The Equation of State of Dense Matter and Neutron Star Masses and Radii

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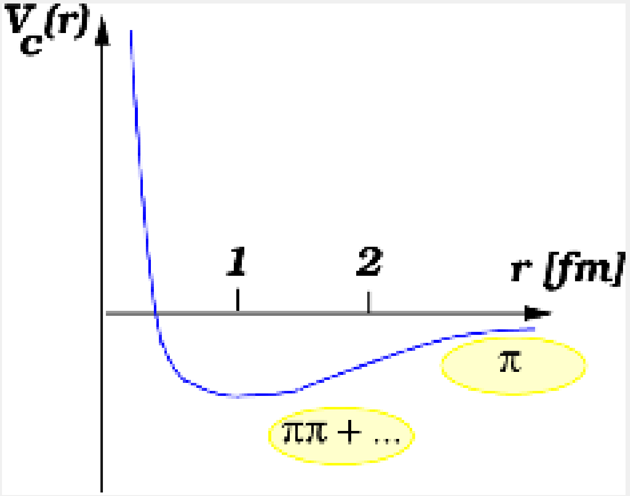
Outline

- More on the n-n interaction
- Back to stars
- Speed of sound
- Two-dimensional fitting
- Bayesian inference
- Quark matter
- Color superconductivity problem

Nucleon-nucleon interaction



- One-pion exchange (attractive) at large distances, repulsion at short distance
- Phenomenological coordinate-space potentials: Argonne-potential

$$\mathcal{H} = \sum_i \frac{-\hbar^2}{2M_i}\nabla_i^2 + \sum_{i<j} V_{ij} + \sum_{i<j<k} V_{ijk}$$

$$V_{ij}(r) = \sum_{p=1,8} v^p(r) O^p$$

$$O^p = (1, \sigma_1 \cdot \sigma_2, S_{12}, L \cdot S) \times (1, \tau_1 \cdot \tau_2)$$

Wiringa et al. (1995), Gandolfi et al. (2015)

- Three-nucleon force
    - Required for saturation

Quantum Monte Carlo

- Large class of methods; unmatched for describing light nuclei

- Diffusion Monte Carlo: project out ground state

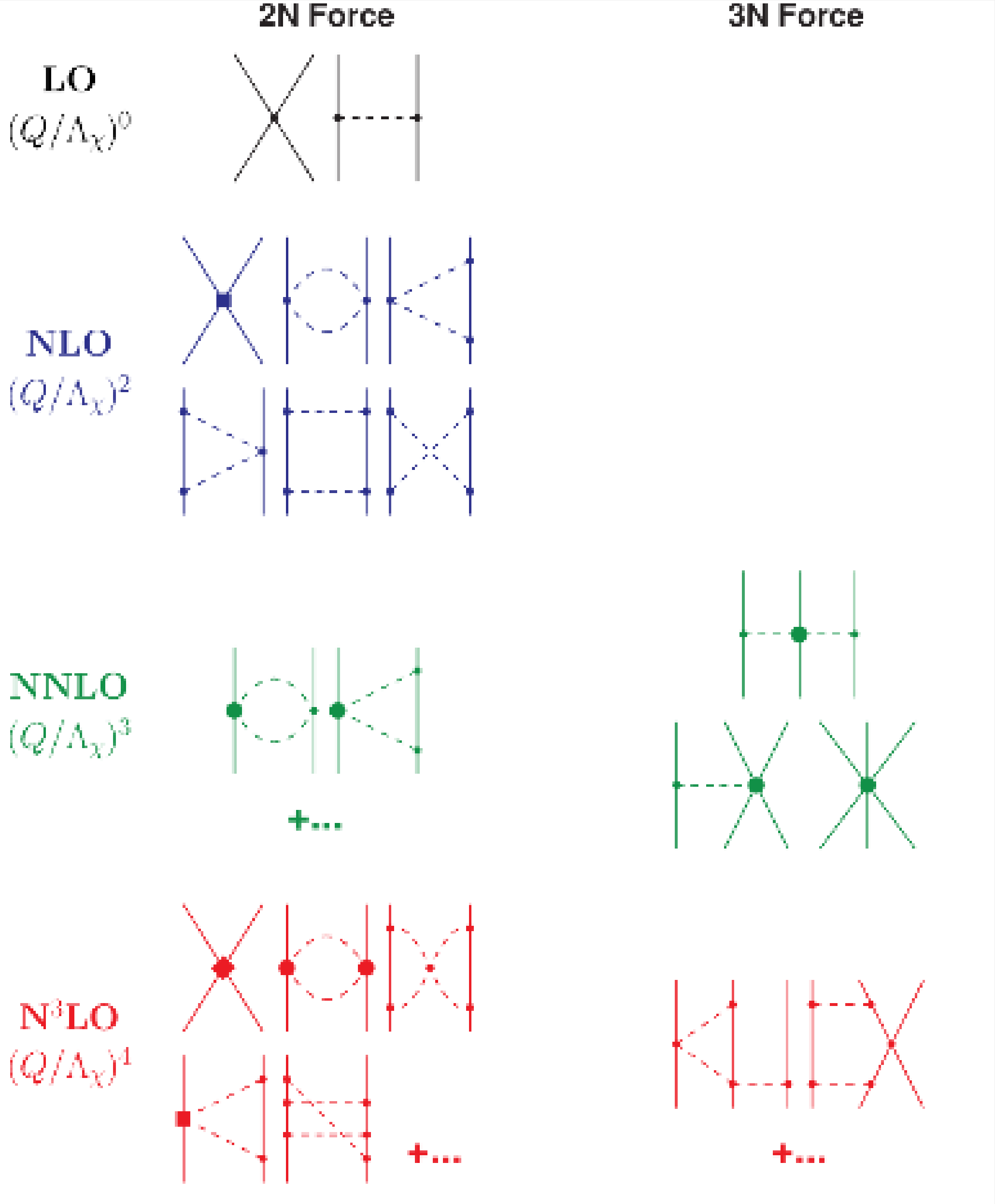
$$|\Psi_0\rangle = \lim_{\tau\to\infty}\left[\exp(\mathcal{H}\tau)\Psi_{\text{trial}}\right]$$

