ETH zürich

Institute for Theoretical
Physics, HPCP lab

Introduction to Lattice QCD

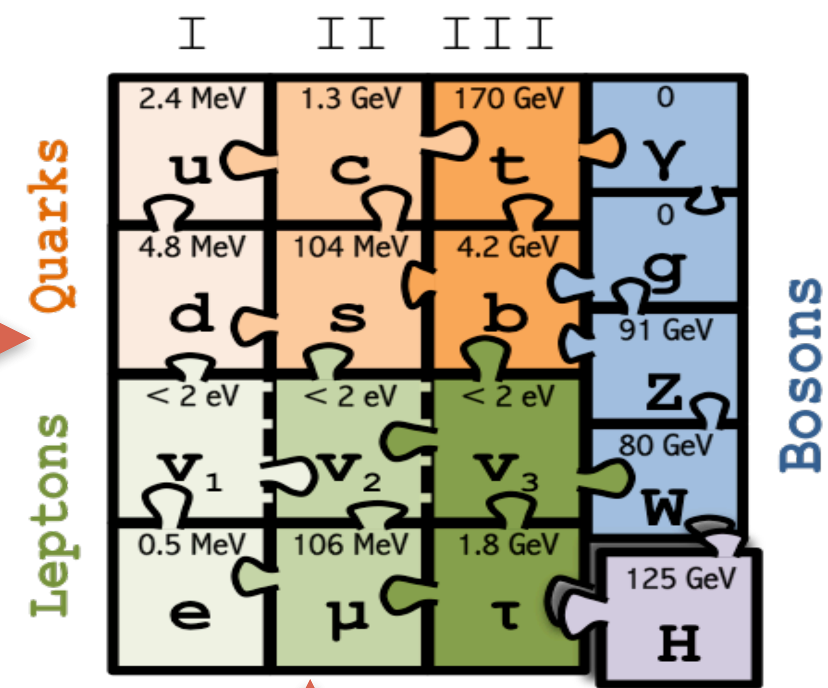
Prof. Marina Krstic Marinkovic, ETH Zurich

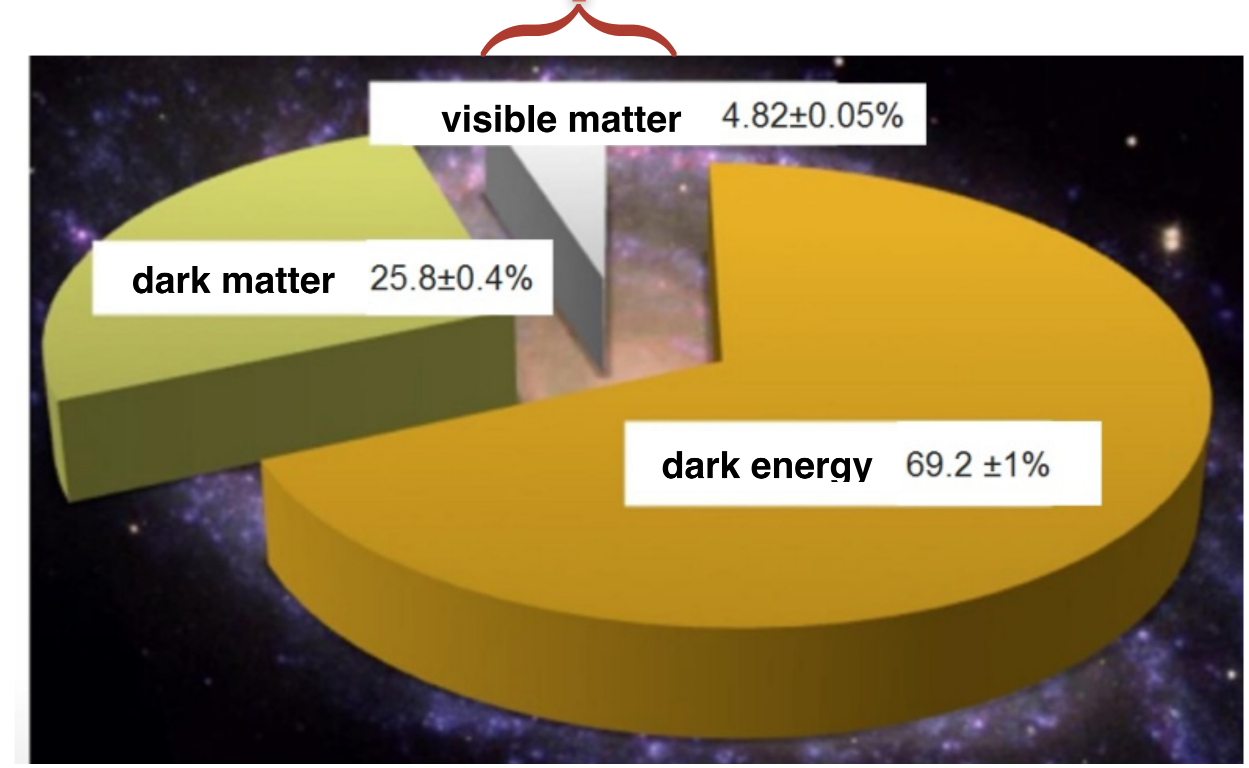
11-22 July 2022, NNPSS, MIT

Building blocks of the universe

- **Four fundamental interactions:**
  - ➡ Electromagnetism
  - ➡ Weak
  - ➡ Strong
  - ➡ Gravity

Standard Model
of Particle Physics





[Planck CMB Data (2013 and 2015)]

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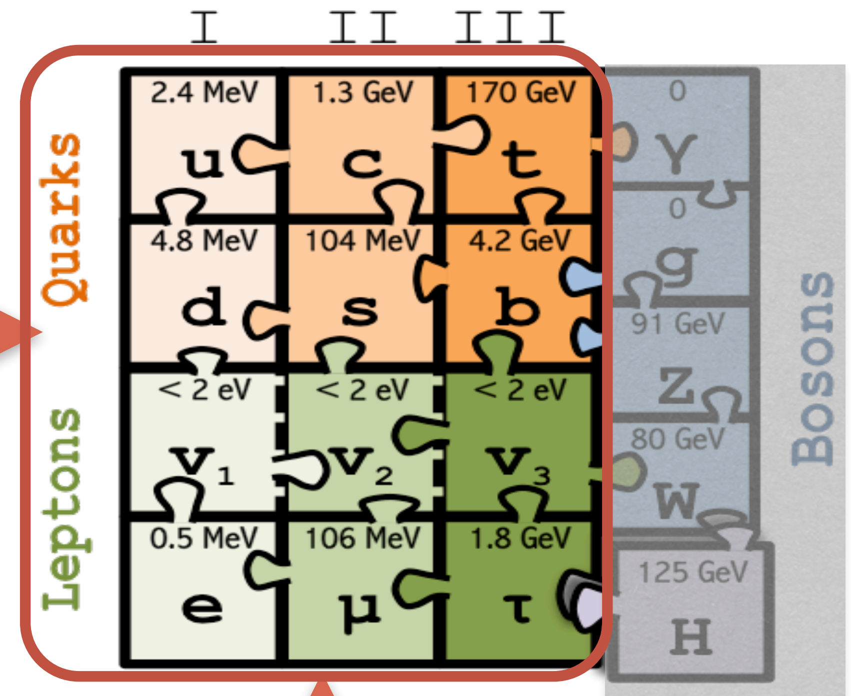

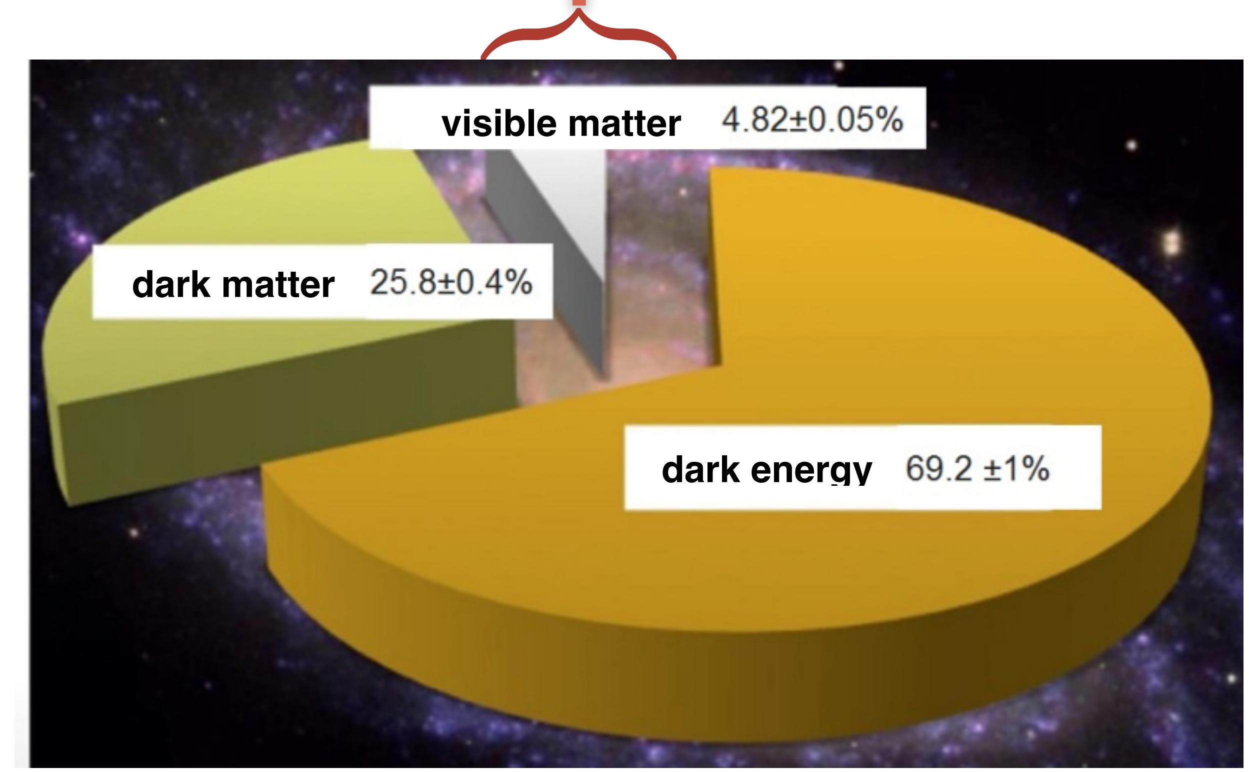

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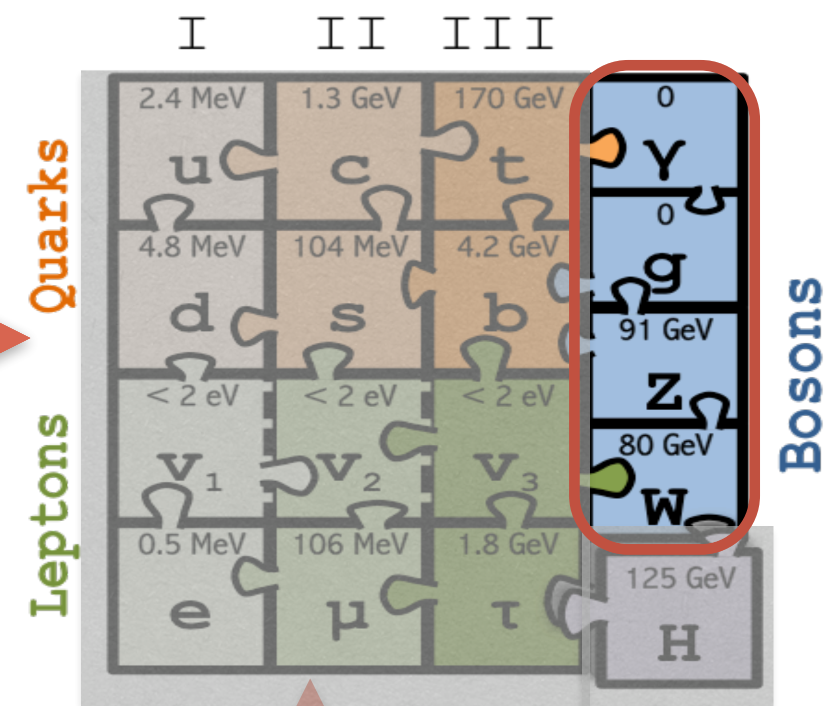

