Metal prices as a function of ore grade

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The study forms part of a larger programme of research aimed at the assessment of resources of minerals, raw materials and food over the next 35 years – the period in which the population of the earth may be expected to double. Working within this time scale it becomes necessary to analyse the long-term changes that have occurred and may be expected in the cost of mining and refining metals and how these changes might affect their market.

Historical trends in the cost and price of minerals

Barnett and Morse[1] examined the economics of natural resource availability using data on capital, labour and output for the USA between 1870 and 1957. They tested the hypothesis that the scarcity of natural resources was increasing by examining the trend of unit cost of extractive products in the agricultural, mineral, forestry and fishing industries. Working in terms of index numbers they defined a measure of unit costs

$$\frac{L + C}{\text{Net } O}$$

where L and C are the labour and capital inputs and Net O is the net output (ie, the gross output adjusted downward to exclude the value of purchased materials used in producing the output).

Barnett and Morse (p. 170) found that from 1890 onwards, costs per unit of net mineral output measured in either labour or labour and capital have declined rapidly and persistently. By 1957 the cost of labour and capital per unit of product was only one-fifth as large as in 1889. The decline is even greater for labour cost alone. The increases in productivity were more rapid in the latter half of the period than in the early half. From 1889 to 1919 it is estimated that the $L+C$ unit cost of minerals declined at the rate of 1·2% per year; from 1919 to 1957 the rate of decline was 3·2% per year.

In summary: instead of increasing costs in the minerals industry as called for by the scarcity hypothesis a declining trend of cost was experienced. And the more sharply increasing returns occurred later, contrary to the hypothesis. This is so with respect to both labour plus capital and labour alone.

If we disaggregate the mineral sector we observe similar trends in the labour cost per unit of output. Each of the mineral fuels has experienced a major decline in unit labour cost and this has been most

[1] **Barnett H.J. and Morse C.**, 'Scarcity and Growth', in *The Economics of Natural Resource Availability*, Resources for the Future, Inc (Johns Hopkins University Press, Baltimore, 1963).

rapid in the case of petroleum and natural gas. The rapid decline in the all-metals series is roughly matched in iron ore and copper; but the decline in lead and zinc is quite small. A similar analysis applied to sand and gravel, stone, phosphate rock, sulphur and fluorspar shows that all have declined in labour cost.

Barnett and Morse also tested the scarcity hypothesis by means of relative prices ie by comparing the unit prices of extractive and non-extractive output. Price data for minerals tend to be 'noisy' due to inelasticities of supply and demand. Mineral prices fluctuate with other prices but with more amplitude. Mineral prices declined from the 1870s to the 1890s and then rose to a peak in the First World War. From this peak they declined to a trough in 1932-33. Since the trough, prices have risen to the present time. Short-term movements aside, the trend of relative mineral prices has been level since the last quarter of the nineteenth century. This finding does not support the scarcity hypothesis.

Nordhaus[2] views the problem of rising materials costs in terms of a simple model of production. If interest rates are relatively constant, costs of production can be expressed in terms of the costs of two primary factors, labour and resources (capital is simply dated labour and resources). Diminishing returns in resource extraction which arise from mining lower grade ores (or deeper or thinner veins) can only be offset if technological progress is sufficiently rapid. A simple index of this process is the movement of the 'labour cost of resources', ie, the ratio of resource price to labour price. Table 1 is reproduced from Nordhaus' article and shows the ratio of the prices of the 11 most important minerals to the price of labour. This indicates that there has been a continuous decline in resource prices for the entire century apart from copper which has shown a slight tendency to rise since 1960.

Historical trends in ore grades

These decreases in costs and prices have been taking place at the same time as very marked changes in the structure of the mining industry. The most important of these is the increased emphasis on big low-grade homogeneous orebodies that can be worked by large-scale open-pit mining methods.

Table 1. Relative price of important minerals to labour (1970 = 100)

	1900	1920	1940	1950	1960	1970
Coal	459	451	189	208	111	100
Copper	785	226	121	99	82	100
Iron	620	287	144	112	120	100
Phosphorus	–	–	–	130	120	100
Molybdenum	–	–	–	142	108	100
Lead	788	388	204	228	114	100
Zinc	794	400	272	256	126	100
Sulphur	–	–	–	215	145	100
Aluminium	3150	859	287	166	134	100
Gold	–	–	595	258	143	100
Crude petroleum	1034	726	198	213	135	100

Source: Values are the price per ton of the mineral divided by the hourly wage rate in manufacturing. Data are from *Historical Statistics, Long Term Economic Growth, Statistical Abstract*. After **Nordhaus**, reference 2.

[2] **Nordhaus, W.D.**, 'Resources as a constraint on growth', (American Economic Association), vol. 64, No. 2, May 1974.

