2. Pinned Joint problem

a) Draw a free-body diagram for the pin. How is it loaded? Does the loading depend on whether the pin is a tight fit for the hole?

The loading depends quite strongly on the tightness of the fit. In the case where the pin is a loose fit, the pin will tilt and contact the joint only at the edges. This results in four points of contact across the pin, transferring loads shown below:

Each force has a location associated with it: \(x_1, x_2, x_3, x_4\). For simplicity we let \(x = 0\) at the pin center.

The conditions of static equilibrium are:

\[
\sum F = 0 \\
\sum M = \sum Fx = 0
\]

From the symmetry of the problem, \(F_1 = -F_4, F_2 = -F_3, x_1 = -x_4, x_2 = -x_3\). Note that the sign convention I am using is that all forces are positive when they point ‘up’.

Because we know that the hinge has to transfer a net load of \(F\) from one member to the other, we know that the pair of forces acting on each side of the hinge must sum to \(F\).

\[F_1 + F_2 = F\]
Finally, to satisfy the second condition of static equilibrium, we can note that:

\[ F_1 x_1 = F_2 x_2 \]

We know that \( x_1 = -\frac{w}{2} \), because it is applied at the end of the pin. The location of \( x_2 \) is unspecified though. It depends on the clearances in the hinge. If the hinge is very loosely fitting, \( x_2 \) will be large; if it’s made finely, then it can be small. But \( x_2 \) cannot, in practice, be made to be zero for this hinge design.

The above case considered a loose fitting hole, in which the forces were point contacts between the hinge and the pin. It should be clear that some bending will occur in the pin, as a result of the loads. If the deformation of the pin is much less than the hole clearance, then the deformation won’t affect the problem significantly. But when the clearance becomes very small, the region of contact will spread out due to the pin elastically deforming.

As the clearance approaches zero, the load in the pin will become a continuous distribution like the one below:

Here the conditions of equilibrium are slightly different, because it is a continuum.

\[
\int_{-w/2}^{w/2} f(x)dx = 0
\]

\[
\int_{-w/2}^{w/2} xf(x)dx = 0
\]

And again, to transfer the load being borne by the hinge, \( \int_{-w/2}^{w/2} f(x)dx = F \)
Assuming a piecewise linear $f(x)$, with some slope $q$,
\[
f(x) = \begin{cases} 
  f_2 - qx, & x < 0 \\
  -f_2 + qx, & x > 0 
\end{cases}
\]

Then, from the integrals above,
\[
q = \frac{3f_2}{w} \\
\frac{1}{2}f_2 + \frac{qw}{8} = -\frac{F}{w} \\
f_2 = -\frac{8F}{7w}
\]

b) How would you choose $d$, $t$, and $w$ to prevent failure? Where are the maximum stresses developed, and what are their values?

There are many, many potential points of failure in this design; not all of them will be analyzed, because this is just a class problem set. For a real product, every likely point of failure should be assessed for safety.

The first, most obvious point of failure will be shear in the pin (1). Although the pin is steel and the hinge is aluminum, there’s no reason that the pin might not break first. The maximum shear load occurs right at $x = 0$, the center of the pin.
The shear force is distributed over the cross-sectional area of the pin, \( A_{\text{pin}} = \frac{\pi d^2}{4} \). The total shear force is \( F \), so the average shear stress at that point in the pin is \( \tau_{\text{avg}} = \frac{F}{A_{\text{pin}}} = \frac{4F}{\pi d^2} \). This is the average shear stress. A theorem of solid mechanics notes that for a cylindrical cross section, the maximum shear stress will exceed the average shear stress by \( 4/3 \). (This subtlety is somewhat advanced knowledge and not expected for this class). So \( \tau_{\text{max}} = \frac{16F}{3\pi d^2} \).