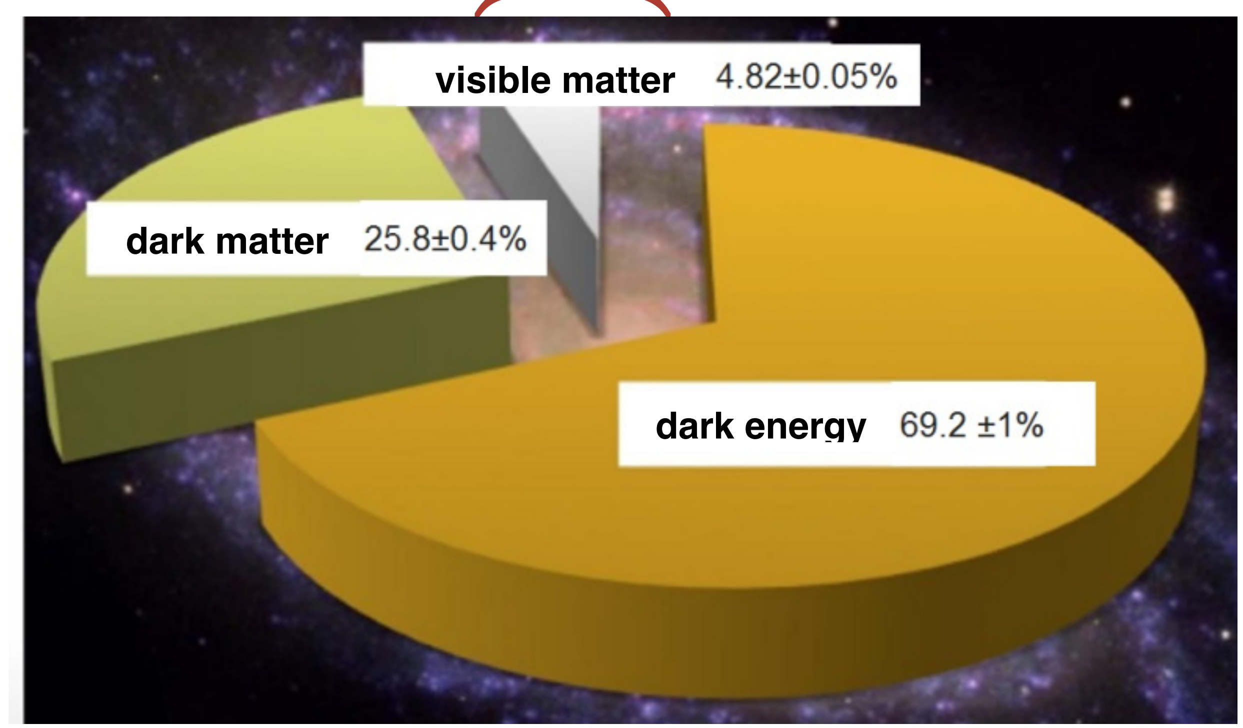

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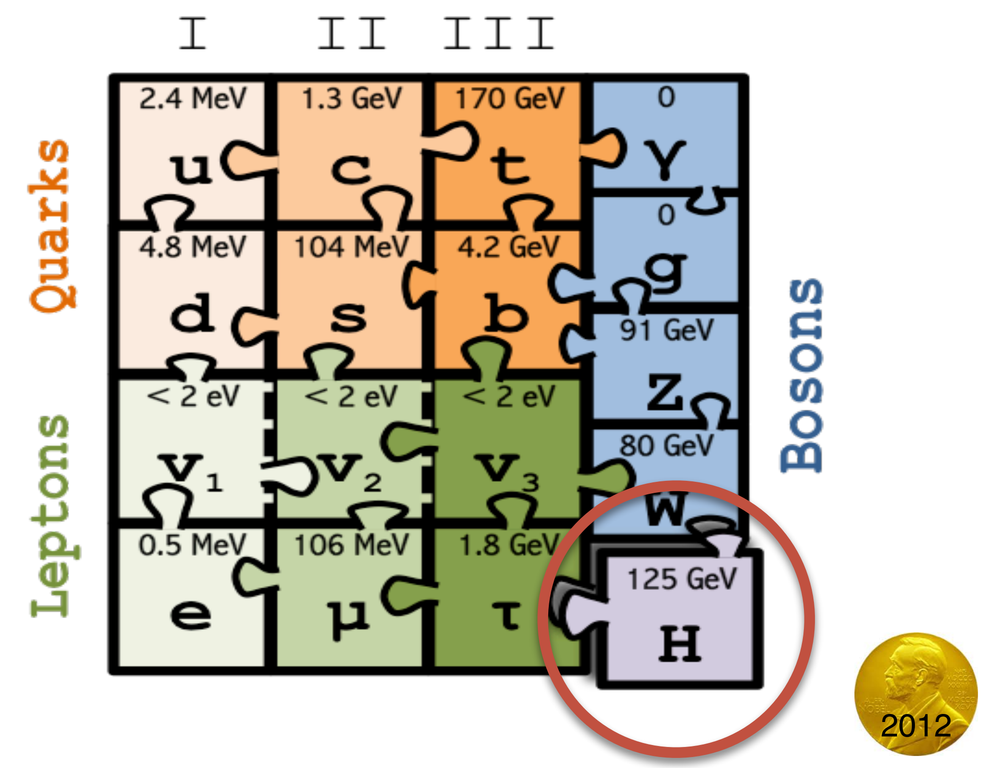

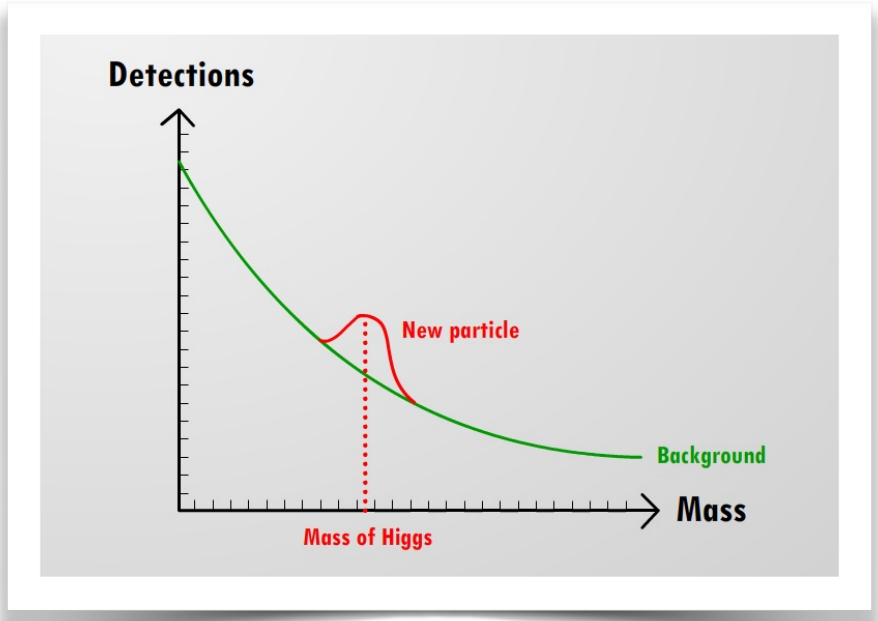

[https://blog.higgshunters.org/]

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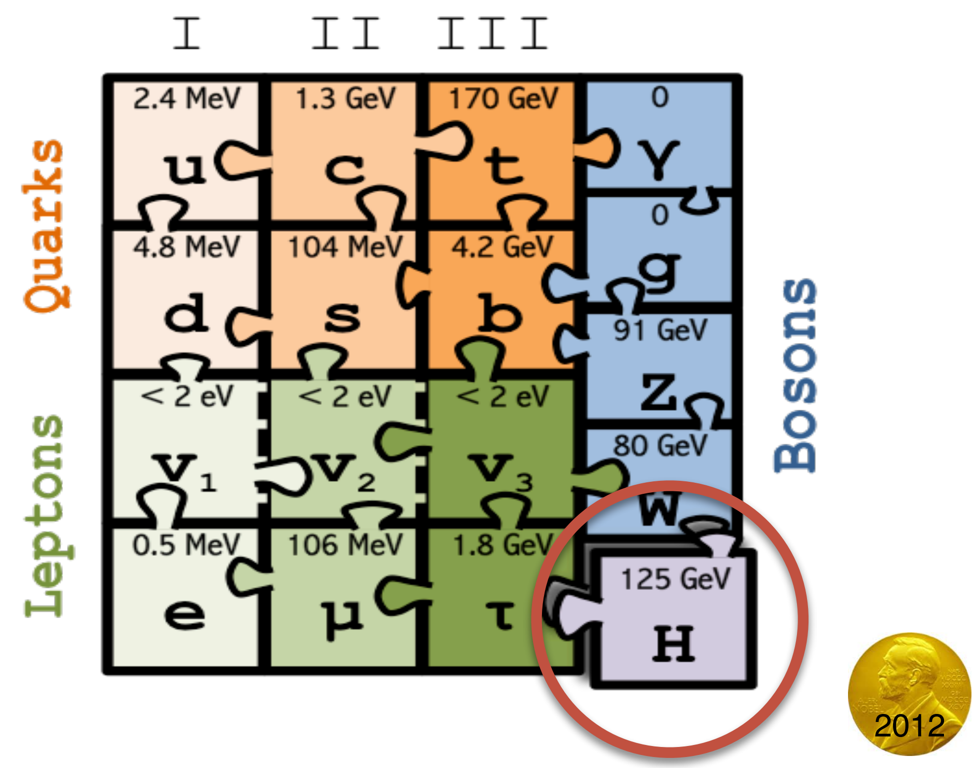

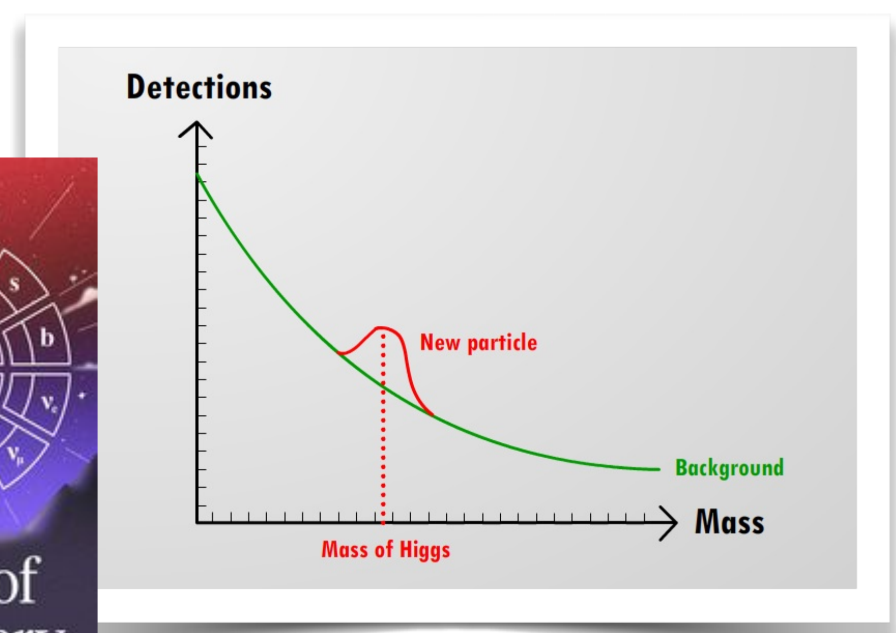

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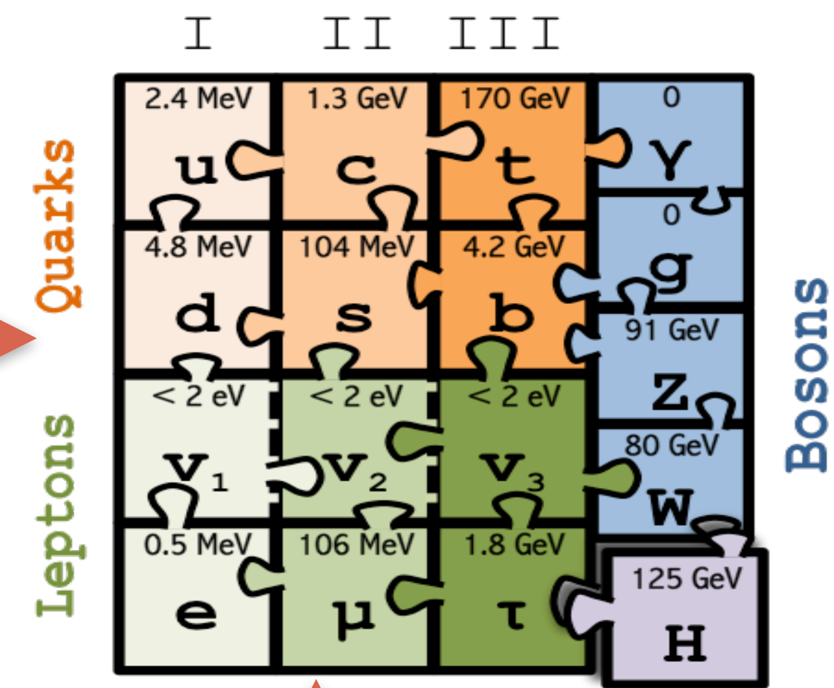
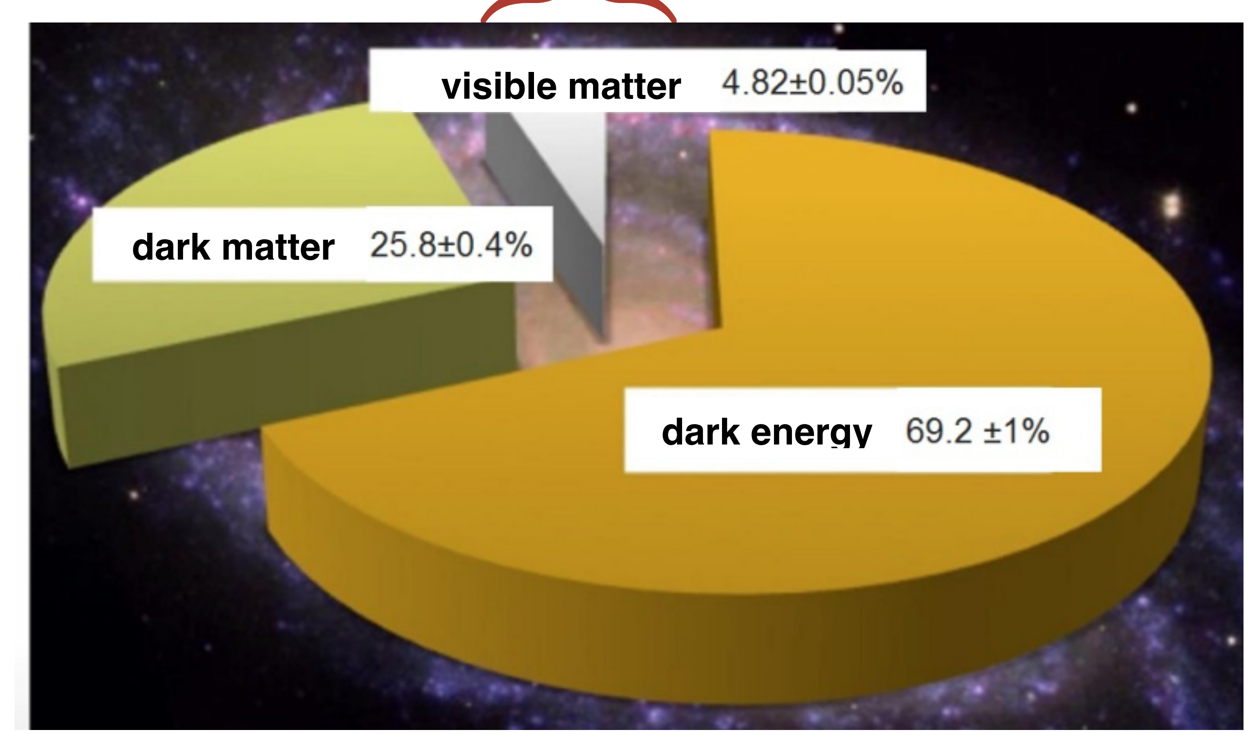