Table 2. **Distribution of ore grades in US copper mines based on cross-sectional data classified by state (% copper metal)**

	Mode	Median	Mean	Standard deviation
1932	1·20	1·57	1·80	1·00
1939	1·00	1·15	1·23	0·47
1944	0·91	0·96	0·98	0·23
1949	0·82	0·87	0·90	0·22
1955	0·81	0·82	0·83	0·11
1960	0·72	0·73	0·73	0·07
1965	0·68	0·69	0·70	0·09
1970	0·60	0·61	0·61	0·06

Source: US Bureau of Mines, *Minerals Yearbook*

Of course all the mines operating at a given time will not be operating on the same grade of ore. However they must all sell their product at about the same price and they must all cover their costs. We are therefore faced by a population of mines covering a spectrum that extends from high-grade mines operating under difficult high-cost conditions to low-grade mines operating economically under optimum conditions. Even though these mines might differ from each other in many ways, we would expect them to make approximately the same return on their assets. It is the average grade of this distribution of mines which is observed to decline with time. At the same time the range of grades mined has become narrower. This thesis can be illustrated by reference to the cross-sectional data on copper mines classified by state which are published by the US Bureau of Mines. A log-normal distribution was used to describe the data of the form

$$f(x) = \frac{1}{\sqrt{2\pi}\, b\, x} \quad \exp \left\{ -\frac{1}{2} \left(\frac{\log x - a}{b} \right)^2 \right\}$$

where x is grade and the maximum likelihood estimators of a and b were calculated from the data at 5-yearly intervals. The results are shown in Table 2 and three representative years are illustrated in Figure 1. These clearly indicate that the high-grade sector of the US copper mining industry has virtually disappeared within the last 40 years due to the effects of depletion and the rising cost of labour-intensive mining from rich but deep and inaccessible ores.

The decline in average grade appears together with a decrease in the variance of the grade of ore mined. There is also a marked tendency for the mean to approach the mode as the skewness of the distribution declines so that the log-normal curve approximates a normal density. This confirms our experience that the copper mining industry now chiefly consists of operations on large low-grade disseminated orebodies most of which are mined by open-cast or block caving methods. The remaining mineral occurrences are becoming more homogeneous as the anomalous high-grade pockets of ore are worked out.

Technological change

We are therefore faced with what appears to be a contradiction. On the one hand prices have remained almost static, and costs have fallen; on the other the amount of rock that must be broken, handled and treated to obtain the same amount of metal has steadily risen.

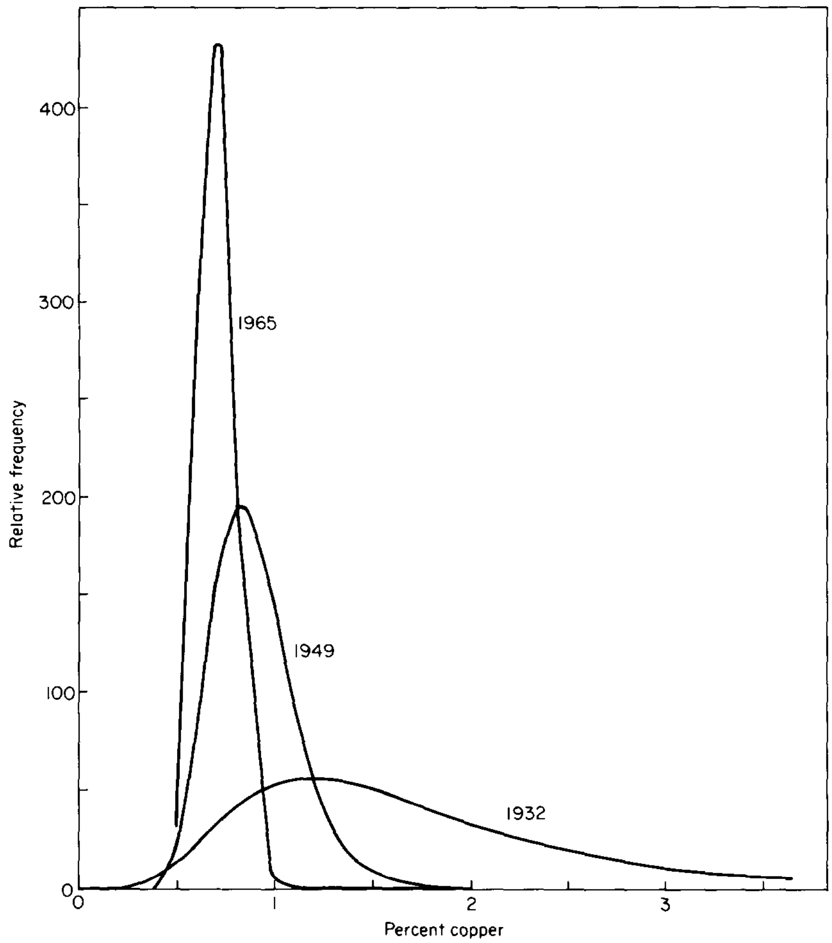

Figure 1. US copper mines: distribution of ore grades. Parameters of lognormal distribution estimated by maximum likelihood methods.

The explanation must lie in the technological progress that has been achieved in the exploitation of metalliferous ores.