Whichever value of shear stress you use, it must be compared to the yield strength of the material, to avoid failure. We will assume that yielding (plastic deformation) of the material counts as failure. We know the stress, we must choose a yield criterion, such as Von Mises or Tresca. For pure shear, Von Mises yield criterion predicts failure when:

\[
\tau > \frac{s_y}{\sqrt{3}}
\]

Where \( s_y \) is the yield strength of the material.

Assuming the pin is made of 304 stainless steel, it has a yield strength of 200 MPa. You could also assume that fracture of the material counts as failure, instead of yielding, in which case it would you would replace \( s_y \) with \( s_{ut} \), the tensile strength, which for 304 stainless is about 500 MPa.

Either way, it is very important at this point to include a safety factor \( F_s \) in the calculation. A safety factor of between 10 and 20 would be reasonable for a load-bearing structure that must support a human; 5 would probably be the absolute minimum. Therefore, we have:

\[
\tau_{\text{max}} < \frac{s_y}{F_s \sqrt{3}}
\]

\[
\frac{16F}{3\pi d^2} < \frac{s_y}{F_s \sqrt{3}}
\]

This results in a pin diameter of:

\[
d > 4 \sqrt[3]{\frac{FF_s}{\sqrt{3}\pi s_y}}
\]

Failure point (2) is rupturing of the aluminum in the hinge due to tensile stress. The force \( F \) is distributed over a total cross-sectional area of \( A_{\text{hinge}} = \frac{(t-d)w}{2} \).

As a very rough approximation, it could be said that the average tensile stress is \( \sigma_{\text{avg}} = \frac{F}{A_{\text{hinge}}} \). But we know that the load is not spread evenly! Look at the free body diagram from part a. The load is much higher toward \( x = 0 \).

Because of Newton’s 3rd law, it is reasonable to expect that the load distribution in the aluminum will be somewhat similar to the load distribution in the pin.
The geometry is too complex to obtain an exact solution for the stress distribution in the aluminum. If you were trying to optimize things, you would want to check the stresses using finite-element modelling for a real design. If you really don’t care, you can just apply a big enough safety factor that it doesn’t matter what the real stress distribution is. But, it would be a fair approximation to say that the stress in the aluminum follows a similar distribution to the load in the pin. Using the result from the integral in part a, the maximum stress exceeds the average stress by a factor of 16/7.

Therefore,

\[ \sigma_{\text{max}} = \frac{16}{7} \frac{F}{A_{\text{hinge}}} = \frac{32}{7} \frac{F}{(t - d)w} \]

Now again, we have to compare that to the yield strength (or tensile strength) of aluminum to check for failure. Don’t forget the factor of safety!

\[ \sigma < \frac{s_y}{F_s} \]

So we can use this to size the aluminum piece:

\[ w(t - d) > \frac{32FF_s}{7s_y} \]

Other stresses that might be considered are bending stresses in the pin, bearing stresses in the aluminum around the hole, and tear-out stresses in other planes of the aluminum part.
c) Think about an alternative construction that accomplishes the same job. Can you improve on this design?

The first main issue with the design is the cantilevered loading on the pin. The asymmetry in the design is responsible for a whole host of its problems. Remember FUNdaMENTTALS: Symmetry! Making the design symmetric will significantly decrease the stresses that the pin has to support. This results in a ‘clevis joint’ design.

An even better design is to realize that, if the clevis pin creates two shear planes to carry the load in the pin, why not have even more shear planes to distribute the load further?

![Diagram of Good and Awesome designs]

In the “Good” design, the shear stresses are cut by half, and the bending stresses (which weren’t analyzed) are reduced much further. In the “Awesome” design, the shear stresses are cut by 1/N, and the bending stresses are reduced nearly to zero. The stresses in the supporting thin plates are also distributed more evenly than in the monolithic piece – no factor of 16/7 to consider. However, the average load-bearing area in the plates does not increase, so no matter how low the stresses in the pin become, there is a minimum diameter for the hole. That implies that the pin is stronger than necessary. If the design had to be made very lightweight, the pin could be replaced by a hollow tube.