Gandolfi et al. (2015)

- Restricted to approximately local potentials
  Gezerlis et al. (2014) making local versions of chiral interactions for QMC

- Fail to simultaneously describe light nuclei and nuclear saturation
  e.g. Akmal et al. (1998) get -12.61 MeV instead of -16 MeV for saturation

Chiral effective theory



- QCD (except for mass terms) has a SU(3)×SU(3) chiral symmetry
- This symmetry is spontaneously broken, and pions are nearly massless Goldstone bosons
- At low momenta, nucleon-nucleon interaction dominated by pions
- This works up to a scale, $\Lambda_\chi \sim 500 - 700$ MeV
- Use an effective interaction with undetermined coupling constants: fixed from light nuclei
- Neutron matter is "perturbative" at low densities

Machleidt and Entem (2011); Epelbaum et al. (2009)

Relativistic stars

- Specify the metric

$$ds^2 = -e^{2\Phi(r)}dt^2 + e^{2\Lambda(r)}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$$

- Now, m is "gravitational mass"

$$\frac{dm}{dr} = 4\pi r^2 \varepsilon; \quad m(r=0) = 0$$

$$\frac{dP}{dr} = \frac{-Gm\varepsilon}{r^2}\left(1 + \frac{P}{\varepsilon}\right)\left(1 + \frac{4\pi P r^3}{m}\right)\left(1 - \frac{2Gm}{r}\right); \quad P(r=R) = 0$$

- The baryonic mass is

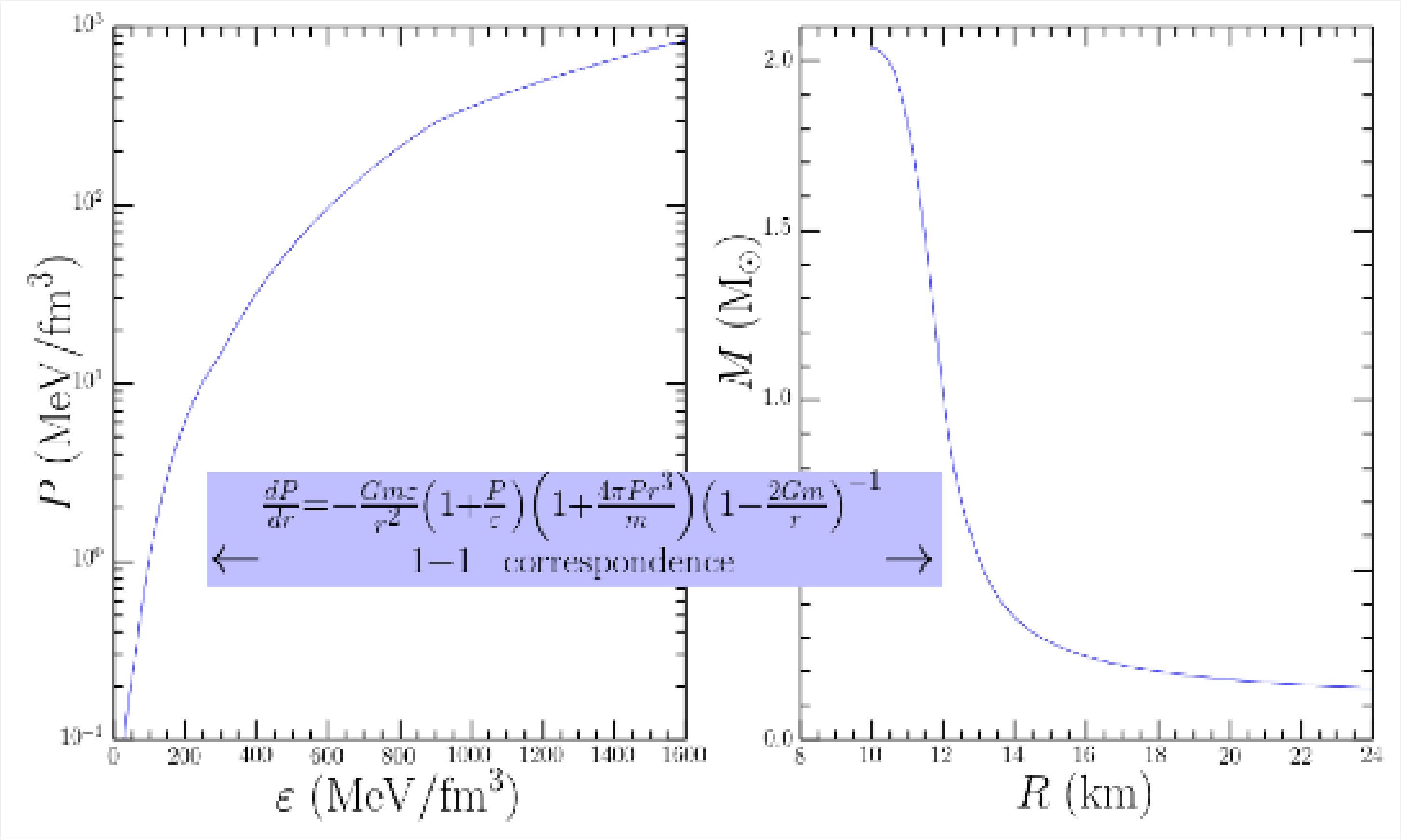
$$M_B = \int_0^R 4\pi r^2 n_B m_B \left(1 - \frac{2Gm}{r}\right)^{-1/2} dr$$