In detail these technical changes are very numerous but they have in common the more efficient application of energy to the materials that are being processed. The following are examples:

Long-life tungsten carbide drilling bits
Light compressed air drills
Flexible self-propelled rock loading and transporting equipment
Raise boring and tunnelling machines
Micro delay detonators
Automatic hoists and winders
Autogenous grinding
Closed circuit crushing and grinding
On-stream analysers, weighing machines etc.
Flotation methods
Hydrometallurgical and ion exchange methods
Magnetic and electrostatic separation methods
Nuclear assay methods
Geophysical and geochemical surveys
Airborne and satellite geological surveys
Deep-hole diamond drilling techniques

The list could obviously be extended but it is clear that investment in this type of capital equipment has effected great savings in labour and capital per unit of metal. It is an open question how far this process can continue. Thermodynamic considerations suggest that an irreducible minimum amount of energy is required to break up the crystal lattice and select the valuable mineral from the waste. Lovering[3] has produced evidence that the horse power installed at metal mines in the USA has increased very rapidly since 1950. Under present conditions energy costs are only a small proportion of mineral costs. However in the long term this situation could change. Lower-grade ore implies more energy per unit of metal; higher energy flows entail larger capital costs. In terms of Barnett and Morse's index this implies that C is increasing and Net O is getting smaller. It follows that the ratio $(L+C)$/Net O could increase, thereby reversing the historical trend. Some commentators feel that there is evidence to suggest that a rise in the cost of copper has taken place.

Prices, costs and grade

It seems clear that the relative prices of metals are not arbitrary and are related in some way to their scarcity. Gold and iron are obvious examples and several writers have suggested that a systematic relationship may exist (Roberts,[4] Cech[5]). The most promising line of approach would seem to be via costs since these are closely related to prices where commodities are produced and traded under conditions of perfect competition. The opportunity cost of producing a unit of metal is the maximum value of goods and services that could have been produced if the resources which were used to produce the metal had been applied elsewhere. Under perfect competition and assuming that profits are maximised, marginal costs, marginal revenue and price must all be equal. Whether or not we can use the cost as a proxy for price depends on how far the free market is distorted by cartels and barriers to entry. This is disputed territory but there is no question that some minerals such as aluminium and diamonds are dominated by a few international companies. However we can reflect that few minerals are free from competition by substitutes. We therefore offer the tentative hypothesis that the relative prices of metals are dictated by the costs of mining and refining them and that these costs are related primarily to the physical conditions under which they are obtained. If supernormal profits are earned they are temporary and soon ended by countervailing market forces.

It seems reasonable to assume that costs are inversely proportional to grade because to win a unit of metal a larger amount of rock must be broken, transported, crushed, ground and beneficiated. Not only must the equipment be larger to handle the greater volume of rock, but it must also become more complex and energy intensive. At the lower grades the valuable mineral exists in the rock either as finely divided grains or as replacement ions in the crystal lattice of the waste mineral. To liberate this mineral may require extremely fine grinding (less than 200 mesh) and/or resort to hydrometallurgical techniques which take the metal into solution. Fine grinding is an expensive process and so are the dewatering techniques that follow because conventional thickeners and settling tanks must be replaced by filters and centrifuges. Since reagent use is proportional to the amount of surface area exposed we would also expect increased consumption of

[3] **Lovering, T.S.**, 'Mineral Resources from the Land', in *Resources and Man*, (W.H. Freeman, San Francisco 1969).
[4] **Roberts, F.**, *The Conservation of Nonrenewable Materials Resources*, (AERE, Harwell, Berks, 1973) AERE Report R 7431, p. 217.
[5] **Cech, R.E.**, 'The price of metals', *Journal of Metals*, December 1970, p. 21.

chemicals per unit output. The difference in technique can be illustrated by comparing a gold cyanidation plant with a base metal flotation plant.

These increased treatment costs are partially offset by the economies of scale and the labour-saving opportunities offered by open-cut and block caving methods. However, grade cannot be the only determinant of costs. In the case of certain common oxide ores such as hematite and bauxite the mining costs are small compared with costs of separating the useful metal from the mineral. Costs might also depend on the difficulty of finding orebodies and thus could be tested by using crustal abundance as a proxy for average prospecting and exploration expenditure.

We therefore propose to establish a relationship between market price and the cost of finding, mining and refining the metal. The actual variables we shall use will be crustal abundance of the metal, the average ore grade and the Gibbs free energy of the ore mineral. We assume that the latter is related to the cost of the plant and energy required to refine the metal from its ore.

Model

For the present work the cost of metal production is divided into two parts: (1) the cost of mining and milling the ore (including transport costs); and (2) the cost of refining and smelting.

