# 5

# **Modification of Electric and Magnetic Fields by Materials**

Certain materials influence electric and magnetic fields through bound charges and currents. Their properties differ from those of metals where electrons are free to move. Dielectric materials contain polar molecules with spatially displaced positive and negative charge. Applied electric fields align the molecules. The resulting charge displacement reduces the electric field in the material and modifies fields in the vicinity of the dielectric. There are corresponding magnetic field effects in paramagnetic and ferromagnetic materials. These materials contribute to magnetic fields through orientation of atomic currents rather than a macroscopic flow of charge as in a metal.

Although the responses of materials to fields differ in scale, the general behavior is similar in form. This is the reason the contributions of dielectric and magnetic materials were singled out in Section 4.5 as  $\rho_2$  and  $j_2$  It is often useful to define new field quantities that automatically incorporate the contributions of bound charges and currents. These quantities are **D** (the electric displacement vector) and **H** (the magnetic field intensity).

The study of the properties of dielectric and magnetic materials (including subsidiary field quantities and boundary conditions) is not conceptually exciting. This is especially true for ferromagnetic materials where there is considerable terminology. Nonetheless, it is essential to understand the properties of dielectric and ferromagnetic materials since they have extensive uses in all types of accelerators.

A partial list of applications of dielectric materials includes the following:

1. Electric field distributions can be modified by adjustment of dielectric-vacuum boundaries. For example, dielectric boundary conditions must be applied to determine optimum shapes of high-voltage vacuum insulators.

2. Dielectrics can store more electrostatic field energy than vacuum. The high-energy storage density of water (80 times that of vacuum) is the basis for much of modern pulsed power technology.

3. Dielectrics reduce the velocity of propagation of electromagnetic waves (or photons). This helps to match the velocities of rf waves and high-energy particles for resonant acceleration. This effect is also important in designing energy storage transmission lines for pulse modulators.

All high-energy accelerators utilize ferromapetic materials. The following are some important applications.

1. Ferromagnetic materials shape magnetic fields. They play a role analogous to electrodes in electrostatics. Shaped iron surfaces (poles) are utilized to generate complex field distributions for focusing and bending magnets.

2. Ferromagnetic materials amplify the flux change produced by a real current. The resulting increased inductance is essential to the operation of transformers. Inductive isolation is the basis of the betatron and linear induction accelerator.

3. Ferromagnetic materials convey magnetic field lines in a manner analogous to the conduction of current by a low-resistivity wire. This effect leads to substantial reductions in power requirements for beam transport magnets.

4. The nonlinear response of a ferromagnetic material to an applied field can be utilized for fast, high-power switching.

The physics of dielectric and ferromagnetic materials is reviewed in this chapter. Special emphasis is placed on the concept of the magnetic circuit. A section is included on permanent magnet circuits.

## **5.1 DIELECTRICS**

Dielectric materials are composed of *polar molecules*. Such molecules have spatially separated positive and negative charge. The molecules may be either bound in one position (solids) or free to move (liquids and gases). Figure 5.1a shows a diagram of a water molecule. The electronegative oxygen atom attracts the valence electrons of the hydrogen atoms, leaving an excess of positive charge at the locations of the hydrogen atoms.



Figure 5.1 Behavior of dielectric materials. (a) Polar molecule (water). (b) Randomly oriented polar molecules. (c) Aligned polar molecules under the influence of an applied electric field. (d) Macroscopic charge on the surface of a dielectric material in an applied electric field from the alignment of polar molecules.

In the absence of an applied electric field, the molecules of a dielectric are randomly oriented (Fig. 5.1b). This results from the disordering effects of thermal molecular motion and collisions. Molecular ordering cannot occur spontaneously because a net electric field would result. With no external influence, there is no source of energy to generate such fields (Section 5.6). The randomized state is the state of lowest net energy (thermodynamic equilibrium).

Applied electric fields act on the charges of a polar molecule to align it as shown in Figure 5.1c. On the average, the molecular distribution becomes ordered as the change in electrostatic potential energy counteracts the randomizing thermal kinetic energy. The macroscopic effect of molecular alignment is shown in Figure 5.1d. Inside the material, a shifted positive charge in one molecule is balanced by a shifted negative charge of a nearby molecule. On the average, there

is no net internal charge to contribute to fields. This balance does not occur on the surfaces. An applied electric field results in positive and negative charge layers at opposite surfaces. The field produced by these layers inside the dielectric is opposed to the applied field.