- Gravitational potential:

$$\text{outside}: e^{2\Phi} = \left(1 - \frac{2GM}{r}\right) \quad \text{inside}: \frac{d\Phi}{dr} = -\frac{1}{\varepsilon}\frac{dP}{dr}\left(1 + \frac{P}{\varepsilon}\right)^{-1}$$

Neutron Star Masses and Radii and the EOS

- Neutron stars (to better than 10%) all lie on one universal mass-radius curve
  (Largest correction is rotation)

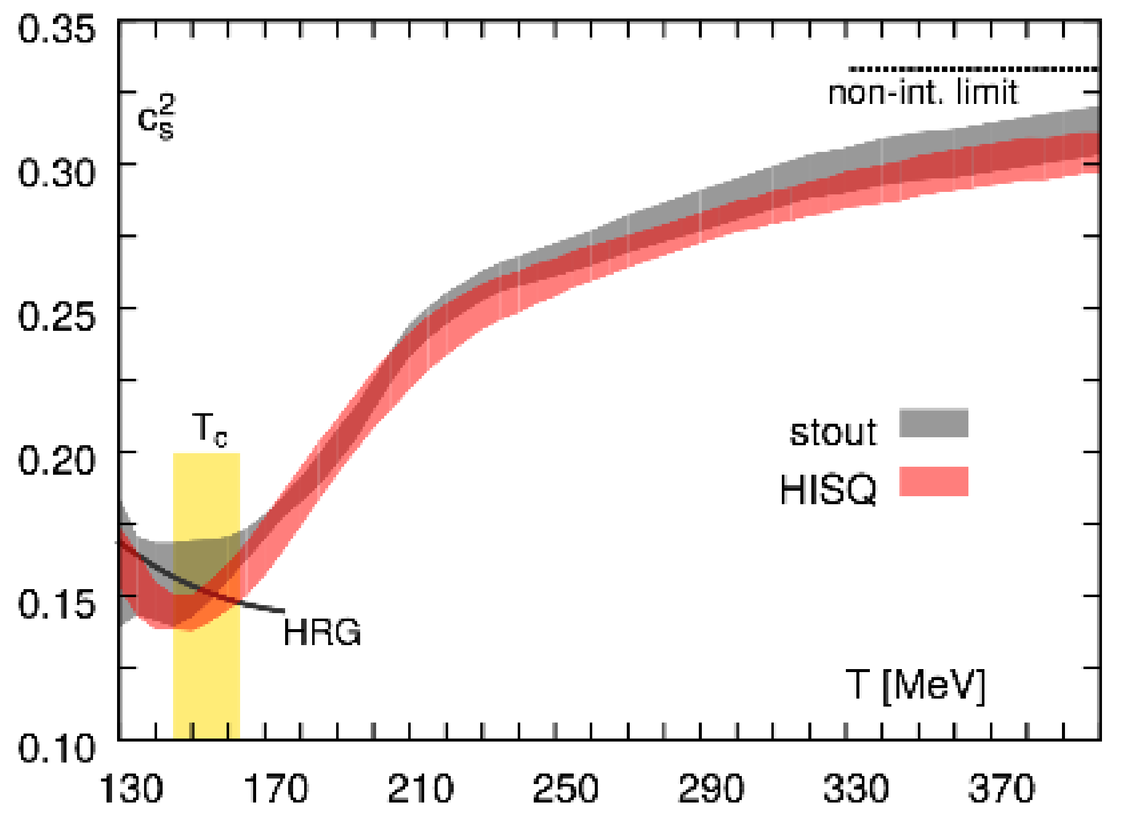


- Two ~ 2 $M_\odot$ neutron stars
  Demorest et al. (2010), Antoniadis et al. (2013)
- This is the most significant constraint on dense QCD outside of perturbation theory
- If we have neutron star observations, we can "connect the dots"

Constraints on the EOS

- Hydrodynamic stability, $dP/d\varepsilon > 0$
- Causality $c_s^2 = dP/d\varepsilon < 1$