For the first part the cost is represented by

$$\hat{p}_1 = \gamma_1\left(\frac{1-c}{c}\right) = \gamma_1\left(\frac{\text{mass of ore treated}}{\text{mass of metal}}\right)$$

where c = representative ore grade for a metal, and
γ_1 = constant of proportionality

As an analysis of any given refining process to assign the cost of the plant and energy required to refine any given metal is far too detailed for the present work, the Gibbs free energy has been employed to estimate the energy required. In diffentials

$$\mathrm{d}G = -S.\mathrm{d}T + V.\mathrm{d}P + \Sigma_k\, \mu_k.\mathrm{d}n_k$$

In the usual notation: G = Gibbs free energy, S = entropy, T = absolute temperature, V = volume, P = pressure, μ_k = chemical potential of species k, n_k = number of gramme molecules of species k present. (For a more detailed treatment see reference 6.)

At constant pressure and temperature dG = energy change caused by the chemical reactions summarised by $\Sigma_k\, \mu_k.\mathrm{d}n_k$ Therefore, ΔG, the total change in G, across a complete refining process at atmospheric pressure and some fixed temperature, is an estimate of the energy required to refine the metal. Because of thermal losses, this estimate is a minimum one.

Some authors, eg, Roberts,[4] use this estimate directly. However, a metal such as copper has more than one method of refining at present, so more than one estimate is obtained. To overcome this, the most common metal compound in the ores has been selected. From tables[7] the Gibbs free energy/kg of the compound could be found. By convention all elements are assigned zero energy, so most ore compounds have a negative energy. ΔG is then assigned the positive value of this energy for the mass of compound containing 1kg metal.

[6] **Zemansky, M.W.,** *Heat and Thermodynamics,* (McGraw-Hill, New York, 1957), 4th ed., p. 484.
[7] **Weast, R.C.** (ed.), *Handbook of Chemistry and Physics,* (Chemical Rubber Co., Cleveland, Ohio, 1969-70), 50th ed.

Hence $p_2 = \gamma_2 \Delta G$

Therefore total estimated cost takes the form:

$$\hat{p} = \gamma_1 \left(\frac{1-c}{c}\right) + \gamma_2 \Delta G \tag{1}$$

This prediction formula implies two assumptions. The market price of a metal is taken to be directly proportional to cost, so that $\hat{p}$ estimates the price with the profit absorbed in γ_1 and γ_2. Second, the energy input of the processing plant per kg metal produced is roughly proportional to the size of the plant and therefore to the capital and running costs. For example aluminium refining and smelting require large energy flows which implies heavy and costly power generation and handling equipment in both stage 1 and stage 2.

Data preparation

The sources of the data are summarised in Tables 4 and 5, while the selected data are given in Table 3.

As some metals are traditionally priced in pounds sterling while others are in US dollars, all prices have been converted to 1968 dollars from their 1973 values. To convert pounds to dollars, a rate of 2·4526[8] has been used while a deflator of 0·78[9] has been used to allow for inflation from 1968 to 1973. The annual average has been used wherever it is available. In cases where several prices are quoted during the year, a mean has been calculated by weighting each price by the number of days for which it applied.

[8] *Annual Abstract of Statistics 1974*, (HMSO, London, 1974), no. 111.

[9] Internal memorandum (W.M. Mayon-White) based on data from the *Financial Times* and the *US Survey of Current Business*.

Table 3. Data used for fitting procedure

Metal	Ore compound	Price \$ (1968)/kg	Ore concn. c	ΔG MJ/kg	Crustal abundance	Annual production tonnes x 10^6
Iron	Fe_2O_3	0·06715	0·42	6·636	$5·8 \times 10^{-2}$	481·9603
Mild steel	Fe_2O_3	0·14402				
Lead	PbS	0·28003	$5·9 \times 10^{-2}$	0·4474	$1·0 \times 10^{-5}$	3·4733
Zinc	ZnS	0·35523	$4·2 \times 10^{-2}$	3·035	$8·2 \times 10^{-5}$	5·5516
Aluminium	Al_2O_3	0·4299	0·42	29·222	$8·0 \times 10^{-2}$	10·8852
Magnesium	$MgCO_3$	0·65815	0·28	42·3	$2·77 \times 10^{-2}$	0·233
Manganese	MnO_2	0·737	0·3	8·487	$1·0 \times 10^{-3}$	8·100
Ferro/mang		0·3173				
Copper	Cu_2S	1·09162	$1·305 \times 10^{-2}$	0·678	$5·8 \times 10^{-5}$	6·4421
Chromium	Cr_2O_3	1·974	0·33	10·07	$9·6 \times 10^{-5}$	2·039
Ferro/chrome		0·5134				
Antimony	$Sb_2S_3(Sb_4O_6)$	1·35	$6·8 \times 10^{-2}$	1·616	$2·0 \times 10^{-7}$	0·0680
Titanium	TiO_2	2·21	$2·8 \times 10^{-2}$	17·81	$8·6 \times 10^{-3}$	1·3789
Nickel	NiS	2·631	$6·0 \times 10^{-3}$	13·18	$7·2 \times 10^{-5}$	0·6683
Tin	SnO_2	3·91237	$8·6 \times 10^{-3}$	4·379	$1·5 \times 10^{-6}$	0·2415
Cobalt	CoS	5·05	$5·9 \times 10^{-4}$	1·41	$2·8 \times 10^{-5}$	0·0235
Cadmium	CdS	6·259	$1·5 \times 10^{-4}$	1·25	$1·8 \times 10^{-5}$	0·0154
Mercury	HgS	6·47628	$3·2 \times 10^{-3}$	0·2435	$2·0 \times 10^{-8}$	0·0105
Molybdenum	MoS_2	6·92	$2·1 \times 10^{-3}$	2·35	$1·2 \times 10^{-6}$	0·0705
Tungsten	$FeMn(WO_4)(WO_3)$	6·94	$3·1 \times 10^{-3}$	4·1538	$1·0 \times 10^{-6}$	0·0366
Uranium	U_3O_8	13·84	$2·0 \times 10^{-3}$	1·608	$1·6 \times 10^{-6}$	0·0189
Niobium	—	36·97	$3·6 \times 10^{-3}$	—	$2·0 \times 10^{-5}$	0·0105
Ferro/niobium		5·07				
Silver	Ag_2S	64·1373	$5·9 \times 10^{-4}$	0·187	$8·0 \times 10^{-8}$	0·0092
Gold	—	3090·81	$6·1 \times 10^{-6}$	—	$2·0 \times 10^{-9}$	0·0014
Platinum	—	3782·57	$2·9 \times 10^{-6}$	—	$5·0 \times 10^{-9}$	0·00008115