The simplest geometry to consider is a one-dimensional dielectric slab in a region of uniform applied electric field. The applied field is produced by a voltage difference between two parallel plates (Fig. 5.1d). The electric field resulting from the charges on the plates (Section 3.2) is denoted  $E_1$ . The surface charges induced on the dielectric produce a field,  $-E_2$ . The total field inside the dielectric is  $E = E_1 - E_2$ . Most dielectrics have the property that the degree of orientation of polar molecules is linearly proportional to the applied field at typical field strengths. Thus, the surface charge density is proportional to applied field, and  $E_2 \sim E_1$ .

The linear response of dielectrics comes about because the degree of alignment of molecules is small at normal temperatures and field strength. Increased applied field strength brings about a proportional increase in the orientation. Nonlinear effects are significant when the dielectric approaches *saturation*. In a saturated state, all molecules are aligned with the field so that an increase in applied field brings about no increase in surface charge. We can estimate the magnitude of the saturation electric field. At room temperature, molecules have about 0.025 eV of thermal kinetic energy. Saturation occurs when the electrostatic potential energy is comparable to the thermal energy. The decrease in potential energy associated with orientation of a polar molecule with charge separation *d* is  $qE_1d$ . Taking q = e and d = 1 Å (10<sup>-10</sup> m),  $E_1$  must be on the order of 250 MV/m. This is much higher than the strongest fields generated in rf accelerators, so that the linear approximation is well satisfied. In contrast, saturation effects occur in ferromagnetic materials at achievable values of applied magnetic field.

The net electric field inside a linear dielectric is proportional to the applied field. The constant of proportionality is defined by

$$E = \frac{E_1}{\varepsilon/\varepsilon_o} = E_1 + E_2. \tag{5.1}$$

The quantity  $\varepsilon/\varepsilon_o$  is the relative dielectric constant. Ordinary solid or liquid dielectrics reduce the magnitude of the electric field, so that  $\varepsilon/\varepsilon_o > 1$ . Equation (5.1) is written in vector notation. This result can be derived from the above one-dimensional arguments by considering a differential cubic volume and treating each component of the field separately. This approach holds if the material is isotropic (liquids, glass). Equation (5.1) may not be valid for some solid materials. For instance, if polar molecules are bound in a crystal lattice, their response to an applied field may vary depending on the orientation of the field with respect to the crystal axes. The dielectric constant depends on the alignment of the field relative to the crystal. Such materials are called bifringent.

Water is a commonly encountered isotropic dielectric medium in electrical energy storage applications. The relative dielectric constant of liquid water is plotted in Figure 5.2 versus temperature and the frequency of an oscillating applied electric field. The low-frequency value is high since water molecules have large charge separation. The dielectric constant decreases with increasing temperature. This comes about because the molecules have higher thermal energy; therefore, they do not align as strongly in an applied electric field. At constant temperature, the relative dielectric constant decreases at high frequency (the microwave regime). This is because the inertia of the water molecules retards their response to the oscillating electric field. The alignment of the molecules lags in phase behind the electric field, so that the medium extracts energy from the field. Thus, water is not an ideal dielectric constant,  $\varepsilon$ ". Higher temperatures randomize molecular motion and lessen the relative effect of the ordered phase lag. This explains the unusual result that the absorption of high-frequency electric fields in water is reduced at higher temperature.

It is useful to define the *displacement vector*  $\mathbf{D}$  when the dielectric is linear. The displacement vector is proportional to the sum of field components excluding the contribution of dielectrics, or (in the notation of Chapter 4)



$$\boldsymbol{D} = \boldsymbol{\varepsilon}_{o} \ (\boldsymbol{E}_{1} + \boldsymbol{E}_{3}). \tag{5.2}$$

Figure 5.2 Real and imaginary parts of the relative dielectric constant of water as a function of the applied electric field frequency. (Adapted from J. B. Halstead, "Liquid Water—Dielectric Properties" in *Water—a Comprehensive Treatise*, F. Franks Ed., Plenum, New York, 1972.)

Thus, **D** arises from free charges (either on electrodes or in the volume). Combining Eqs. (5.1) and (5.2) the electric displacement is related to the net field inside a dielectric region by

$$\boldsymbol{D} = \boldsymbol{\varepsilon} \boldsymbol{E}. \tag{5.3}$$

If a dielectric is inserted into a vacuum field region (Section 4.1), the following equations hold:

$$\nabla \cdot \boldsymbol{D} = 0, \quad \nabla \cdot \boldsymbol{E} \neq 0. \tag{5.4}$$

The meaning of these equations is illustrated in Figure 5.3. A thin differential volume element that includes a vacuum-dielectric boundary is illustrated. There is no flux of **D** lines out of the volume since there are no free charges to act as sources. The divergence of **E** is nonzero because the magnitude is different on both faces. The volume includes a net positive charge in the form of the dielectric surface charge.