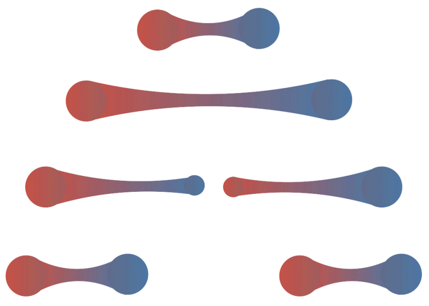


See lectures in this school:
[Ronald Garcia Ruiz, Week1, Mon-Tue]
[Maria Piarulli, Week1, Mon-Wed]

[Planck CMB Data (2013 and 2015)]

The strong interaction

- **Commonly accepted theory of strong interaction is Quantum Chromodynamics (QCD)**
- Fundamental parameters of **QCD:**
  - ➡ gauge coupling: g
  - ➡ quark masses: $m_u, m_d, m_s, \ldots$



- **Two exciting properties of QCD:**
  - ➡ color confinement
  - ➡ asymptotic freedom
- Strong coupling "constant":

$$\alpha_s(Q^2) = \frac{\bar{g}^2(Q^2)}{4\pi}$$

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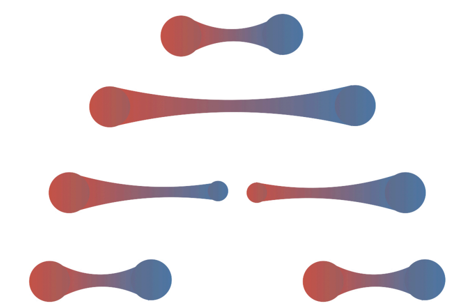

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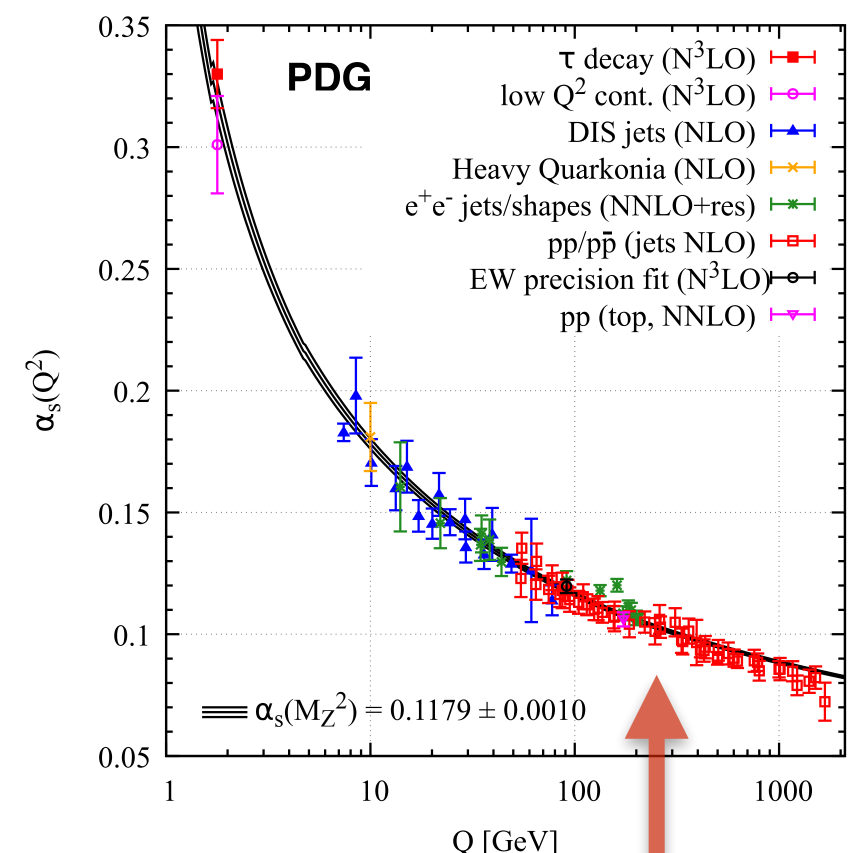
[D. Gross, F. Wilczek; D. Politzer (1973)]



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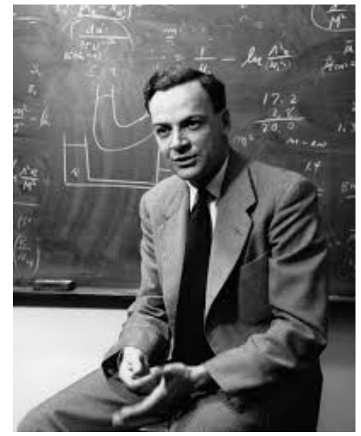
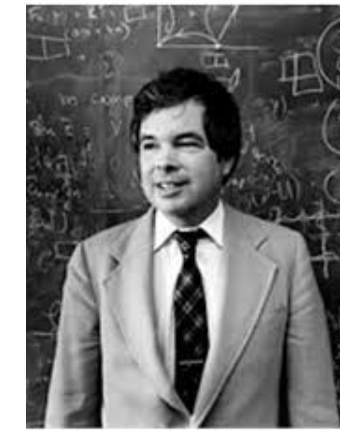
Non-perturbative treatment of QCD

[R. P. Feynman, "Space-Time Approach to Non-Relativistic Quantum Mechanics" Rev. Mod. Phys. 20, 367 (1948)]



[K. Wilson, "Confinement of quarks" *Phys. Rev.D.* **10** (8): 2445–245 (1974)]

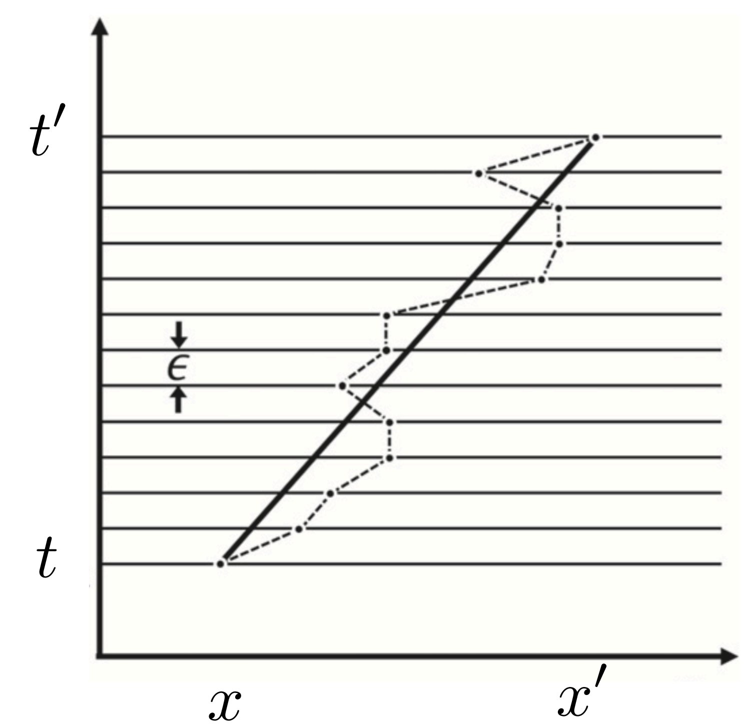


(1) Path Integral quantization

(2) Continuation to Euclidean time

(3) Lattice regularization

$$\langle x'|U(t',t)|x\rangle = \int dx_1 \, \langle x'|U(t',t_1)|x_1\rangle \, \langle x_1|U(t_1,t)|x\rangle$$



$$\langle x'|U(t',t)|x\rangle \; = \; \int \mathcal{D}x \; e^{\frac{i}{\hbar} S[x]}$$

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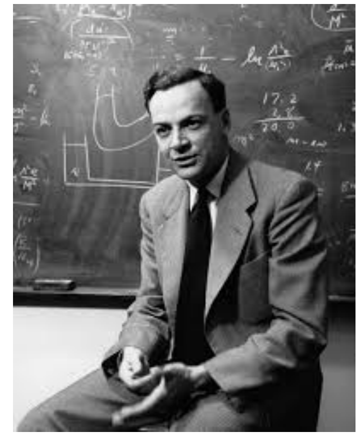

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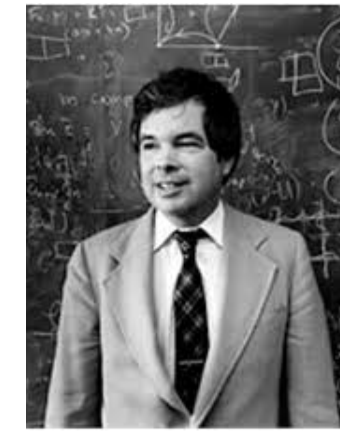

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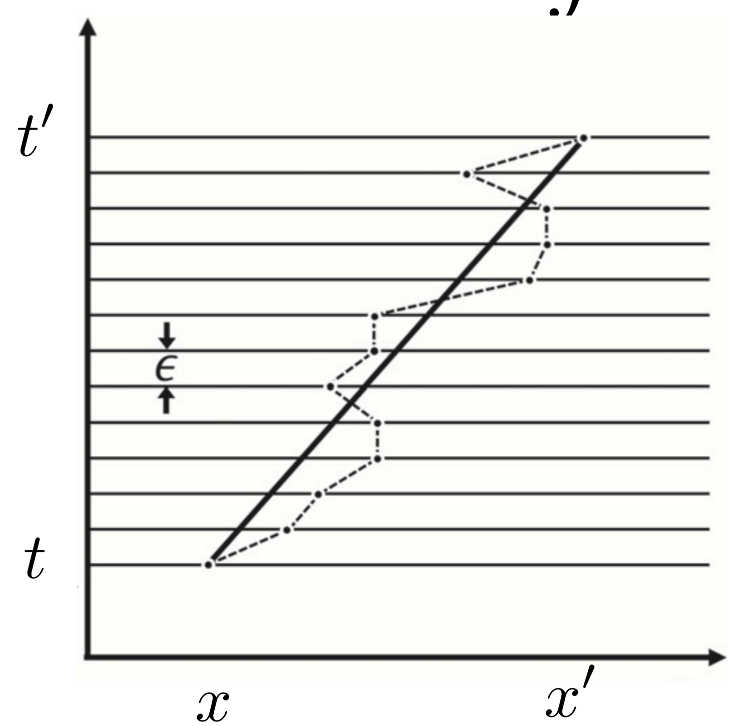

$$\epsilon \longrightarrow -i\, a$$

$$Z = \int \mathcal{D}x \, e^{-\frac{1}{\hbar} S_E[x]}$$

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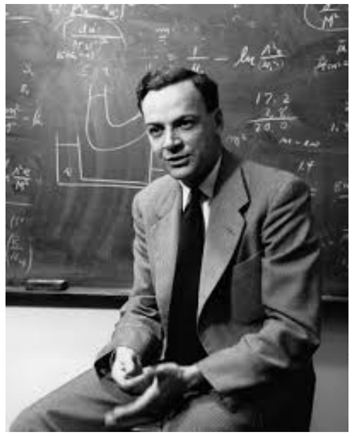

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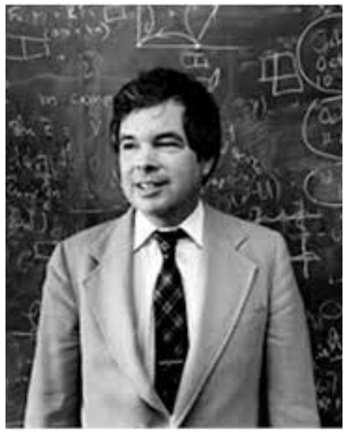