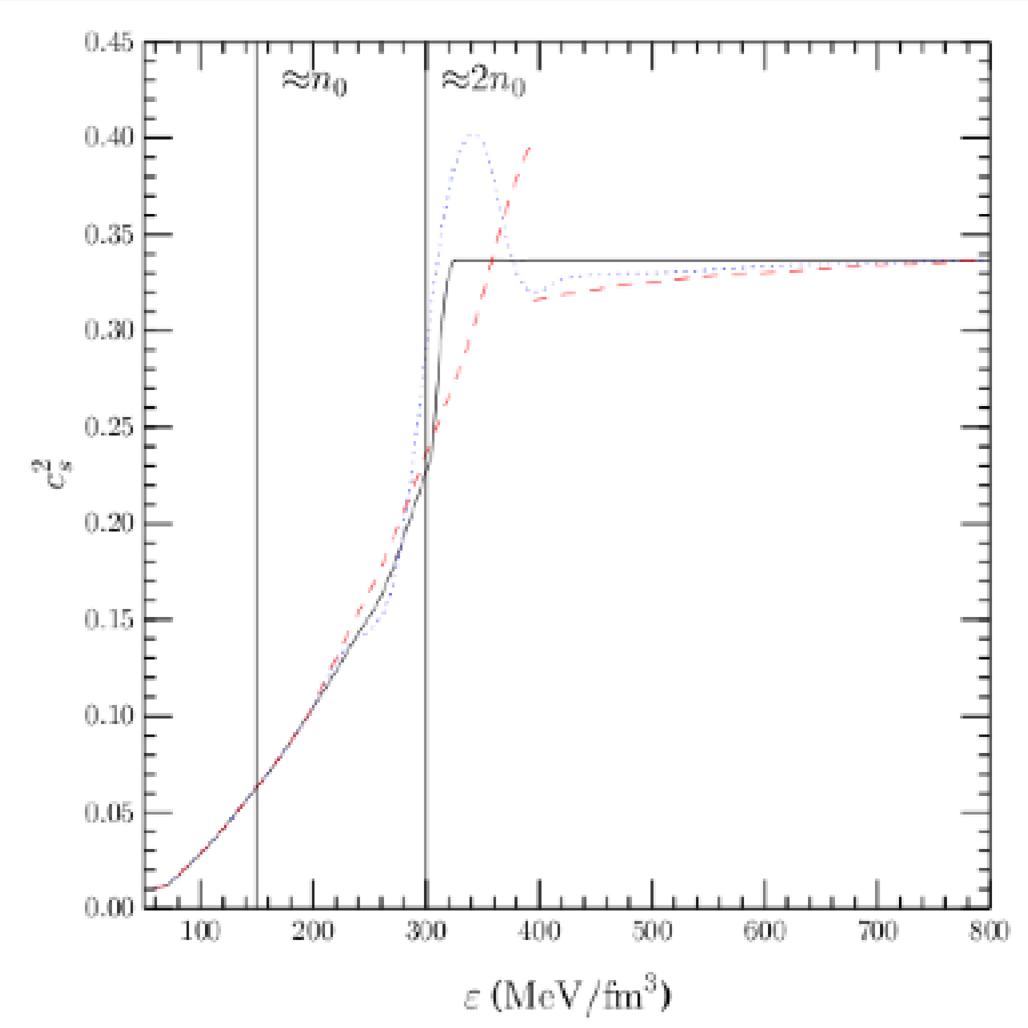
Speed of sound

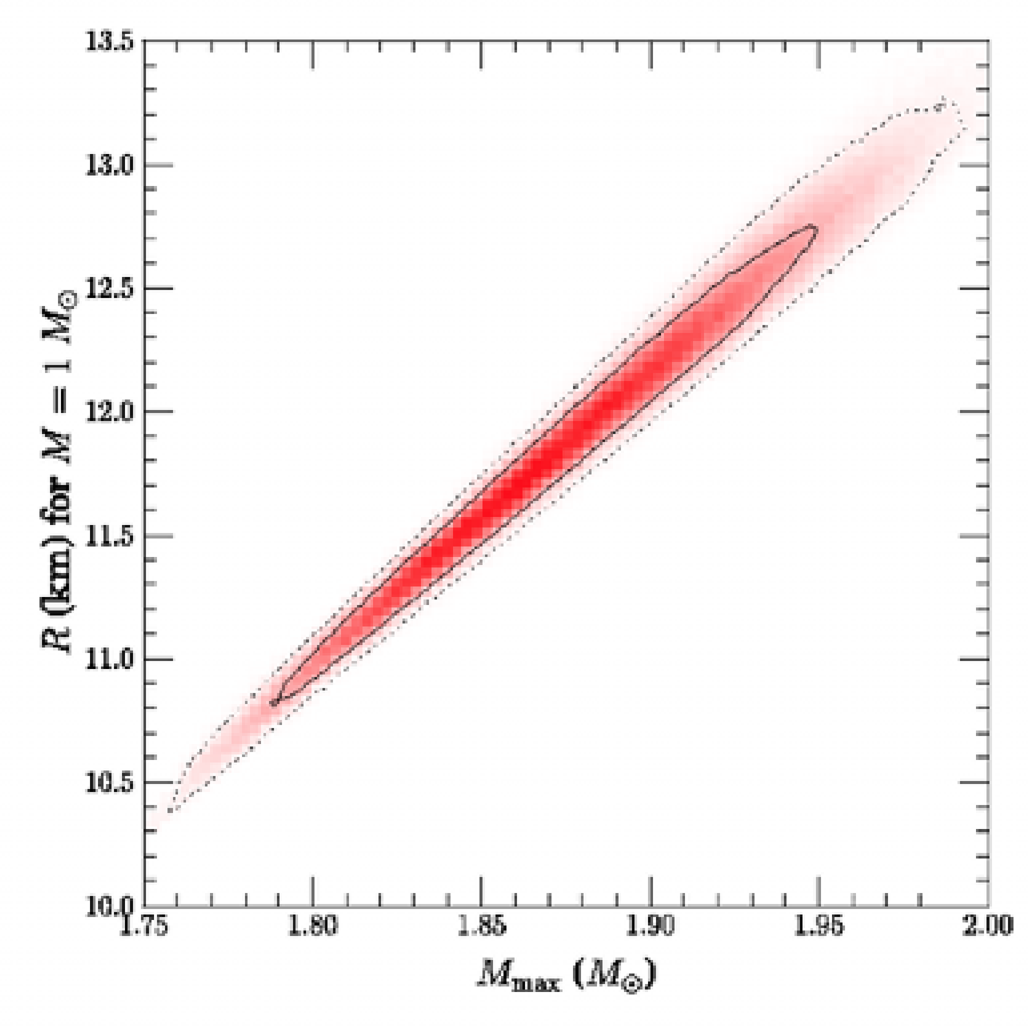


Bazavov et al. (2015)