Table 4. Data sources used to obtain metal price, Gibbs free energy, crustal abundance and annual production quoted in Table 2

Metal	Price	ΔG	Crustal abundance	Annual production
Iron	1A	2	5	6
Mild steel	4A			
Lead	1			
Zinc	1			
Aluminium	1			
Magnesium	1			6
Manganese	1A			7B
Ferro/mang	4C			6
Copper	1			6
Chromium	1A			7A
Ferro/chrome	4B	2		6
Antimony	1A	2A		6
Titanium	1A			6A
Nickel	1			6
Tin	1			
Cobalt	1A			
Cadmium	1			
Mercury	1			
Molybdenum	1A			
Tungsten	1A	2B		6
Uranium	3	2C		6B
Niobium	1A			6
Ferro/Niobium	4D			
Silver	1			
Gold	1A			6
Platinum	1A			6C

Sources

1 Annual average for 1973 from **Bauer, W.** (ed.) *Metal statistics 1963-1973* (Metallgesellschaft AG, Frankfurt, 1974), 61st ed.
1A June 1973 price (for Pt cheapest source is used). *Ibid.*
2 Value for the mass of compound, containing 1kg of metal.
2A Used Sb_4O_6 as no value for SB_2S_3
2B Used WO_3 as no value for FeMn (WO_4)
2C Approx $\Delta G(U_3O_8) \approx \frac{1}{3}(2\Delta G(UO_3) + \Delta G(UO_2))$
Weast, R.C. (ed.) *Handbook of Chemistry and Physics* (Chemical Rubber Co., Cleveland, Ohio, 1969-70), 50th ed.
3 Average price of 1kg U contained in U_3O_8 from Mining Journal, *Mining Annual Review, 1974* (London)
Annual average for 1973
4A Hot rolled mild steel bars
4B Price/kg of Cr in 2% C, 68-70% Cr Fe/Cr alloy
4C Price/kg of Mn in 2% C, 78% Mn Fe/Mn alloy
4D Price/kg of Nb in 60% Nb Fe/Nb Ta alloy, from **Packard, R.** (ed.) *Metal Bulletin Handbook, 1974,* 7th ed.
5 **Skinner, B.J.** *Earth resources* (Prentice-Hall, Englewood Cliffs, NJ, 1969) p. 149.
6 1971 production figures for mine production
6A 1970 production Ti content in $FeTiO_3 + TiO_2$
6B 1971 production U content in U_3O_8
6C 1971 production 0·64 of Pt metals production, from *World Metal Statistics*, **26**, no. 3, March 1973 (World Bureau of Metal Statistics, London).
7A Cr content in Cr_2O_3 production for 1971
7B Mn content in Mn ores production for 1971, from *Statistical Year Book 1972* (United Nations, New York, 1973)