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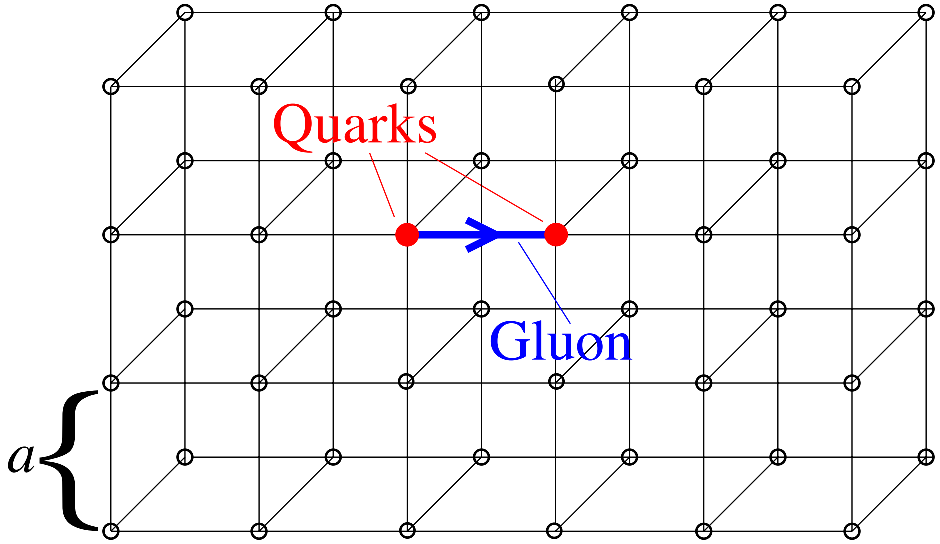

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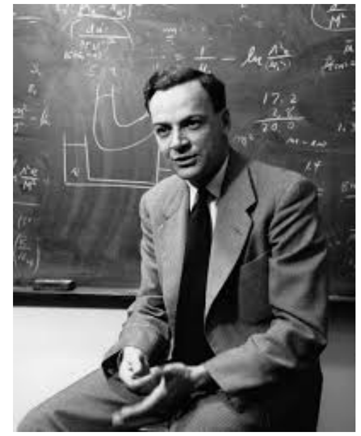

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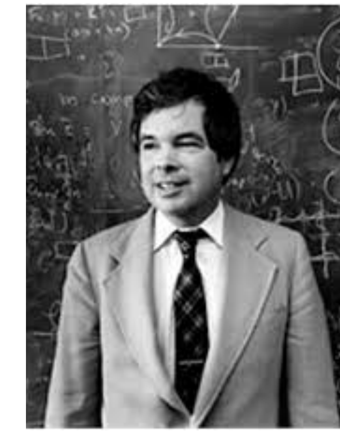

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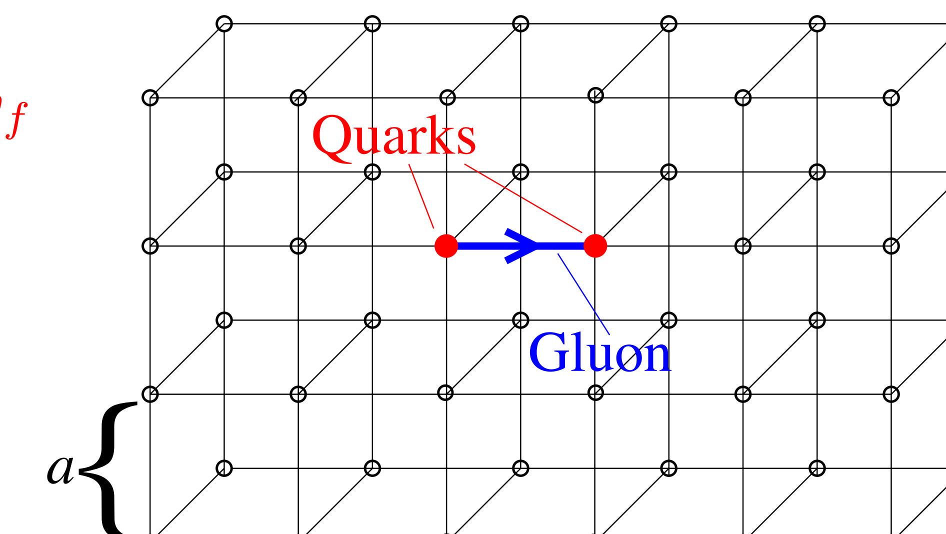
$$\mathcal{L}^E_{QCD} = \frac{1}{2g} F^a_{\mu\nu}F^a_{\mu\nu} + \sum_{f=u,d,s,\dots} \overline{\psi}_f \left\{ \gamma_\mu \left(\partial_\mu + i A^a_\mu T^a\right) + m_f \right\} \psi_f$$

$$S^E_{QCD} = \int d^4x \, \mathcal{L}^E_{QCD}$$



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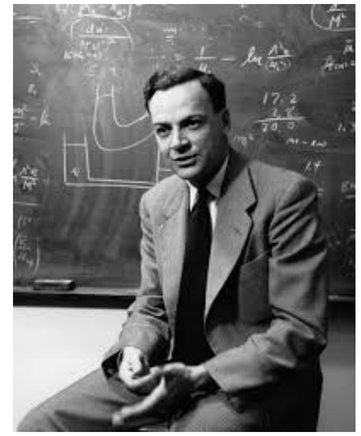

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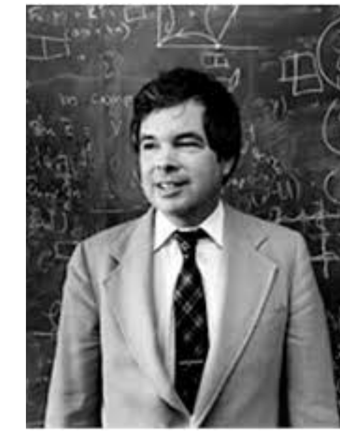

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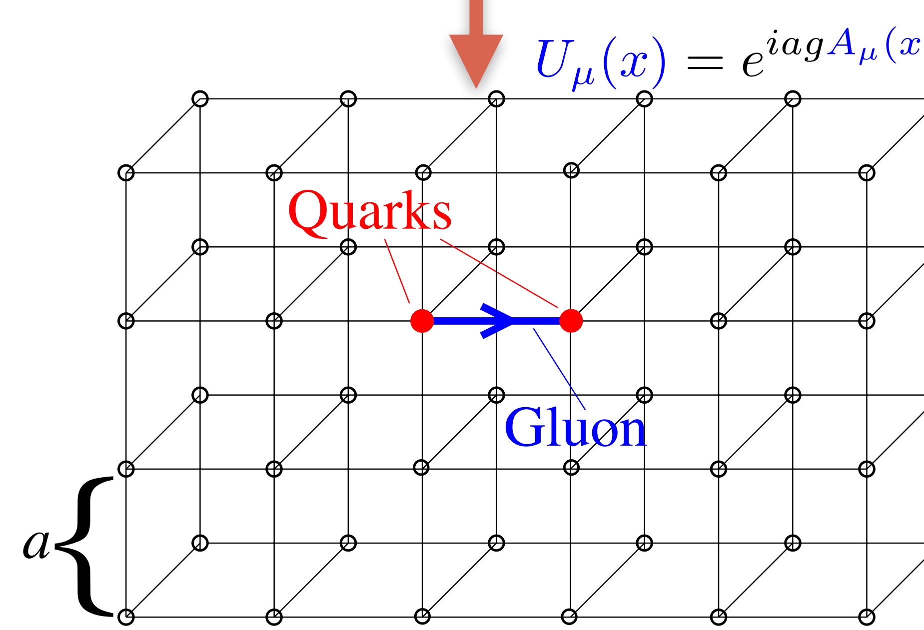
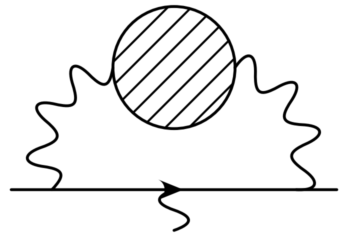
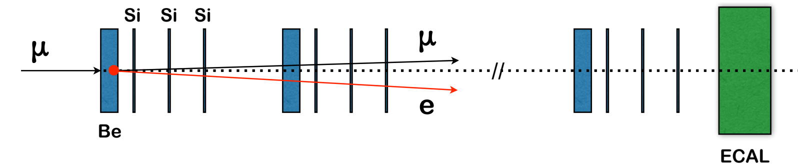
→ CLASSICAL STATISTICAL MECHANICS

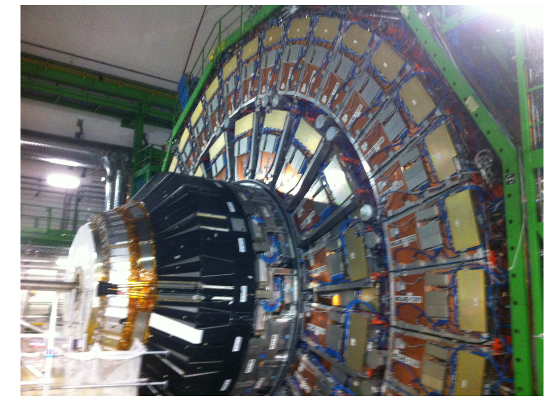
$$Z = \int \mathcal{D}U \, \mathcal{D}\psi \, \mathcal{D}\overline{\psi} \, e^{-S^E_{QCD}}$$



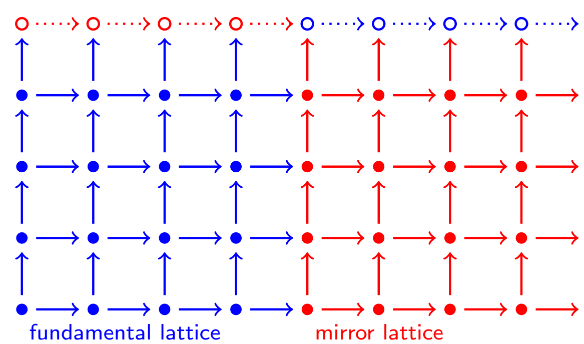


Physics
(Theory&Experiment)