- The speed of sound at zero density and finite temperature: $c_s^2 \to 1/3$ as $T \to \infty$
- What happens at high density and zero temperature?
- Perturbation theory suggests c_s^2 increases to 1/3 from below
  Kurkela et al. (2010)
- $c_s^2 \approx 1/12$ in neutron matter at the saturation density
- Is $c_s^2 > 1/3$ anywhere in the universe?

Assume $c_s^2 < 1/3$ everywhere





Bedaque and Steiner (2015)

- Assume the speed of sound as large is maximal, but < 1/3 (black curve)
- No! Not unless R is large. c_s^2 must be non-trivial at high densities. Why?
- Implies a phase transition at high-density, or a some new length scale

Models and Phenomenology

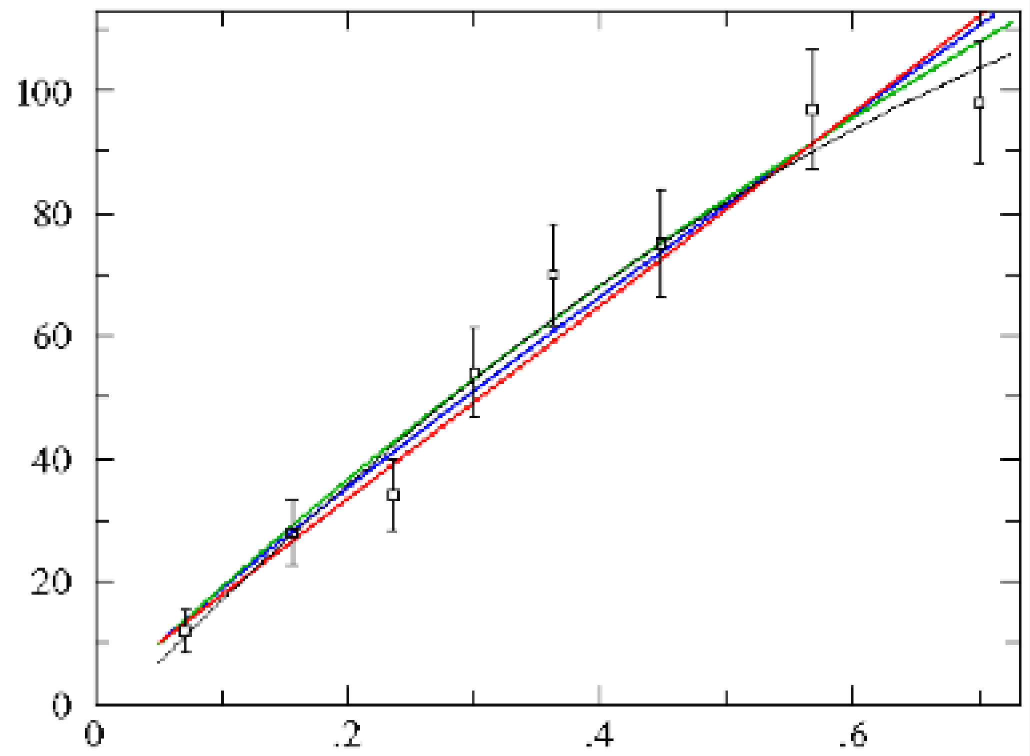
- Except for maybe lattice QCD and perturbation theory, we're almost all doing (at least a little) phenomenology/modeling

- The ultimate test of our models is our ability to match and/or predict experiments and/or observations

- Often, better physics input (e.g. more grounding in QCD) leads to better descriptions of the data and better predictions

- But, simpler and/or analytical models can be extremely helpful in creating understanding