Table 5. Data sources used to obtain ore grades, quoted in Table 2

Metal	Ore grade	Data source
Iron	0·42	US Bureau of Mines, *Minerals Yearbook 1972*
Lead	$5{\cdot}9 \times 10^{-2}$	*ibid.*
Zinc	$4{\cdot}2 \times 10^{-2}$	*ibid.*
Aluminium	0·42	*ibid.*
Magnesium	0·28	Chemical composition of pure magnesite ($MgCO_3$)
Manganese	0·30	Groote Eylandt manganese project, Australia
Copper	$1{\cdot}305 \times 10^{-2}$	Median of published data on copper mines
Chromium	0·33	Assuming 48% Cr_2O_3
Antimony	$6{\cdot}8 \times 10^{-2}$	Brunswick Reef, South Africa
Titanium	$2{\cdot}8 \times 10^{-2}$	US Bureau of Mines, *Minerals Yearbook 1972*
Nickel	$6{\cdot}0 \times 10^{-3}$	International Nickel, Ontario, *Annual Report 1971*
Tin	$8{\cdot}6 \times 10^{-3}$	Average of three Tasmanian tin-lode mines
Cobalt	$5{\cdot}9 \times 10^{-4}$	Western Mining Corporation, Nickel mines, Australia
Cadmium	$1{\cdot}5 \times 10^{-4}$	Broken Hill lead-zinc mines, Australia
Mercury	$3{\cdot}2 \times 10^{-3}$	US Bureau of Mines, *Minerals Yearbook 1972*
Molybdenum	$2{\cdot}1 \times 10^{-3}$	Climax Molybdenum, Colorado, USA
Tungsten	$3{\cdot}1 \times 10^{-3}$	Average of two Tasmanian tungsten-tin mines
Uranium	$2{\cdot}0 \times 10^{-3}$	US Bureau of Mines, *Minerals Yearbook 1972*
Niobium	$3{\cdot}6 \times 10^{-3}$	Oka Pyrochlore Mines, Ontario, Canada
Silver	$5{\cdot}9 \times 10^{-4}$	US Bureau of Mines, *Minerals Yearbook 1972*
Gold	$6{\cdot}1 \times 10^{-6}$	Witwatersrand, South Africa
Platinum	$2{\cdot}9 \times 10^{-6}$	Merensky Reef, South Africa

For many of the metal prices we have depended on *Metallgesellschaft, The Metal Bulletin* and other journals. We are aware that not all transactions in metals take place at the published price. However we do not think that the discrepancy between the transaction price and the price we have used is great enough to affect our results since we are trying to establish a relationship that holds over five orders of magnitude. Of course it may be possible to obtain discounts or make contracts at prices that differ from the London Metal Exchange quotation. However, because there are many buyers and sellers there is a persistent tendency for the distribution of prices to tend towards a central limit. On the other hand those metals that are controlled by a few large companies usually have a well defined price structure that changes infrequently. Here again the published price is a good guide to the transaction price.

Ore grades

The choice of representative average ore grades is clearly a key decision but unfortunately it is impossible to be completely precise. However, because the relationship we are studying holds over several orders of magnitude, minor errors in the data will not affect the results significantly. The ideal method would be to take a weighted average grade for all the mines currently operating in the world. However the data are not nearly complete. The best figures exist for the USA where the US Bureau of Mines publishes figures for crude ore mined and units of marketable product sold. However, in some cases the US figures are not representative or the relevant figures are not published. In these cases we have given figures for a mine or mines that are representative or dominate the industry. In particular the grade of copper ore gave us difficulty and we finally settled on the median value of all mines who published their grade figures in Canada, South and Central Africa, South America and the USA. Our figure for antimony is based on two small producers in the West. The dominant source is in China and we have been unable to obtain reliable data (see Table 5).

In the actual use of the data, a few special adjustments have been made. Little pig iron is actually sold for use, so it is more appropriate to use the price of mild steel. The metals manganese, chromium and niobium have few uses in their pure form. (Some chromium is used in electroplating.) The majority of the output of these metals is prepared and sold as ferro-alloys for use in steel making. Therefore it is more realistic to take the price of the alloys, calculated on the amount of metal contained in the alloy.

Method of fitting

As Formula 1 is linear, γ_1 and γ_2 could be obtained from least squares fitting to the data. The prices vary by nearly 10^4 and least squares minimise the residual function

$$R = \sum_{i=1}^{n} (p_i - \hat{p}_i)^2$$

R will thus be dominated by the high prices, so that the solution will be ill conditioned. The most obvious difficulty is that the resulting values for γ_1 and γ_2 might yield negative predicted prices, $\hat{p}$ for the low price metals. To overcome this difficulty, $ln\hat{p}_i$ has been fitted against $ln\hat{p}_i$ so

$$R = \sum_{i=1}^{n} (lnp_i - ln\hat{p}_i)^2$$ where n metals are used.

The sum of squares is non-linear in γ_1 and γ_2, so that R has to be minimised directly. For this particularly simple case, a steepest descent method has proved adequate. However, it does not follow that this numerical method would be satisfactory for a different estimating function. (One reason for the choice was that a more sophisticated method was not available in coded form at the start of the work.) For a detailed discussion of minimising methods, see reference 10.

Results

Preliminary results showed that the accuracy of the prediction was much improved by weighting the points.

$$R = \sum_{i=1}^{n} w_i (\ln p_i - \ln \hat{p}_i)^2$$

Of the simple choices for w_i that have been tried, $w_i = \ln$ (annual consumption of metal i), gave the best results.

In all the fitting attempts, the three metals, cobalt, cadmium and antimony had their prices predicted very badly. As they are so far removed from the rest, the fitting procedure was used without them and then the resulting parameters were used to predict their prices.