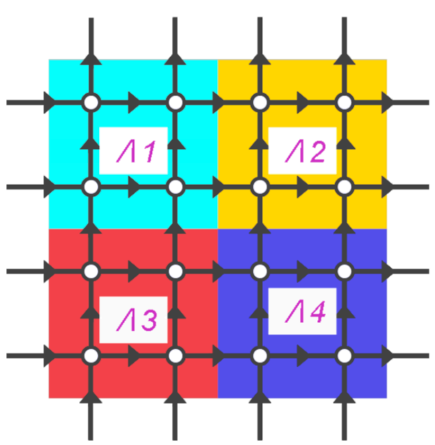




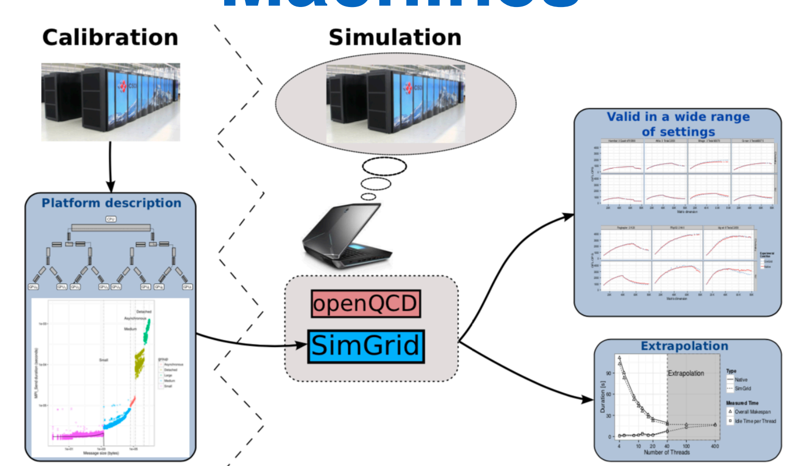
Lattice QCD



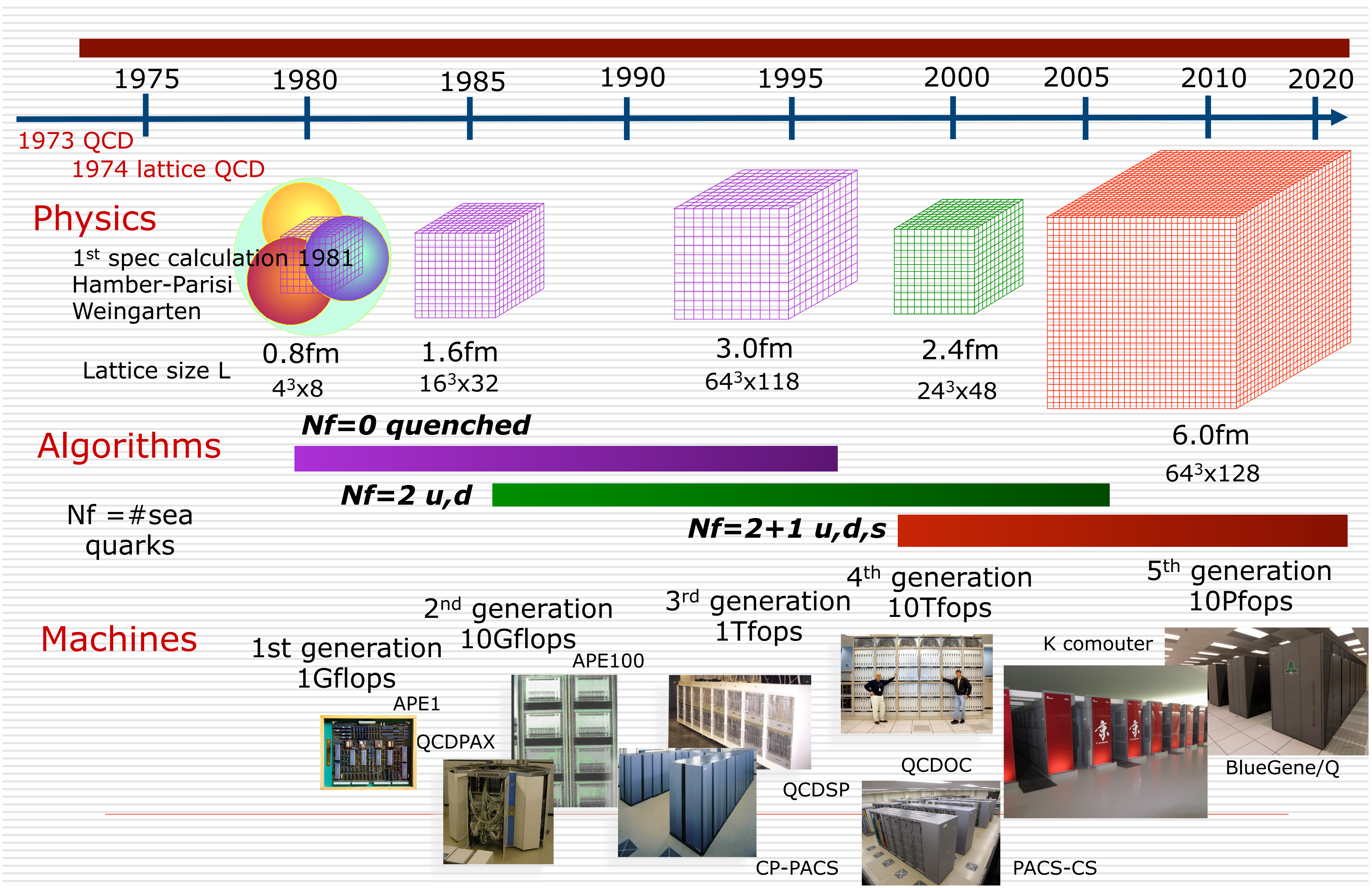
Computational Methods



Machines

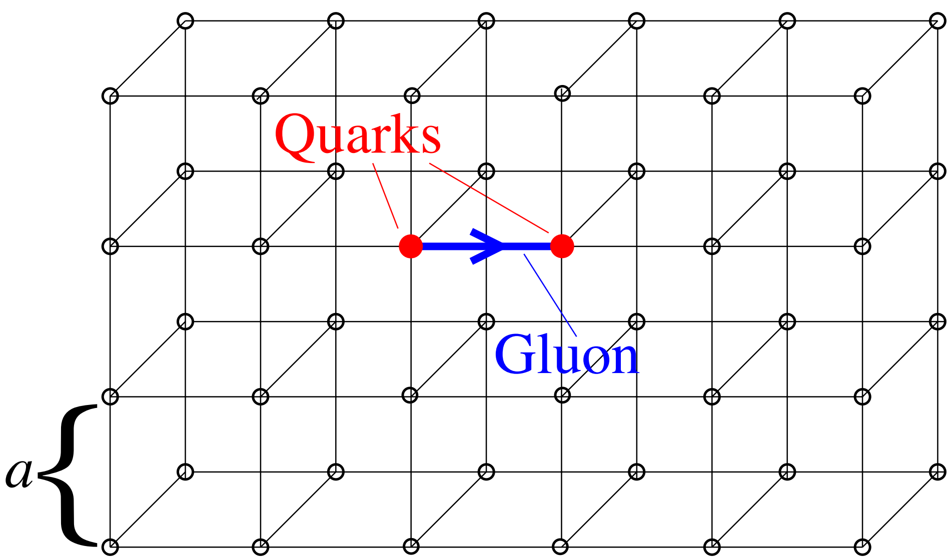


Development of Lattice QCD



[Credit: A. Ukawa, HPC Summer School (2013)]

Typical Lattice QCD Computation



https://www.cscs.ch/publications/photo-gallery/]

Altamira@IFCA, Santander

Wilson Cluster at Mainz U.

- physical lattice size: **~6fm**, spacing **0.05-0.1fm**
- **64^3 x 128** lattice —> **34 x 10^6** points
- Operator dimensions: **10^7 x 10^7** matrices
- **Advanced computational methods** are needed
- **Large computer resources**: multiple TFlop years!

Backstage of LGT Calculations

[bbc tyne in pictures backstage at theatre royal; image by www.bbc.co.uk]

Plan of the Lectures:

- **Lecture 1: Path Integral Quantization and scalar fields on the lattice;** *Why do we need to consider discrete space/time?*

- **Lecture 2: QCD on the lattice; Computational methods for lattice field theories;** *Why is LQCD so comp. expensive? What are the limitations?*

- **Lecture 3: Lattice QCD phenomenology: selected topics** (Muon g-2, flavor physics, machine learning, quantum computing applications and more) *Where can we move the needle with LQCD?*

Part 1:
Path Integral Quantization and scalar fields on the lattice

Outline

- Point Mechanics vs. Classical Field Theory
- Path Integral in Quantum Mechanics (real and Euclidean time)
- Scalar Field Theory on the lattice
- Analogy with Statistical Mechanics and continuum limit
- Spectrum of the lattice Scalar Theory

Classical Point Mechanics

- **Point Mechanics — 2nd Newton's law:**

$$m\frac{d^2x}{dt^2} = m\,\partial_t^2 x = F(x) = -\frac{dV(x)}{dx}$$

m **— mass of the classical non-relativistic point particle**

$V(x)$ **— external potential**

- **The principle of least action:**

$$S[x] = \int dt\ \mathcal{L}(x, \partial_t x)$$

- **Lagrange function:**

$$\mathcal{L}(x, \partial_t x) = \frac{m}{2}(\partial_t x)^2 - V(x)$$

- **Euler-Lagrange equation:**

$$\partial_t \frac{\delta \mathcal{L}}{\delta(\partial_t x)} - \frac{\delta \mathcal{L}}{\delta x} = 0$$

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Classical Field Theory

- **Generalization to systems with infinitely many d.o.f. (field values $\phi(\vec{x})$)**
- **Classical field e. o. m. for neutral scalars — Klein-Gordon eq.:** $\partial_\mu \partial^\mu \phi = -\frac{dV(\phi)}{d\phi}$
- **Again, the classical e. o. m obtained by minimizing the action:**

$$S[\phi] = \int d^4x \; \mathcal{L}(\phi, \partial_\mu \phi)$$

$$\partial_\mu \frac{\delta \mathcal{L}}{\delta(\partial_\mu x)} - \frac{\delta \mathcal{L}}{\delta x} = 0$$

- **A simple example of interacting scalar field theory (`$\phi^4$ — *theory*`)**

$$V(\phi) = \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4$$

m **— mass of the scalar field**

λ **— coupling strength of its self-interaction**

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Point Mechanics vs. Classical Field Theory

Point Mechanics	Field Theory
time t	space-time $x = (t, \vec{x})$
particle coordinate x	field value ϕ
particle path $x(t)$	field configuration $\phi(x)$
action $S[x] = \int dt \ L(x, \partial_t x)$	action $S[\phi] = \int d^4x \ \mathcal{L}(\phi, \partial_\mu \phi)$
Lagrange function $L(x, \partial_t x) = \frac{m}{2}(\partial_t x)^2 - V(x)$	Lagrangian $\mathcal{L}(\phi, \partial_\mu \phi) = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - V(\phi)$
equation of motion $\partial_t \frac{\delta L}{\delta(\partial_t x)} - \frac{\delta L}{\delta x} = 0$	field equation $\partial_\mu \frac{\delta L}{\delta(\partial_\mu \phi)} - \frac{\delta L}{\delta \phi} = 0$
Newton's equation $\partial_t^2 x = -\frac{dV(x)}{dx}$	Klein-Gordon equation $\partial_\mu \partial^\mu \phi = -\frac{dV(\phi)}{d\phi}$
kinetic energy $\frac{m}{2}(\partial_t x)^2$	kinetic energy $\frac{1}{2}\partial_\mu \phi \partial^\mu \phi$
harmonic oscillator potential $\frac{m}{2}\omega^2 x^2$	mass term $\frac{m^2}{2}\phi^2$
anharmonic perturbation $\frac{\lambda}{4}x^4$	self-interaction term $\frac{\lambda}{4}\phi^4$

[Credit: U.-J. Wiese https://inspirehep.net/literature/946884]

- **Point Mechanics vs. Classical Field Theory**

Path Integral in Quantum Mechanics: real time (I)

- **Time-dependent Schrödinger Eq.**

$$i\hbar\partial_t \left|\Psi(t)\right\rangle \;=\; H\left|\Psi(t)\right\rangle$$

- **Time evolution operator**

$$U(t',t) = e^{-\frac{i}{\hbar}H(t'-t)}; \qquad \left|\Psi(t')\right\rangle \;=\; U(t',t)\left|\Psi(t)\right\rangle$$

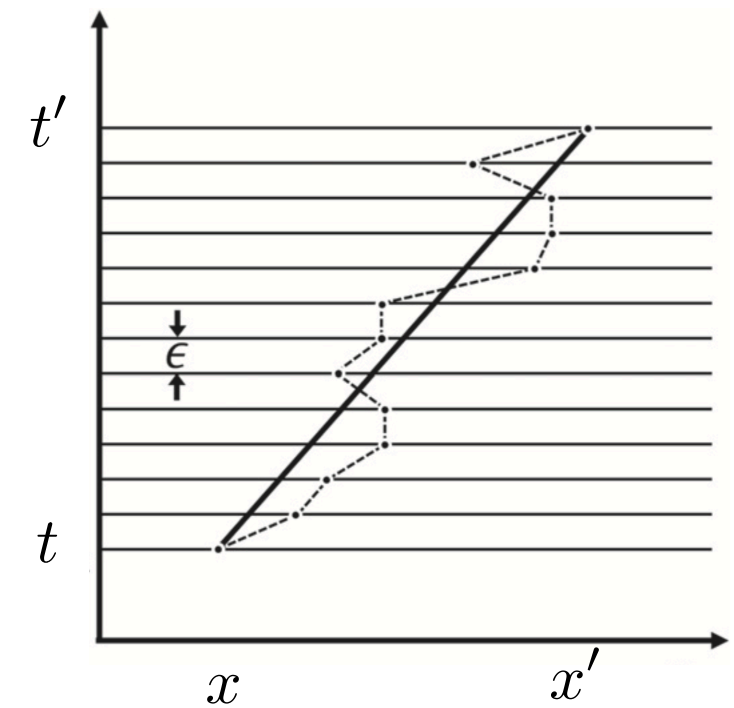
H – **time independent Hamilton operator**

- **Transition amplitude of a non-relativistic point particle — a propagator:**

$$\langle x'|U(t',t)|x\rangle$$

- **Contains information about the energy spectrum of the theory.**

Path Integral in Quantum Mechanics: real time (II)



- **Propagator from x to x'**

$$\langle x'|U(t',t)|x\rangle = \int dx_1 \, \langle x'|U(t',t_1)|x_1\rangle \, \langle x_1|U(t_1,t)|x\rangle$$

- **Divide time interval into N elementary steps of size**

$$[t',t]; \qquad t'-t = N\,\epsilon$$

- **Repeat previous procedure at all intermediate times**

$$\langle x'|U(t',t)|x\rangle = \int dx_1 \int dx_2 \, \ldots \int dx_{N-1} \, \langle x'|U(t',t_{N-1})|x_{N-1}\rangle \times \, \ldots \, \times \langle x_2|U(t_2,t_1)|x_1\rangle \times \langle x_1|U(t_1,t)|x\rangle$$

- **Take factor** $\langle x_{i+1}|U(t_{i+1},t_i)|x_i\rangle$**; assume single non-relativistic point particle**

$$H = \frac{p^2}{2m} + V(x)$$

- **Insert complete set of states + BCH formula:**

$$\langle x_{i+1}|U(t_{i+1},t_i)|x_i\rangle = \frac{1}{2\pi} \int dp \; e^{-\frac{i\epsilon p^2}{2m\hbar}} \; e^{-\frac{i}{\hbar}p(x_{i+1}-x_i)} \; e^{-\frac{i\epsilon}{\hbar}V(x_i)}$$

Path Integral in Quantum Mechanics: real time (III)

$$\langle x_{i+1}|U(t_{i+1},t_i)|x_i\rangle = \frac{1}{2\pi}\int dp\; e^{-\frac{i\epsilon p^2}{2m\hbar}}\; e^{-\frac{i}{\hbar}p(x_{i+1}-x_i)}\; e^{-\frac{i\epsilon}{\hbar}V(x_i)}$$

- $\int dp$ **ill-defined: integrand rapidly oscillating f-on**

- **For it to be well-defined:**

  1. **replace:** $\epsilon \longrightarrow \epsilon - i\,a; \qquad a \in \mathbb{R}$
  2. **evaluate** $\int dp\,\ldots$
  3. **take** $a \to 0$ **limit, one gets:**

$$\langle x_{i+1}|U(t_{i+1},t_i)|x_i\rangle = \sqrt{\frac{m}{2\pi i\hbar\epsilon}}\; e^{\frac{i}{\hbar}\epsilon\left[\frac{m}{2}\left(\frac{x_{i+1}-x_i}{\epsilon}\right)^2 - V(x_i)\right]}$$