When dielectrics are included in a vacuum region, the Laplace equation can be written

$$\nabla \cdot \left(\frac{\varepsilon(\boldsymbol{x})}{\varepsilon_o} \nabla \varphi\right) = 0.$$
 (5.5)

Equation (5.5) proceeds from Eq. (5.4), which implies that  $\nabla \cdot \varepsilon \mathbf{E} = 0$ . The potential is still given by  $-\nabla \phi = \mathbf{E}$  since the force on a particle depends on the net electric field, independent of the sources of the field. Numerical methods to solve Eq. (5.5) are similar to those of Section 4.2. A



Figure 5.3 Electric fields at a vacuum-dielectric boundary.

value of the relative dielectric constant is associated with each point and must be included in the finite difference formulation of the Laplacian operator.

The concept of the dielectric constant often leads to confusion in treating plasmas. A plasma is a relatively dense region of equal positive and negative free charges (Fig. 5.4). The clearest approach to describe the interactions of plasmas and electric fields is to include the electron and ion space charge as contributions to  $\rho_3$  (free space charge). Nonetheless, it is a common practice to introduce the concept of a plasma dielectric constant to describe phenomena such as the refraction of optical radiation. This permits utilization of familiar optical definitions and equations. Referring to the plasma slab illustrated in Figure 5.4a, the plasma dielectric constant is clearly undefined for a steady-state applied field since positive and negative charges are free to move in opposite directions. At very low frequency, plasmas support real currents, as in a metal conductor. When a medium-frequency ac electric field is applied, the heavy ions are relatively immobile. The electrons try to move with the field, but

displacements lead to charge separation. The space charge tield acts to cancel the applied field. The electrons are thus bound to the ions. The result is that at medium frequency, electric fields are excluded from plasmas. Alternatively, the plasma can be described by a relative dielectric constant much greater than unity. Note the geometric similarity between Figure 5.4b and 5.1d. At high frequencies, electron inertia becomes an important factor. At high frequencies (such as the optical regime), the electron motion is 180° out of phase with applied electric fields. In this case, the electron space charge (oscillating about the immobile positive charge) adds to the applied field,



**Figure 5.4** Response of particles in a plasma slab to an oscillating applied electric field. (*a*) Direct-current field (zero frequency). (*b*) Low-frequency ac field. (*c*) High-frequency ac field.



**Figure 5.5.** Infrared laser interferogram of plasma in dense neutral backgrounds. (a) Exploding wire: note that the dense expanding neutrals cause an upward shift of fringes ( $\varepsilon/\varepsilon_0 > 1$ ) while the electrode plasma causes a downward fringe shift ( $\varepsilon/\varepsilon_0 < 1$ ). (b) Spark in atmospheric air: dense plasma causes a downward shift. (Photographs by the author.)

so that  $\varepsilon/\varepsilon_o < 1$ . Plasmas are a very unusual dielectric material at high frequencies. This effect is important in laser interferometry of plasmas. Figure 5.5 shows far-infrared holographic interferograms of an exploding wire and a plasma spark in atmospheric air. The direction of displacement of the fringes shows the dielectric constant relative to vacuum. Note that there are displacements in both directions in Figure 5.5a because of the presence of a dense shock wave of neutrals ( $\varepsilon/\varepsilon_o > 1$ ) and an electrode plasma ( $\varepsilon/\varepsilon_o < 1$ ).

## **5.2 BOUNDARY CONDITIONS AT DIELECTRIC SURFACES**

Methods for the numerical calculation for vacuum electric fields in the presence of dielectrics were mentioned in Section 5.1. There are also numerous analytic methods. Many problems involve uniform regions with different values of  $\varepsilon/\varepsilon_0$ . It is often possible to find general forms of the solution in each region by the Laplace equation, and then to determine a general solution by

matching field components at the interfaces. In this section, we shall consider how electric fields vary passing from a region with  $\epsilon/\epsilon_o \neq 0$  to a vacuum. Extensions to interfaces between two dielectrics is straightforward.

The electric fields at a dielectric-vacuum interface are divided into components parallel and perpendicular to the surface (Fig. 5.6). The magnitude of the electric field is different in each region (Section 5.1); the direction may also change. The relationship between field components normal to the interface is demonstrated by the construction of Figure 5.6b. A surface integral is taken over a thin volume that encloses the surface. The main contributions come from integration over the faces parallel to the surface. Using Eq. (5.3) and the divergence theorem,



Figure 5.6 Boundary conditions for electric field components at a vacuum-dielectric boundary. (a) Electric field lines at boundary. (b) Geometry to find the relationship of normal field components. (c) Geometry to find the relationship of parallel field components.