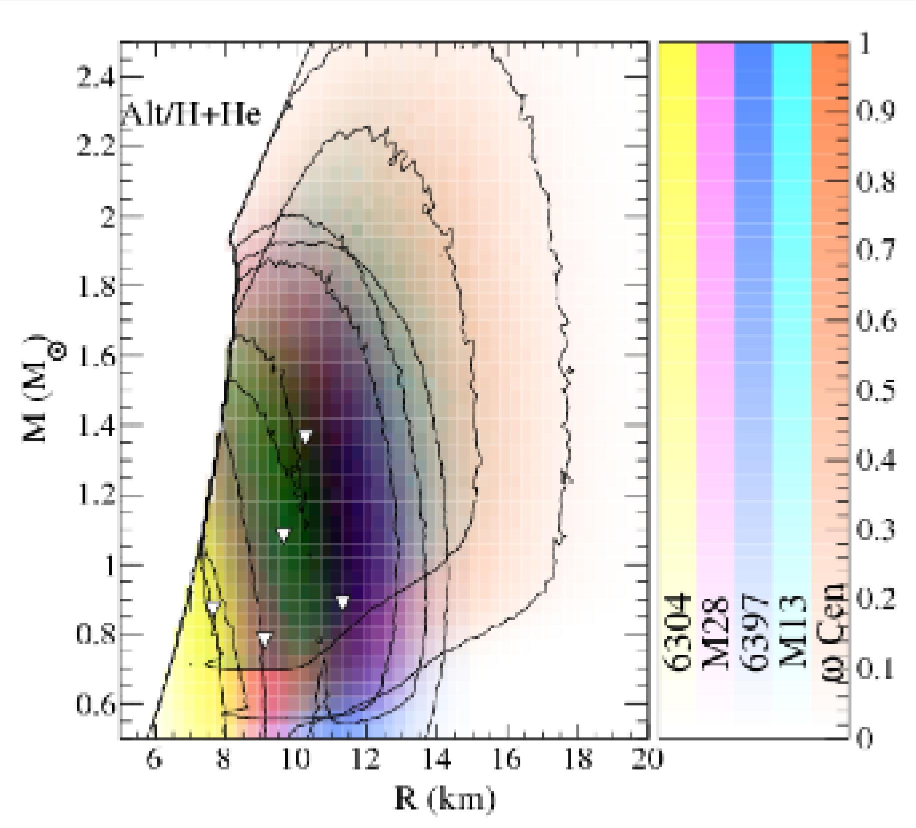
χ^2 fitting



$$\chi^2 = \sum_i \left[\frac{(\text{data})_i - (\text{model})_i}{(\text{err})_i} \right]^2$$

- Traditional χ^2 fit works great when:
    1. Uncertainties are independent Gaussian distributions
    2. There are more data points than parameters
    3. Uncertainties are dominant in one "direction"
    4. Model is not extremely nonlinear and thus the likelihood is nearly Gaussian
- You can minimize χ^2 or maximize the likelihood function $\mathcal{L} = \exp(-\chi^2/2)$
- Look at covariance matrix to determine parameter uncertainties

Two-dimensional fitting problems



Some of the data

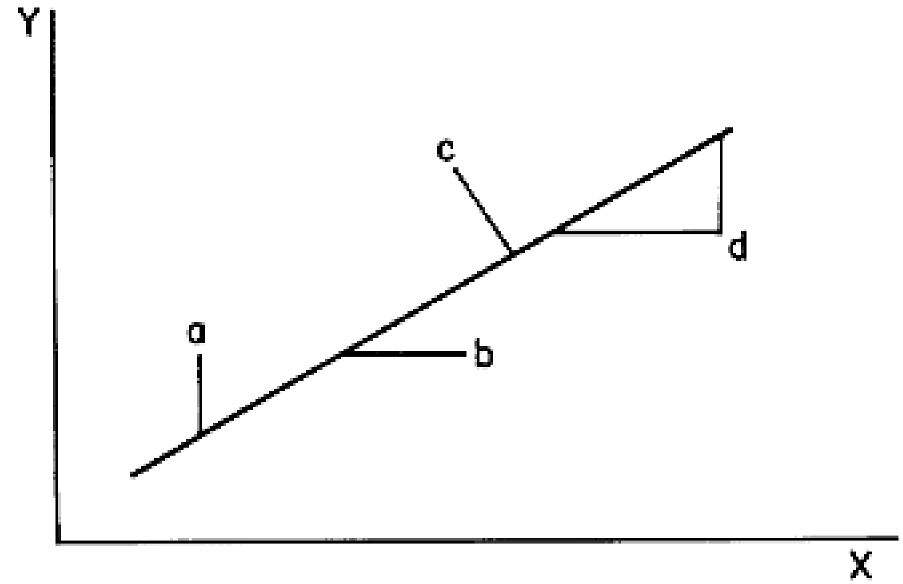


FIG. 1.—Illustration of the different methods for minimizing the distance of the data from a fitted line: (*a*) OLS($Y|X$), where the distance is measured vertically; (*b*) OLS($X|Y$), where the distance is taken horizontally; (*c*) OR, where the distance is measured vertically to the line; and (*d*) RMA, where the distances are measured both perpendicularly and horizontally. No illustration of the OLS bisector is drawn in this figure.

Isobe et al. 1990

- Several frequentist approaches to two-dimensional fitting problems, which go by several names:
  - Reduced major axis regression
  - Geometric mean regression
  - Orthogonal least squares
  - Deming regression

Bayesian Inference

- Bayes theorem:
  $P[\mathcal{M}_i|D] \propto P[D|\mathcal{M}_i]P[\mathcal{M}_i] = \mathcal{L} \times \text{prior}$
- Prior distribution must be specified by the user: initial probability distribution before looking at the data
- Prior distribution is scary for frequentists. It's irrelevant when data is plentiful. When it's important, it helps us quantify the limitations of the data
- Determine parameters through marginalization, i.e.

$$P(\mathcal{M}_i^0) = \int \delta(\mathcal{M}_i - \mathcal{M}_i^0) P[D|\mathcal{M}_i]P[\mathcal{M}_i]\, d\mathcal{M}$$

- Integrals can be computationally demanding
- Reproduces traditional χ^2 fit in the appropriate limits

Fitting two-dimensional data

- Assume Gaussian uncertainties in both x and y, a set of N points $(x_i \pm \delta x_i, y_i \pm \delta y_i)$
- Integrate the model, $y(x, \{p_j\})$, over the data; ambiguity in definition of length (line element, s)

$$\mathcal{L} \propto \prod_i^N \sqrt{\left[\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2\right]} \exp\left\{\frac{-[x - x_i]^2}{2\delta x_i^2}\right\} \exp\left\{\frac{-[y(x, \{p_j\}) - y_i]^2}{2\delta y_i^2}\right\}$$