- **Insert into original propagator:**

$$\langle x'|U(t',t)|x\rangle \;=\; \int \mathcal{D}x\; e^{\frac{i}{\hbar}S[x]}$$

$$\int \mathcal{D}x \;=\; \lim_{\epsilon\to 0}\sqrt{\frac{m}{2\pi i\hbar\epsilon}}^{\,N-1}\int dx_1\int dx_2\,\ldots\int dx_{N-1}$$

Path Integral in Quantum Mechanics: real time (IV)

$$\langle x'|U(t',t)|x\rangle \;=\; \int \mathcal{D}x \, e^{\frac{i}{\hbar}S[x]}$$

$$\int \mathcal{D}x \;=\; \lim_{\epsilon\to 0} \sqrt{\frac{m}{2\pi i\hbar\epsilon}}^{\,N-1} \int dx_1 \int dx_2 \, \ldots \int dx_{N-1}$$

- **The action is continuum limit of the discretised action:**

$$S[x] = \int dt \, \Big[\frac{m}{2}(\partial_t x)^2 - V(x)\Big] = \lim_{\epsilon\to 0} \sum_i \epsilon \Big[\frac{m}{2}\Big(\frac{x_{i+1}-x_i}{\epsilon}\Big)^2 - V(x_i)\Big]$$

- **Summary:**

  (i) **Integrate over all particle positions for each intermediate time t_i**

  (ii) **Amounts to integrating over all possible paths of a particle starting at x and ending at x'**

  (iii) **Each path weighted with oscillating phase**

  (iv) **Classical path: the smallest oscillations $\Rightarrow$ largest contribution to the P.I. ($\hbar \to 0$ class. limit)**

- **Note: definition of the path integral required an analytic continuation in time!**

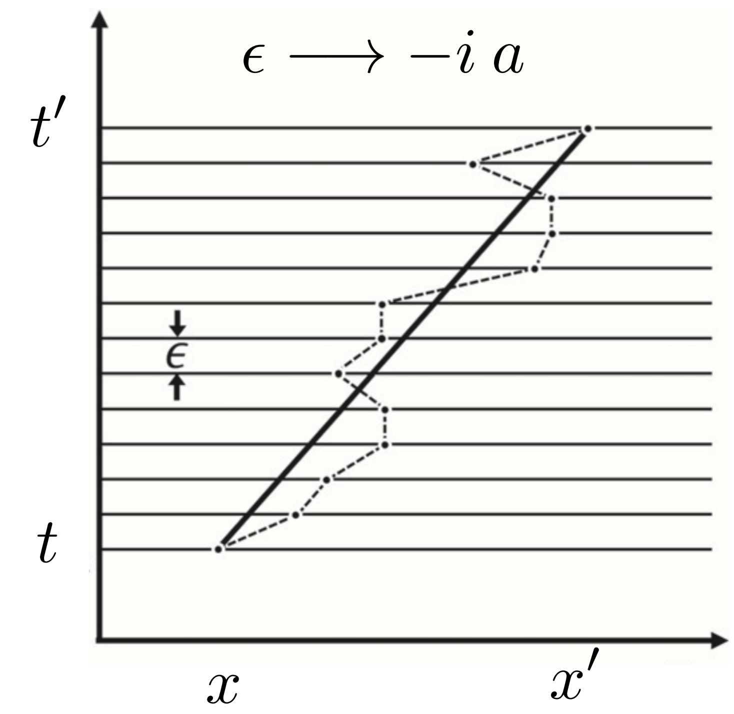
Euclidean Path Integral in Quantum Mechanics

- **Statistical partition function:** $Z = e^{-\beta H}; \quad \beta = \frac{1}{T}$
- $\beta = \frac{i}{\hbar}(t' - t) \quad \Longrightarrow \quad e^{-\beta H} \Leftrightarrow U(t,t')$

System at finite temperature T

$\Updownarrow$

System propagating in purely imaginary time



- **Repeat all the steps from the derivation in real time:**

  (i) **Divide Euclidean time interval into N time steps:** $\beta = \frac{Na}{\hbar}$

  (ii) **Insert complete set of position eigenstates**

- **The Euclidean Path Integral:**

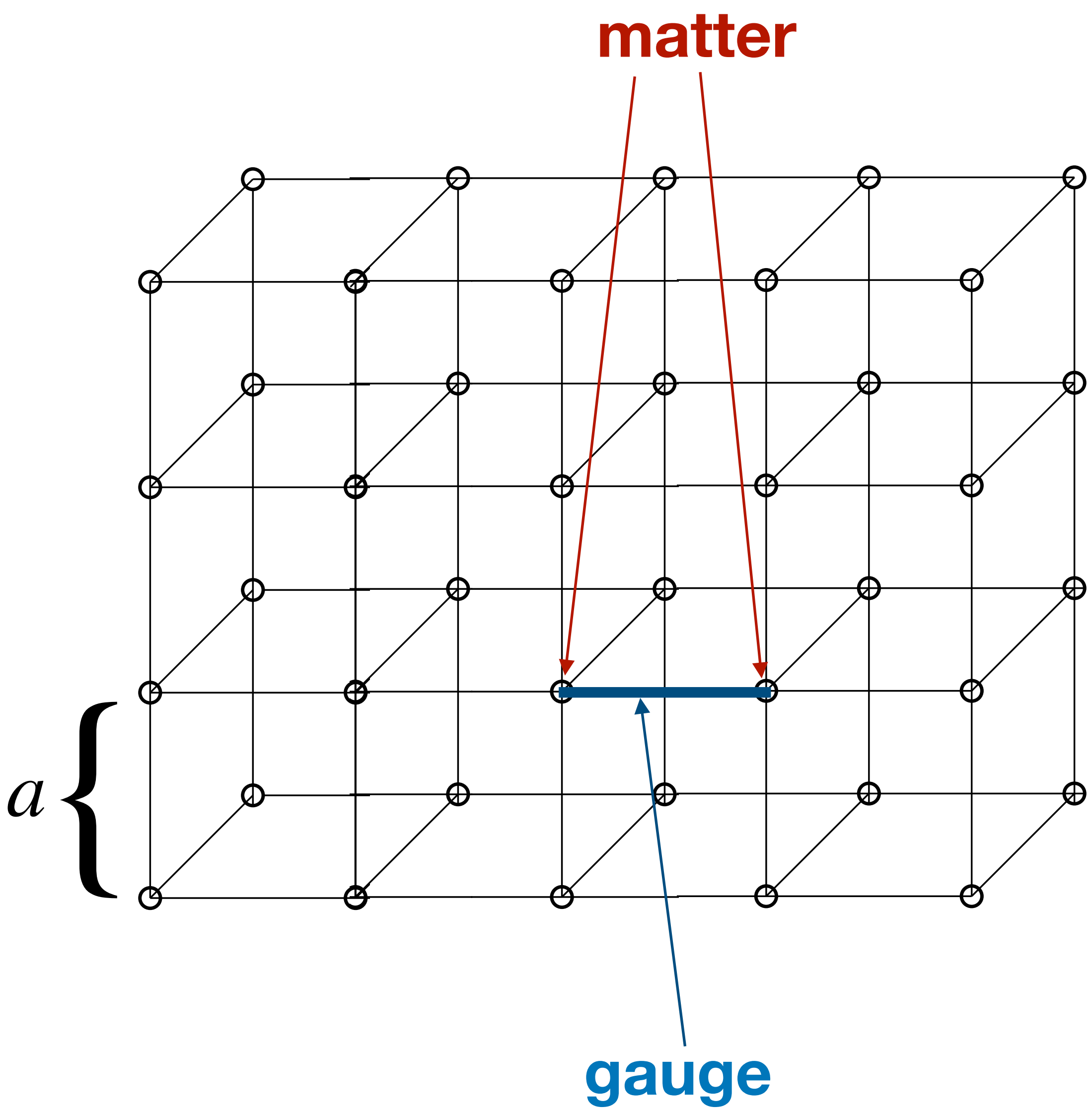
$$Z = \int \mathcal{D}x\, e^{-\frac{1}{\hbar}S_E[x]}$$

$$S_E[x] = \int dt\, \left[\frac{m}{2}(\partial_t x)^2 + V(x)\right] = \lim_{a\to 0} \sum_i a\left[\frac{m}{2}\left(\frac{x_{i+1}-x_i}{a}\right)^2 + V(x_i)\right]$$

$$\int \mathcal{D}x = \lim_{a\to 0} \sqrt{\frac{m}{2\pi i\hbar a}}^{\,N} \int dx_1 \int dx_2 \ldots \int dx_N$$

Scalar fields on the lattice

- Quantum field theories beyond tree level are plagued by UV divergencies
- Defining the theory on a lattice introduces a minimum length



- Lattice spacing acts as an UV cut-off: $\Lambda \sim \frac{1}{a}$

$$Z = \int \mathcal{D}\phi \; e^{-S_E[\phi]}$$

$$\int \mathcal{D}\phi = \prod_x \int d\phi(x)$$

$$S_E[\phi] = \int d^D x \left[\frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!}\phi^4\right]$$

Scalar field theory

- **Discretize space-time and define matter fields on the sites of the lattice:**

$$\begin{aligned} x &\longrightarrow an = (an_1, an_2, an_3, an_4); \quad n_\mu \in \mathbb{Z} \\ \phi(x) &\longrightarrow \phi(an) \end{aligned}$$

- **Momentum integrals are restricted to the first Brillouin zone:**

$$\begin{aligned} \int d^4x &\longrightarrow a^4 \sum_{n_\mu} \equiv \sum_x \\ \int \frac{d^4k}{2\pi} &\longrightarrow \int_{|k|<\frac{2\pi}{a}} \frac{d^4k}{2\pi} \equiv \int_k \end{aligned}$$

- **Discretize the derivatives:**

$$\begin{aligned} \partial_\mu \phi(x) &\longrightarrow \nabla_\mu \phi(x) = \frac{1}{a}\left[\phi(x + a\hat{\mu}) - \phi(x)\right] \\ &\longrightarrow \nabla^*_\mu \phi(x) = \frac{1}{a}\left[\phi(x) - \phi(x - a\hat{\mu})\right] \end{aligned}$$

Lattice action

- **Discretized action (bare parameters)**

$$S[\phi] = \sum_x \left[\frac{1}{2}\nabla_\mu \phi(x) \nabla_\mu \phi(x) + \frac{1}{2} m_0^2 \phi(x)^2 + \frac{\lambda_0}{4!}\phi(x)^4\right]$$

- **Scalar propagator**

$$\Delta(k)^{-1} = \sum_\mu \left[\frac{2}{a} sin(\frac{k_\mu a}{2})\right]^2 + m_0^2$$
$$= k^2 + m_0^2 + \mathcal{O}(a^2)$$

- **Space time symmetry:** $O(4) \longrightarrow H(4)$

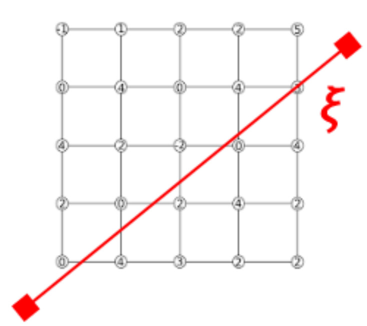
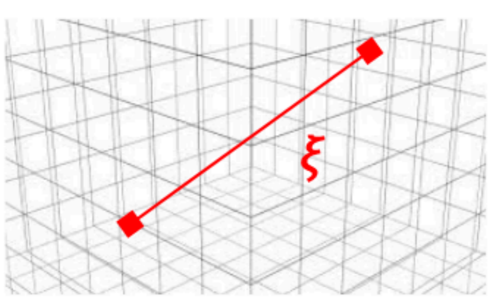
- **Rotation symmetry recovered in the continuum limit**

From P.I. in Quantum Mech. to Statistical Stat. Mech.