As it seemed possible that the global scarcity might affect the cost of a metal, several trials were made with a third term $\ln(c/c_a)$(where c_a = crustal abundance) added to the prediction formula. However, the results from this formula were no better than from the two-term one; in some cases they were worse. Within the scope of the work described here, the overall scarcity seemed to have no detectable

Table 6. Predicted metal prices for runs A and B

Metal	Price \$ (1968)/kg	Run A Predicted price p	Run A 10 log 10 (p/p̂)	Run B Predicted price p	Run B 10 log10 (p/p̂)
Mild steel	0·14402	0·1352	0·275	0·1739	−0·818
Lead	0·28003	0·3297	−0·709	0·3274	−0·679
Zinc	0·35523	0·5102	−1·573	0·521	−1·664
Aluminium	0·4299	0·5003	−0·659	0·672	−1·94
Magnesium	0·65815	0·7358	−0·484	0·9842	−1·747
Ferro/manganese	0·3173	0·1844*	2·358	0·2336	1·329
Copper	1·09162	1·54	−1·495	1·521	−1·439
Ferro/chrome	0·5134	0·2038*	4·012	0·2625	2·913
Antimony	1·35	0·3033*	6·484	0·3086*	6·41
Titanium	2·21	0·9898	3·489	1·084	3·094
Nickel	2·631	3·563	−1·316	3·589	−1·348
Tin	3·91237	2·402	2·12	2·392	2·138
Cobalt	5·05	34·27*	−8·316	33·76*	−8·251
Cadmium	6·259	134·8*	−13·33	132·7*	−13·27
Mercury	6·47628	6·302	0·119	6·207	0·185
Molybdenum	6·92	9·645	−1·442	9·512	−1·382
Tungsten	6·94	6·569	0·239	6·494	0·289
Uranium	13·84	10·11	1·362	9·97	1·425
Ferro/niobium/ tantalum	5·07	5·596*	−0·429	5·51	−0·362
Silver	64·1373	34·25	2·724	33·73	2·791
Gold	3090·81	3314·0	−0·303	3264·0	−0·236
Platinum	3782·57	6972·0	−2·656	6865·0	−2·588

Notes:
In run A, antimony, cobalt, cadmium and the ferro alloys are omitted.
In run B, only antimony, cobalt and cadmium are omitted.
* Metals not included in the fitting procedure

[10] **Murray, W.** (ed.), *Numerical Methods for Unconstrained Optimisation*, (Academic Press, London, 1972), p. 144.

influence on price. This is determined by the local abundance of the metal, ie, the grade of particular orebodies.

Two final runs for the two-term formula are presented in Table 6 and Figures 2 and 3. The residuals of the fit are displayed as the logarithm of the ratio $p/\hat{p}$, ie, $10 \log_{10} (p/\hat{p})$. This equals the ratio $p/\hat{p}$ expressed in decibels, a terminology frequently used in engineering applications. For run *A* both the outlier metals and the special ferro-alloys have been omitted from the fitting procedure. For run *B* the ferro-alloys have been included. There is little difference in the two sets of predictions, so set *B* has been preferred for two reasons. The data set for the fitting includes more metals and the histogram of the residuals, Figure 2b, is slightly more symmetric. With the parameters given by this run $\gamma_1 = 1{\cdot}991 \times 10^{-2}$ and $\gamma_2 = 2{\cdot}206 \times 10^{-2}$. Formula (1) predicts 10 metal prices within the band $\pm$ 1·5 decibels, ie

$$\frac{p}{\sqrt{2}} \leqslant \hat{p} \leqslant p\sqrt{2}$$

with a further 7 within the band $\pm$3 decibels, ie

$$\frac{p}{2} \leqslant \hat{p} \leqslant 2p$$

To illustrate the use of this relationship consider a decline in the world average grade of copper ore from 1·3% to 1·2% Cu. Using the