- Line element

$$d\ell = \sqrt{\left[\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2\right]} ds$$

- Must specify prior distribution over s (nuisance variable) and model parameters $\{p_j\}$
- Limiting forms imply traditional χ^2 fit

A Worked Example

- There are analytical solutions to the TOV equations. Define $\beta = GM/R$. Given the EOS:

$$\varepsilon = 12\sqrt{p_* P} - 5P$$

- The solution to the TOV equations is

$$R = (1 - \beta)[288 p_* G\,(1 - 2\beta)\,/\pi]^{-1/2}$$

- Denote the solution to this equation as $R_{\mathrm{sol}}(p_*, M)$
- Presume one bivariate normal data point:

$$\mathcal{D}(R, M) = \exp\left\{-(R - 10\ \mathrm{km})^2/\left[2 \cdot (1\ \mathrm{km})^2\right]\right\}$$
$$\times \exp\left\{-(R - 1.4\ \mathrm{M}_\odot)^2/\left[2 \cdot (0.1\ \mathrm{M}_\odot)^2\right]\right\}$$

- Choose $s = M$, then integrate

$$P[\mathcal{M}|D] = \mathcal{D}[R_{\mathrm{sol}}(p_*, M), M] \times P_{\mathrm{prior}}(p_*, M)$$

- For example:

$$P(p_*) = \int dM\ \mathcal{D}[R_{\mathrm{sol}}(p_*, M), M] \times P_{\mathrm{prior}}(p_*, M)$$

Quark Matter at Finite Density

- The MIT Bag model

$$P(T=0) = -B + \sum_{i=u,d,s} P_{i,\text{non-interacting}}(T=0)$$

- Bag parameter B models confinement

- Can add density-independent superconducting gaps as well
  e.g. Alford et al. (2005)

- Basic picture: Transition to quarks at high density when their pressure becomes larger than that of nucleons
  Collins and Perry (1975)

- It is not known if this transition to deconfined quark matter is attained in neutron stars

Nambu Jona-Lasinio model

$$\mathcal{L} = \bar{q}_{i\alpha} \left(i\partial\!\!\!/ \delta_{ij}\delta_{\alpha\beta} - m_i\delta_{ij}\delta_{\alpha\beta} - \mu_{ij,\alpha\beta}\gamma^0 \right) q_{j\beta}$$
$$+ G \sum_{a=0}^{8} \left[\left(\bar{q}\lambda^a q\right)^2 + \left(\bar{q}i\gamma_5\lambda^a q\right)^2 \right]$$

Klevansky (1992); Hatsuda and Kunihiro (1994); Buballa (2005); notation from Steiner et al. (2002)

- i and j are flavor indices, α and β are color indices, a is an index over the SU(3) matrices
- Exhibits a chiral phase transition
- Can be formulated in a way that has the same symmetries as QCD
- Often used in the mean-field approximation

$$\bar{q}_1 q_2 \bar{q}_3 q_4 \rightarrow \langle \bar{q}_1 q_2 \rangle \bar{q}_3 q_4 + \bar{q}_1 q_2 \langle \bar{q}_3 q_4 \rangle - \langle \bar{q}_1 q_2 \rangle \langle \bar{q}_3 q_4 \rangle$$

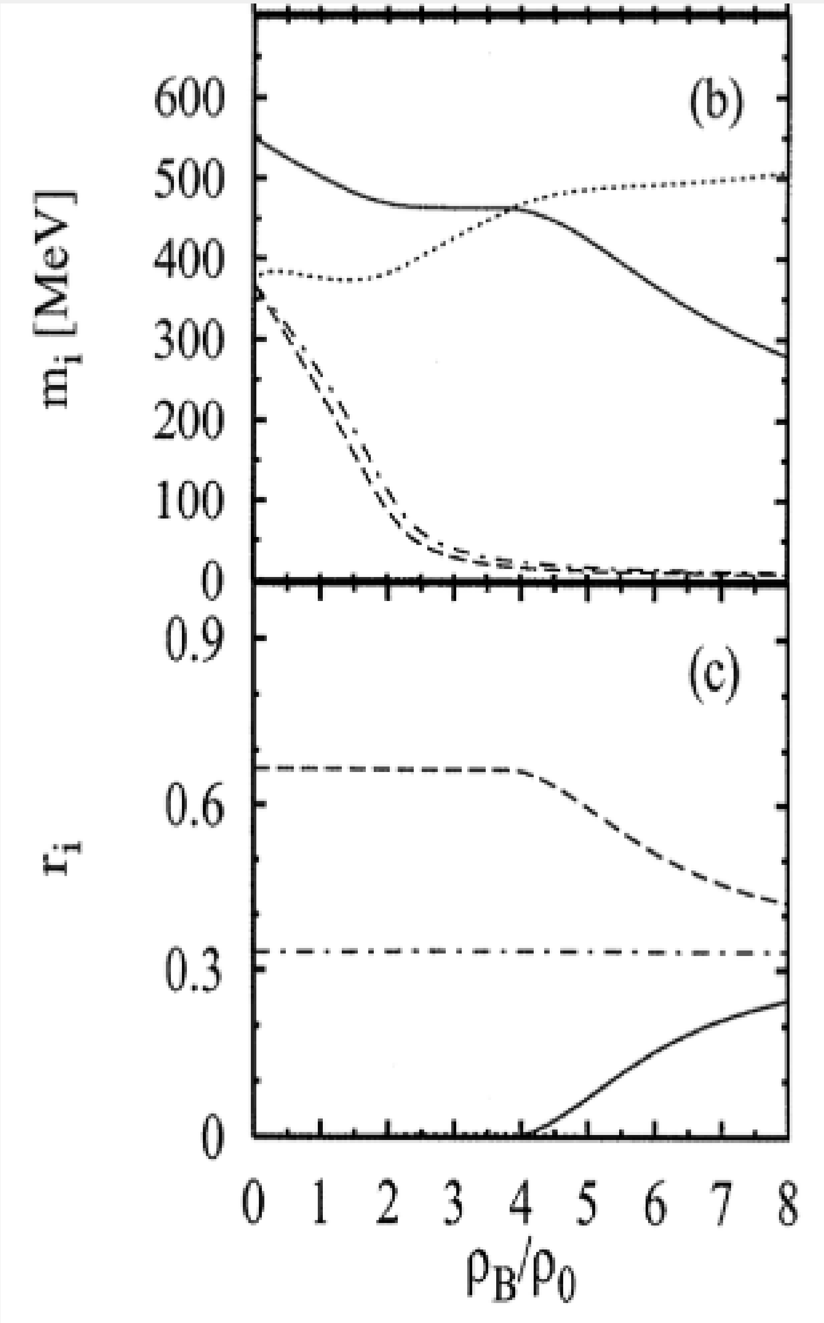
- Maximize pressure w.r.t. $\langle \bar{q} q \rangle$, and $\mu_{ij,\alpha\beta}$
- Non-renormalizable, UV cutoff at $p = \Lambda$

