_b = \frac{V_1 + V_2}{2} + \left(\frac{V_2 - V_1}{2}\right) \frac{2R_1 + R_{gain}}{R_{gain}}$$

$$V_{out} = \frac{R_3}{R_2} \left[\frac{V_1 + V_2}{2} + \left(\frac{V_2 - V_1}{2}\right) \frac{2R_1 + R_{gain}}{R_{gain}} - \frac{V_1 + V_2}{2} + \left(\frac{V_1 - V_2}{2}\right) \frac{2R_1 + R_{gain}}{R_{gain}} \right]$$

Step 2. Combine like terms and cancel. The $\frac{V_1 + V_2}{2}$ terms cancel. The difference terms combine.

$$V_{out} = \frac{R_3}{R_2} \left[\frac{2R_1 + R_{gain}}{R_{gain}} \right] (V_2 - V_1)$$

The result depends only on the difference between $V_2$ and $V_1$, not on either of their actual voltages. The effect of $R_{gain}$ is shown here too; decreasing it increases the gain of the amplifier. The answer can also be written as:

$$V_{out} = \frac{R_3}{R_2} \left[1 + 2 \frac{R_1}{R_{gain}} \right] (V_2 - V_1)$$
4. The following circuit is called an LR filter. It uses an inductor instead of a capacitor to filter a signal.

![LR Filter Circuit](image)

An inductor has the voltage-current relationship \( V = L \frac{dl}{dt} \), which can also be expressed as an impedance \( V = j \omega L \).

a) Considering this, what kind of filter is shown above?

Solution:
It's a low-pass filter. The inductor will be like a short circuit for low frequencies, but an open circuit for high frequencies.

Write the transfer function for the circuit \( \frac{V_{out}}{V_{in}} \) as a function of frequency \( \omega \).

Solution:
Treat it as a voltage divider with a frequency-dependent impedance \( X_L \):

\[
V_{out} = V_{in} \left( \frac{R}{R + X_L} \right)
\]

\[
X_L = j \omega L
\]

The transfer function is therefore:

\[
\frac{V_{out}}{V_{in}} = \frac{R}{R + j \omega L}
\]

For low frequencies, \( \omega \to 0 \) and \( V_{out} = V_{in} \). For high frequencies, \( \omega \to \infty \) and \( V_{out} = 0 \).

b) Choose reasonable values of \( L \) and \( R \) in order to filter out a frequency of 60 Hz.

Solution:
"Filter out" 60 Hz is subjective and up to the student to decide. We can choose to reduce the magnitude of the voltage by about a factor of 10 at 60 Hz. First note that 60 Hz = \( \frac{120\pi}{s} \). Let \( \frac{120\pi}{377} = \omega \).

\[
\frac{1}{10} = \left| \frac{R}{R + j \omega L} \right| = \left| \frac{1}{1 + \frac{j \omega L}{R}} \right|
\]

\[
\left| 1 + \frac{j \omega L}{R} \right| = 10
\]

Students can either do a more-than-good-enough approximation by discarding the \( j \) and just saying...
\[
\frac{\omega L}{R} \approx 10
\]

So

\[
\frac{L}{R} = \frac{10}{\omega} = 0.0265 \text{ s}
\]

Or jumping through the hoops of adding the complex numbers properly, in which case

\[
\sqrt{1 + \frac{\omega^2 L^2}{R^2}} = 10
\]

\[
\frac{\omega L}{R} = \sqrt{99} = 9.95
\]

\[
\frac{L}{R} = 0.0264 \text{ s}
\]

Either way it’s still more accurate than you need.

Next, choose values for \(L\) and \(R\). There’s some freedom here, but \(L\) should be not larger than 500 mH for practical values, and not much smaller than 1 \(\mu\)H. Likewise, \(R\) should be greater than 1 \(\Omega\) and less than 10 \(M\)\(\Omega\). For example

\[
L = 100 \text{ mH}
\]

\[
R = 3.8 \text{ } \Omega
\]

Is acceptable.

c) In what kind of practical application would inductors be used in a real filter?

Solution:

Not in an LR filter for signals!

Inductors, when they’re used, would always be used in combination with a capacitor to make an LC resonant filter, which has a very narrow peak.

Inductors will always be found in power electronics, where filters can’t have resistors because they would dissipate too much power. So all power electronics filters are basically LC filters.

In the olden days, LC filters would be used for radio tuning and telecom and other applications that required a sharp, narrowband filter. But now inductors are rarely used for this because they have too many imperfections (stray capacitance, parasitic resistance, nonlinearities).