Quantum mechanics	Classical statistical mechanics
Euclidean time lattice	d-dimensional spatial lattice
elementary time step a	crystal lattice spacing
particle position x	classical spin variable s
particle path $x(t)$	spin configuration s_x
path integral $\int \mathcal{D}x$	sum over configurations $\prod_x \sum_{s_x}$
Euclidean action $S_E[x]$	classical Hamilton function $\mathcal{H}[s]$
Planck's constant $\hbar$	temperature T
quantum fluctuations	thermal fluctuations
kinetic energy $\frac{1}{2}(\frac{x_{i+1}-x_i}{a})^2$	neighbor coupling $s_x s_{x+1}$
potential energy $V(x_i)$	external field energy $\mu B s_x$
weight of a path $\exp(-\frac{1}{\hbar} S_E[x])$	Boltzmann factor $\exp(-\mathcal{H}[s]/T)$
vacuum expectation value $\langle \mathcal{O}(x) \rangle$	magnetization $\langle s_x \rangle$
2-point function $\langle \mathcal{O}(x(0)) \mathcal{O}(x(t)) \rangle$	correlation function $\langle s_x s_y \rangle$
energy gap $E_1 - E_0$	inverse correlation length $1/\xi$
continuum limit $a \to 0$	critical behavior $\xi \to \infty$

[Credit: U.-J. Wiese https://inspirehep.net/literature/946884]

From Statistical Mech. to Quantum Field Theories

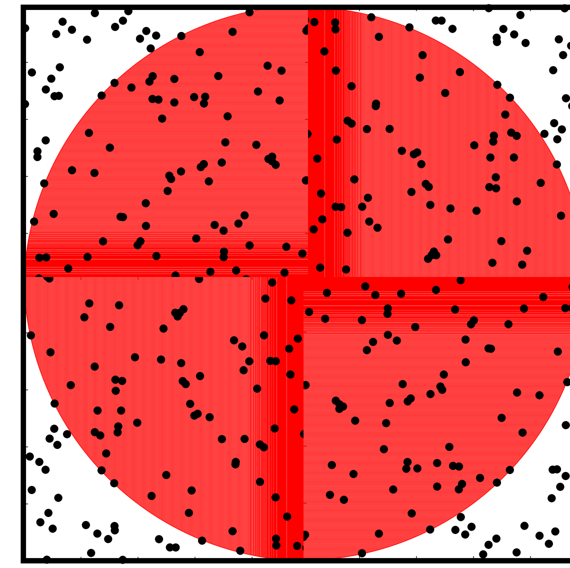
Classical Statistical Mechanics		Quantum Field Theories	
Partition function	$Z_\beta = \sum_\sigma e^{-\beta H(\sigma)}$	Feynman Path Integral	$\mathcal{Z} = \int D[U]\, e\, [(-1/g^2)\, S(U)]$
Inverse temperature	$\beta \sim 1/T$	Inverse gauge coupling	$\sim 1/g^2$
Correlation functions	$< \sigma(x)\, \sigma(y) > \sim e^{-\lvert x-y\rvert/\xi}$	2-point functions	$< Tr(U(p)_x)\, TrU(p)_y > \sim e^{-\lvert x-y\rvert r}$
Inverse correlation length	$1/\xi$	Particle mass: **m**	
2nd order phase transition $\xi/a \to \infty$ with **a** fixed		Continuum limit: $m\,a \to 0$ with **m** fixed	

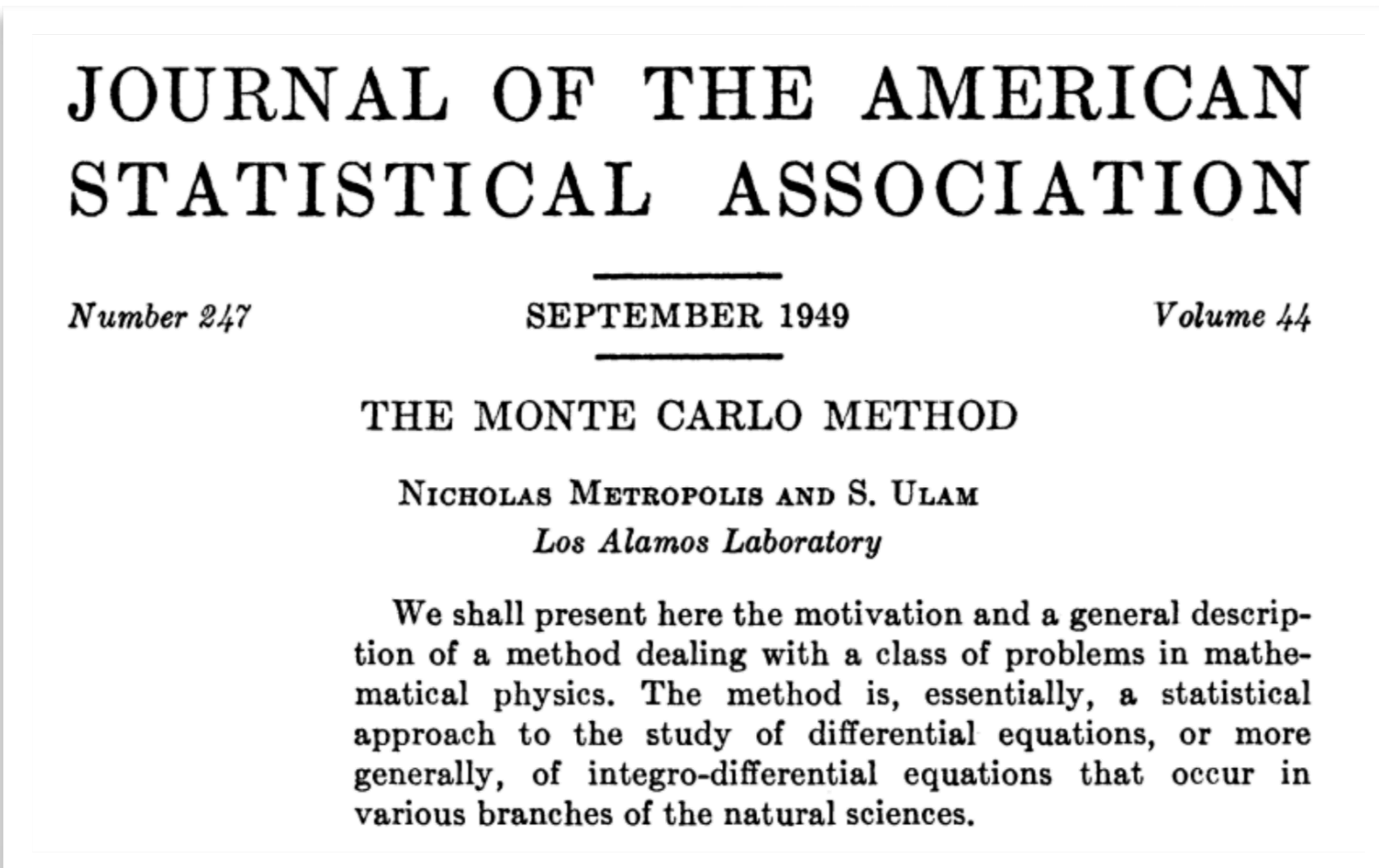
2-d spatial lattice with physical lattice spacing ***a***

4-d space-time lattice with unphysical cut-off ***a***

A tool for Lattice Simulations: *Monte Carlo* methods

- **Monte Carlo: a numerical method for estimating high-dimensional integrals by random sampling**





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THE MONTE CARLO METHOD

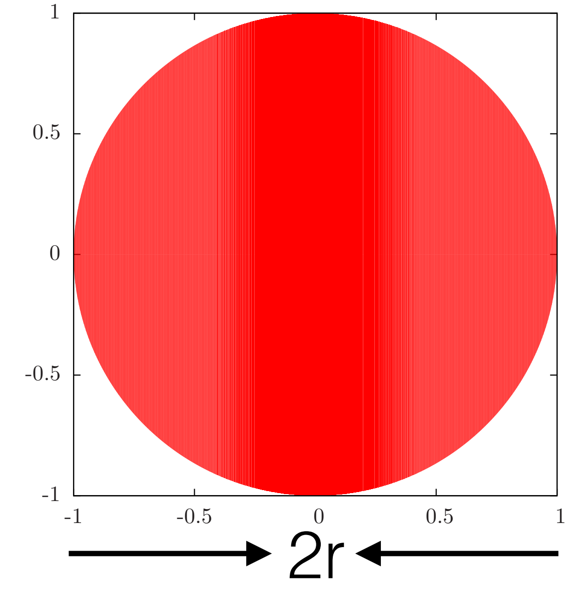
NICHOLAS METROPOLIS AND S. ULAM
Los Alamos Laboratory

We shall present here the motivation and a general description of a method dealing with a class of problems in mathematical physics. The method is, essentially, a statistical approach to the study of differential equations, or more generally, of integro-differential equations that occur in various branches of the natural sciences.

A tool for Lattice Simulations: Monte Carlo methods

- **Ratio of the surface area of the circle and the square:**

➡ $$\frac{S_{circle}}{S_{square}} = \frac{\pi r^2}{4r^2}$$



- **We can then express:**

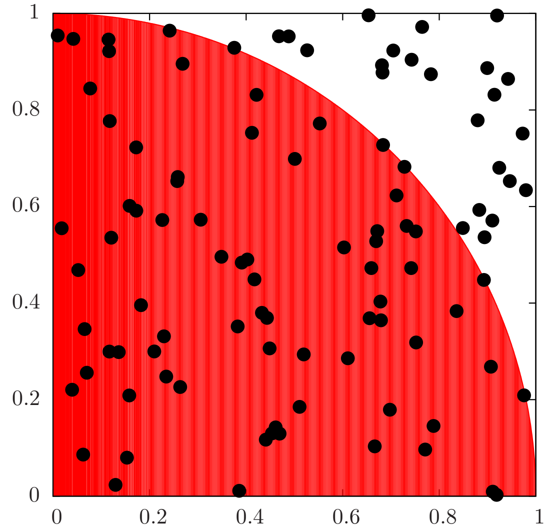
➡ $$\pi = \frac{4S_{circle}}{S_{square}}$$

- **If we know S_{circle}/S_{square}, we know the value of π**

- Throw N random points on the surface of the square

➡ $$\pi = \frac{4S_{circle}}{S_{square}} \approx \frac{N_{inside}}{N}$$

- For N - large, the value of **π** will be very well approximated



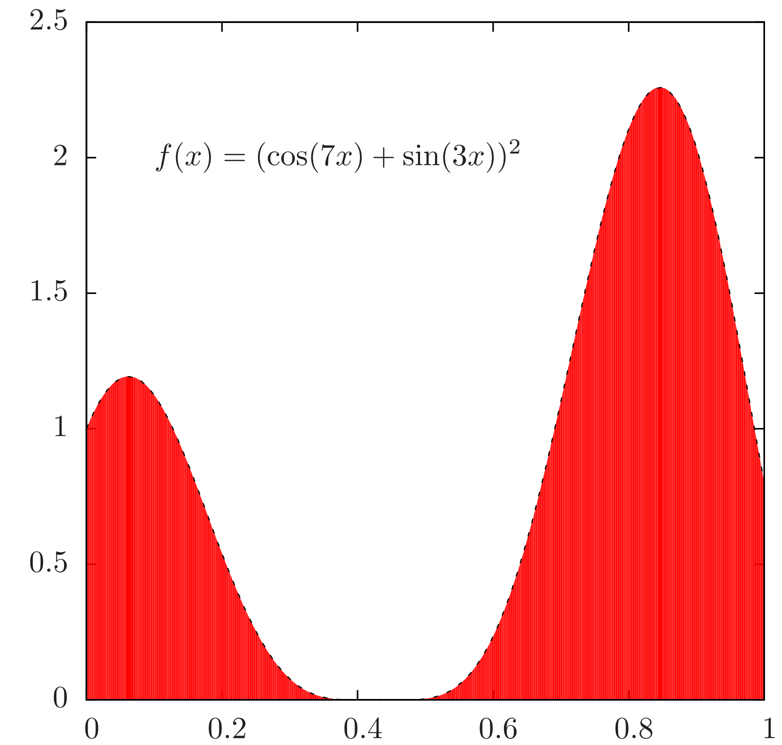
A tool for Lattice Simulations: Monte Carlo methods

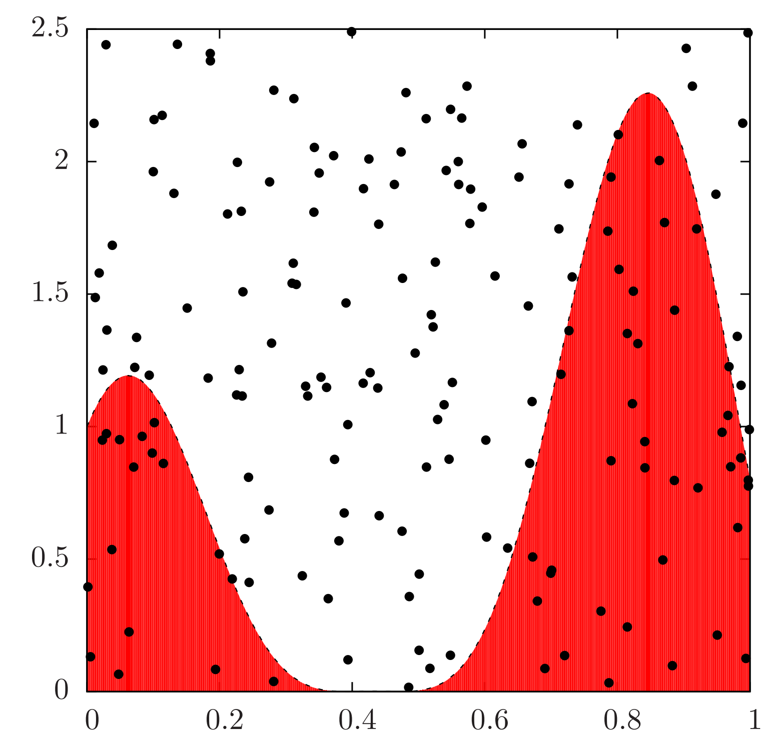
- **Calculating 1-dim integral:**

  ➡ $$I = \int_0^1 f(x)dx$$

- **Again, throw N random points on the rectangular surface:**
- **Count those under the value of the function** $f(x)$
- **Then the value of the integral is obtained by:**

  ➡ $$I \approx \langle I \rangle_N = 2.5 \times 1 \times \frac{N_{inside}}{N}$$





- **For larger N, better and better approximation of the integral**
- **Monte Carlo error in d-dim:** $\approx \frac{1}{\sqrt{N}}$
- **Numerical integration error in d-dim:** $\approx \frac{1}{N^{2/d}}$
- **For d>4, Monte Carlo is better than Numerical Integration!**