Dynamically Generated Mass

- $\langle \bar{q}q \rangle$ plays the role of a new thermodynamic parameter
- Maximize the pressure
- Minimization with respect to $\langle \bar{q}q \rangle$ leads to a "dynamically generated mass"
- This is the so-called "mass-gap" equation

$$m_i^* = m_i - 4G \langle \bar{q}_i q_i \rangle$$

$$\langle \bar{q}q \rangle = -\frac{3}{\pi^2} \int_{k_{Fi}}^{\Lambda} dk\, k^2 \frac{m_i^*}{\sqrt{m_i^{*2} + k^2}}$$



Buballa and Oertel (1999)

Color superconductivity

- Model for color superconductivity

$$\mathcal{L}_\Delta = G_\Delta \sum_k \sum_\gamma \left(\bar{q}_{i\alpha}\epsilon_{ijk}\epsilon_{\alpha\beta\gamma}q^C_{j\beta}\right)\left(\bar{q}^C_{i'\alpha'}\epsilon_{i'j'k}\epsilon_{\alpha'\beta'\gamma}q_{j'\beta'}\right)$$

$$+G_\Delta \sum_k \sum_\gamma \left(\bar{q}_{i\alpha}i\gamma_5\epsilon_{ijk}\epsilon_{\alpha\beta\gamma}q^C_{j\beta}\right)\left(\bar{q}^C_{i'\alpha'}i\gamma_5\epsilon_{i'j'k}\epsilon_{\alpha'\beta'\gamma}q_{j'\beta'}\right)$$

Notation from Steiner et al. (2002)

- Leads to a phase transition, analagous to chiral phase transition

$$\Delta^{k\gamma} = 2G_\Delta \left\langle \bar{q}_{i\alpha}i\gamma_5\epsilon^{ijk}\epsilon^{\alpha\beta\gamma}q^C_{j\beta}\right\rangle$$

- Similar mean field approximation, but with anomalous propagators
- Nambu-Gorkov formalism
- Leads to new "dispersion relation", $E(k) \neq \sqrt{k^2 + m^{*2}}$

Dispersion relation from a Dirac-like Lagrangian

- Thermal field theory: partition function is just related to the determinant of the inverse propagator $\log Z = \log \det D$

- Determinant operation carried out over Dirac indices and momentum-frequency space

- Matrix representing inverse propagator ($\beta = 1/T$)

$$D = -i\beta \left[(-i\omega_n + \mu) - \gamma^0 \vec{\gamma} \cdot \vec{p} - m\gamma^0 \right]$$

$$\log Z = 2 \sum_n \sum_{\vec{p}} \log\{\beta^2 \left[(\omega_n + i\mu)^2 + p^2 + m^2 \right] \}$$

- Leads to (see details here)

$$PV = \log Z = 2V \int \frac{d^3k}{(2\pi)^3} \{\log[1 + e^{-\beta(\omega-\mu)}] + \log[1 + e^{-\beta(\omega+\mu)}]\}$$

Problem 4

- Superconductivity in the Nambu-Gorkov formalism leads to an inverse propagator of the form

$$D = -i\beta \begin{bmatrix} -i\omega_n - \gamma^0\vec{\gamma}\cdot\vec{p} - m\gamma^0 + \mu & i\Delta\gamma^0\gamma_5 C \\ i\Delta\gamma^0 C\gamma_5 & -i\omega_n - \gamma^0\vec{\gamma}^T\cdot\vec{p} + m\gamma^0 - \mu \end{bmatrix}$$

- where $C = i\gamma^0\gamma^2$ is the charge conjugate matrix in the Dirac representation

- Compute the determinant and, in analogy to the normal case, determine the new dispersion relation in the case where $m = 0$