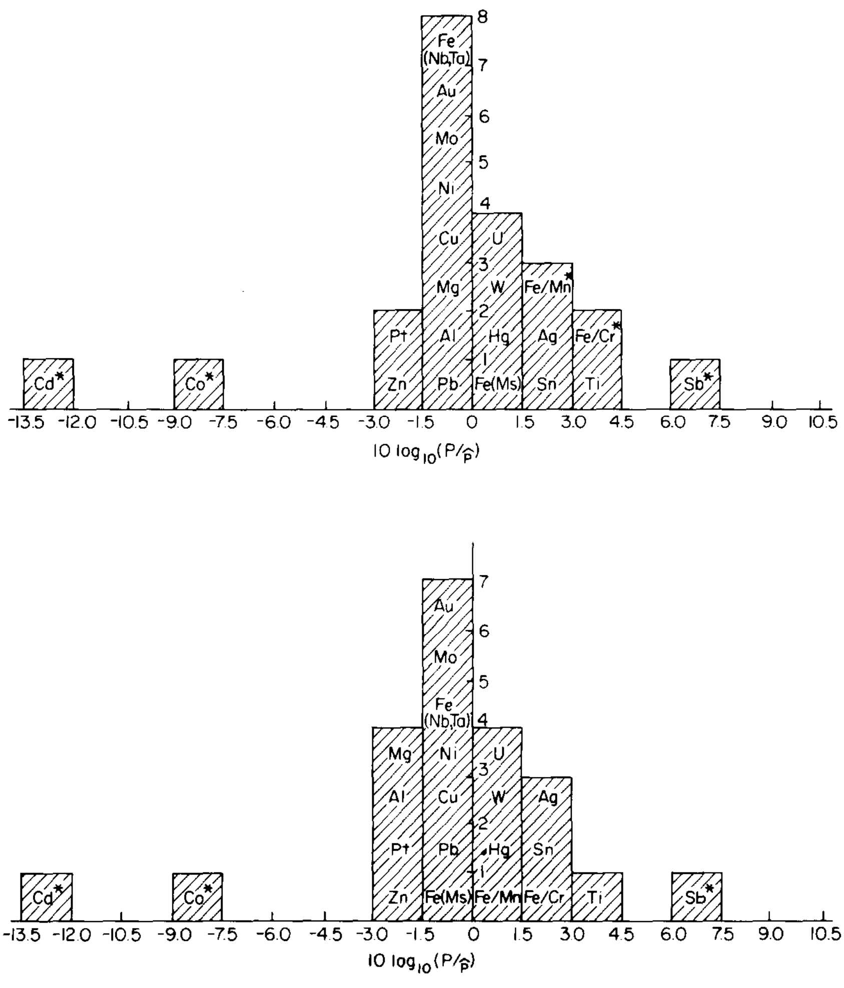

Figure 2. Predicted prices for the two trials A and B, plotted as a histogram: the price ratio $p/\hat{p}$ is expressed in decibels. Metals marked with an asterisk were omitted from the fitting procedure.

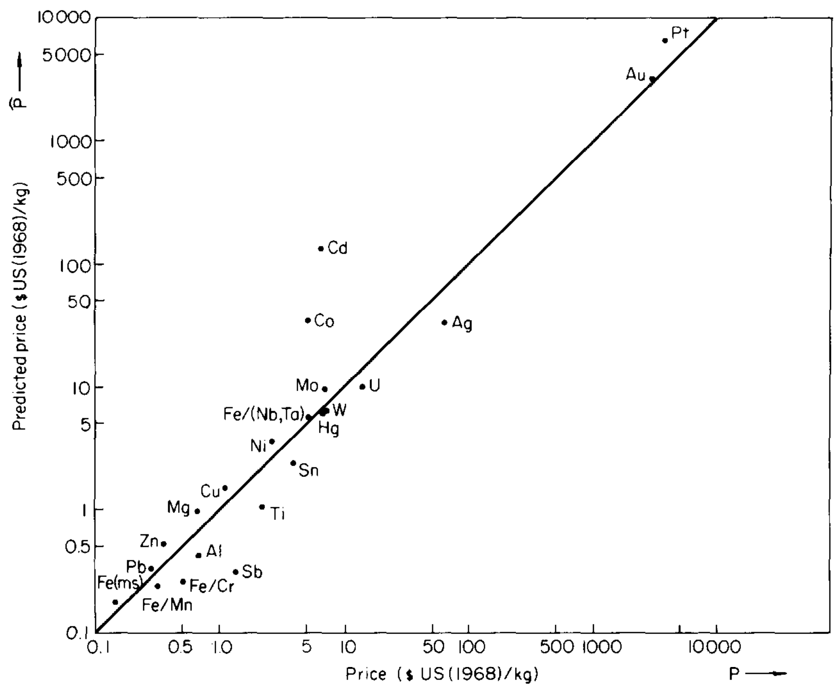

Figure 3. Predicted prices from trial B plotted against actual price

estimated values of γ_1 and γ_2 we calculate that the price of copper metal would rise from 69 cents per lb to 75 cents per lb. Such a small amount could be concealed by the random fluctuations which affect the markets for base metals. Put in another way, a 7·7% decline in grade is equivalent to an 8·3% rise in the price of copper. It follows that the elasticity of price with respect to grade is −1·1. This calculation assumes that the technology has remained static – an unlikely assumption in view of the findings of Barnett and Morse.

The poor prediction for the outliers cobalt and cadmium can be explained from their status as by-products. Cobalt is a by-product of nickel refining and cadmium of zinc refining. By-product pricing depends largely on commercial considerations, which are outside the scope of the formula, so a poor prediction for these two is not surprising. The prediction for antimony uses the ore grade quoted for the Western producers, while the price is governed by the major Chinese source. Once again a poor prediction might be expected.

We intend to make use of the work described in this paper in our analysis of the future availability of resources. However, a relationship between grade and price is not sufficient for our purposes. We will also need to know the distribution of grades of metalliferous ores, and the future demand for metals. The story is not complete even then because technical progress can be assumed to continue and therefore the price might well be lower than this simple prediction would indicate. Not only the capital and labour requirements for exploiting metals are relevant but also the costs of other inputs (particularly energy) so that a realistic prediction of price should take into account each of the separate factors.

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