A tool for Lattice Simulations: Monte Carlo methods

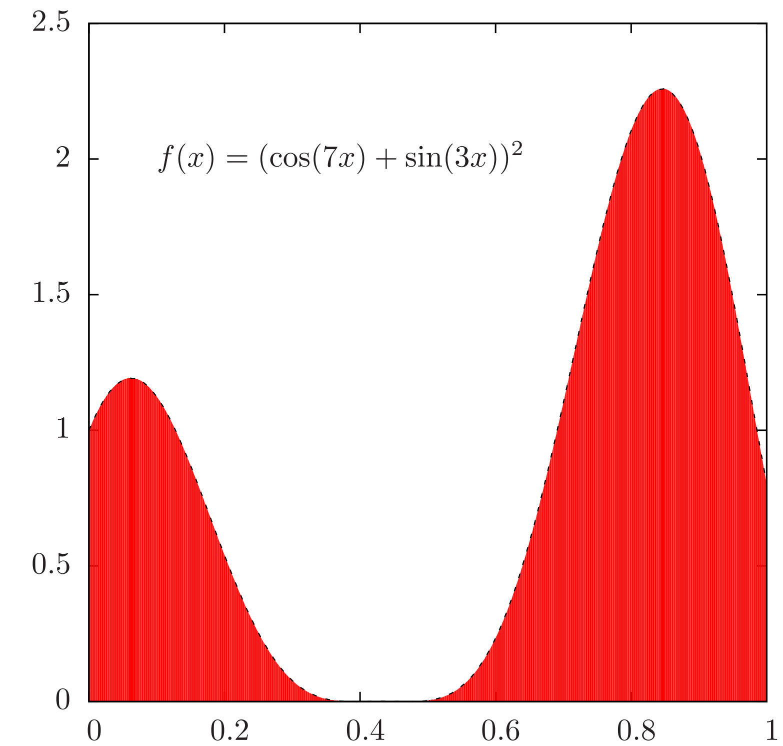
- **Calculating 1-dim integral:**

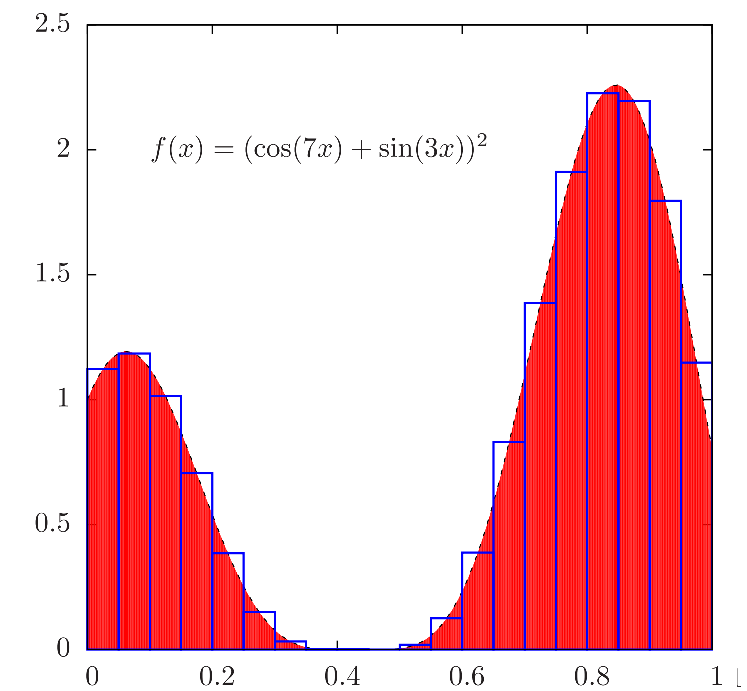
  ➡ $$I = \int_0^1 f(x)dx$$

- **Again, throw N random points on the rectangular surface:**
- **Count those under the value of the function** $f(x)$
- **Then the value of the integral is obtained by:**

  - $$I \approx \langle I \rangle_N = 2.5 \times 1 \times \frac{N_{inside}}{N}$$

- **For larger N, better and better approximation of the integral**
- **Monte Carlo error in d-dim:** $$\approx \frac{1}{\sqrt{N}}$$
- **Numerical integration error (midpoint rule) in d-dim:** $$\approx \frac{1}{N^{2/d}}$$
- **For d>4, Monte Carlo is better than Numerical Integration!**





Expectation values in Monte Carlo

- **Ising model Hamiltonian (without external magnetic field):**

$$H = J \sum_{\langle x,y \rangle} s_x s_y; \qquad \langle x, y \rangle - \text{nearest neighbours}$$

- **Generate configurations of spins with probability** $\sim e^{-\frac{H}{T}}$ **(importance sampling)**

- **Expectation value of the observables as averages over ensemble of spin configurations**

$$\langle O \rangle \approx \bar{O} = \frac{1}{N_{cnfg}} \sum_{k=1}^{N_{cnfg}} O[\bar{s}_k] + \mathcal{O}(\frac{1}{\sqrt{N_{cnfg}}})$$

- **How do we generate the ensemble** $\{\bar{s}_k\}$ **?**

Numerical Simulations

- **Numerical approach exploits the analogy:**

Euclidean QFT ⟷ **Statistical Mechanics**

$$\langle O[\phi] \rangle = \frac{1}{Z} \int \mathcal{D}\phi \; e^{-S_E[\phi]} \; O[\phi]$$

- **Path integrals are computed by importance sampling**

$$\mathcal{P}[\phi_i] \propto e^{-S_E[\phi]}$$

- **Generate ensemble of field configurations** $\{\phi_i\}$

- **Expectation values** $\langle O[\phi] \rangle$ **are averages over the ensemble**

$$\langle O \rangle \approx \bar{O} = \frac{1}{N_{cnfg}} \sum_{i=1}^{N_{cnfg}} O[\phi] + \mathcal{O}(\frac{1}{\sqrt{N_{cnfg}}})$$

- **How do we generate ensemble** $\{\phi_i\}$ **with the correct probability distribution?**

Markov processes

- Recursive procedure that generates $\{\phi_i\}$ with specific algorithm s.t. aimed distribution is asymptotically obtained

$$\{\phi_0\} \longrightarrow \{\phi_1\} \longrightarrow \{\phi_2\} \longrightarrow \cdots \longrightarrow \{\phi_i\} \longrightarrow \{\phi_{i+1}\} \longrightarrow \ldots$$

- Markov chains that converge exponentially to the equilibrium distr.

- The configurations $\{\phi_i\}$ are correlated by construction

$$Var\,[\bar{O}] \;=\; Var[O]\,\Big(\frac{2\tau_O}{N_{cnfg}}\Big)$$

— integrated autocorrelation time

- The error of the estimator scales as $\dfrac{1}{\sqrt{N_{cnfg}}}$ (Monte Carlo)

- The variance of the actual observable

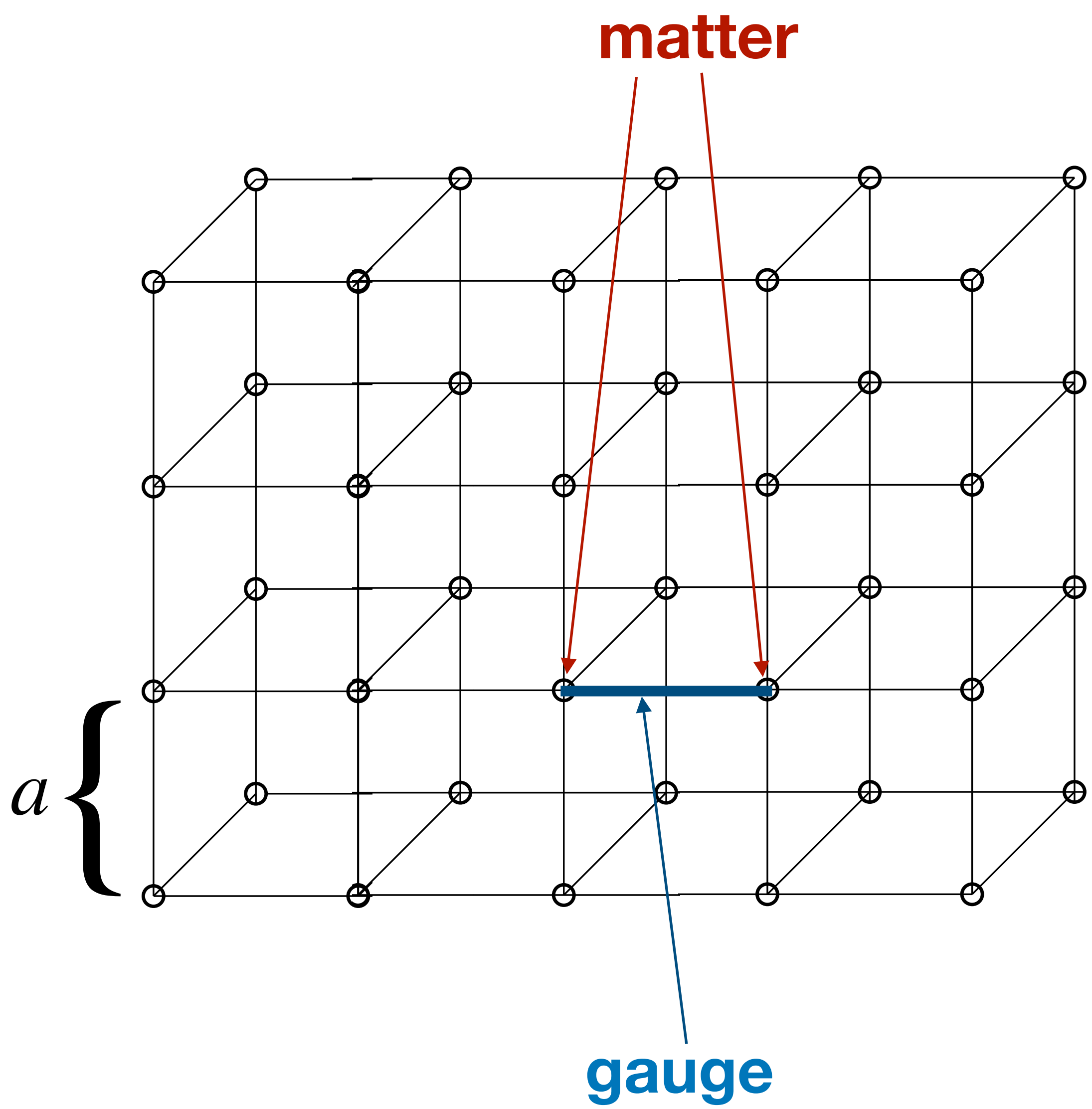
$$Var\,[O] \;=\; \langle \big(O - \langle O\rangle\big)^2 \rangle$$

→ property of QFT itself, should not depend of the Markov chain.

QFT on the lattice: symmetries

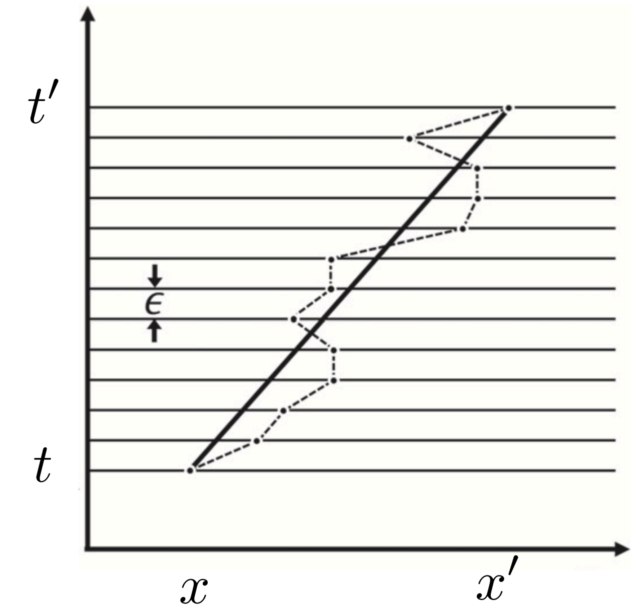
- **Translational Symmetry:**
  Broken to discrete symmetry, but nicely restored in the continuum limit

- **Rotational Symmetry:**
  Similar to translational symmetry. Finite number of irreducible representations (quantum numbers) instead of spin, but correct states are obtained in the continuum limit



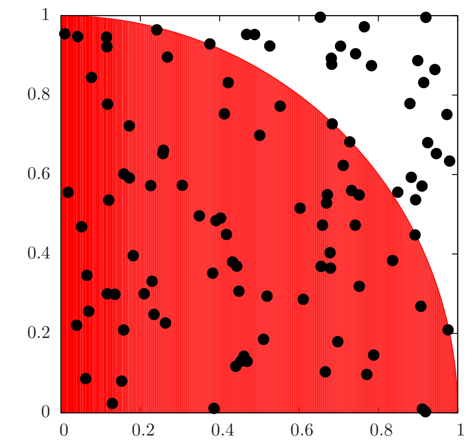
In today's lecture:

- We have learned/reminded ourselves how to quantize:



  - ➡ Quantum Mechanics
  - ➡ Scalar Field Theory

- Mapping to Classical Statistical Mechanics

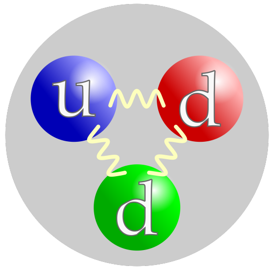


- Monte Carlo Sampling of Spin Systems $\longrightarrow$ Quantum Fields

- Monte Carlo Errors, other sources of errors

In tomorrow's lecture:

- We shall generalize this approach to Quantum Chromodynamics (QCD)



- We shall see how the numerical sampling is done in practice
  - ➡ for scalar field theory
  - ➡ for QCD

- Why is QCD numerically so expensive?

- Practical examples