This gives the matching condition for perpendicular field components,

$$E_{\perp\beta}/E_{\perp\alpha} = \varepsilon_o/\varepsilon, \tag{5.6}$$

Matching conditions for the parallel field components can be determined from the construction of Figure 5.6c. A slab of dielectric extends between two parallel metal plates at different voltages. The dielectric-vacuum interface is normal to the plates. The geometry of Figure 5.6c is the simplest form of capacitor. The charges that produce the electric field must be moved against the potential difference in order to charge the plates. During this process, work is performed on the system; the energy can be recovered by reversing the process. Thus, the capacitor is a storage device for electrostatic energy.

In the absence of the dielectric, electric field lines normal to the plates are produced by positive and negative surface charge layers on the plates. When the dielectric is introduced, polarization charge layers are set up that try to reduce the electric field inside the dielectric. Because there is no net charge between the plates or inside the dielectric, the condition  $\nabla \cdot \mathbf{E} = 0$  holds everywhere between the plates. Field lines are thus straight lines parallel to the dielectric-vacuum interface. The integral  $-\int \mathbf{E} \cdot d\mathbf{x}$  has the constant value  $V_0$  on any path between the equipotential plates. In particular, the integral can be taken just inside and outside the dielectric surface. This implies that the parallel electric field is the same inside and outside the dielectric surface. This fact can be reconciled with the presence of the polarization charge by noting that additional surface charge is distributed on the plates. The extra charge cancels the effect of polarization charge on the electric field, as shown in Figure 5.6 c.

The matching condition for parallel components of electric field at a dielectric-vacuum surface is

$$E_{\parallel \alpha} = E_{\parallel \beta}. \tag{5.7}$$

The construction also shows that a dielectric fill allows a capacitor to store more plate charge at the same voltage. Since the electrostatic energy is proportional to the charge, the energy density is proportional to  $\epsilon/\epsilon_0$ . This explains the predominance of water as a medium for high-power density-pulsed voltage systems. Compact high-voltage capacitors are produced using barium titanate, which has a relative dielectric constant which may exceed  $10^4$ .

The combined conditions of Eqs. (5.6) and (5.7) imply that the normal components of electric field lines entering a medium of high  $\varepsilon/\varepsilon_0$  from vacuum are small. Inside such a medium, electric field lines are thus bent almost parallel to the interface. Figure 5.7 shows an example of applied dielectric boundary conditions. Equipotential lines are plotted at the output of a high-power, water-filled pulser. The region contains water ( $\varepsilon/\varepsilon_0 = 80$ ), a lucite insulator ( $\varepsilon/\varepsilon_0 \approx 3$ ), and a vacuum region ( $\varepsilon/\varepsilon_0 = 1$ ) for electron beam acceleration. The aim of the designer was to distribute equipotentials evenly across the vacuum side of the insulator for uniform field stress and to shape boundaries so that field lines enter the surface at a 45' angle for optimum hold-off (see Section



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Figure 5.7 Equipotential lines near the vacuum insulator of a high-power pulse generator. (Courtesy J. Benford, Physics International Company.)

9.5). Note the sharp bending of equipotential lines at the lucite-water boundary. The equipotentials in the water are evenly spaced straight lines normal to the boundary. They are

relatively unaffected by the field distribution in the low dielectric constant region.

## **5.3 FERROMAGNETIC MATERIALS**

Some materials modify applied magnetic fields by alignment of bound atomic currents. Depending on the arrangement of electrons, atoms may have a magnetic moment. This means that the circulating electrons produce magnetic fields outside the atom. The fields, illustrated in Figure 5.8, have the same form as those outside a circular current loop (Section 4.7); therefore, the circular loop is often used to visualize magnetic interactions of atoms. The magnetic moment  $p_m$  of a loop of radius a carrying a current *I* is

$$p_m = I (\pi a^2).$$
 (5.8)



Figure 5.8 Magnetic field lines near an atom with a magnetic moment.

In classical physics, the atomic  $p_m$  is visualized to originate from the circular current loop of a valenc electron rotating about the atom. The current *I* is about  $qev/2\pi a$  (where *v* is the orbital velocity) and *a* is about 1 Å. Although this gives a rough estimate of a typical atomic  $p_m$  the microscopic problem must be approached by quantum mechanics. The correct result is that the magnetic moment is quantified, and can have values

$$p_m = \pm meh/4\pi m_e, \tag{5.9}$$

where *h* is the Planck constant

$$h = 6.63 \times 10^{-34} J - s$$

and m is an integer which depends on the arrangement of electrons in the atom.

On a macroscopic scale, when two fixed adjacent current loops have the same orientation, the magnetic forces act to rotate the loops to opposite polarity (Fig. 5.9a). This is a consequence of the fact that when the magnetic moments are aligned antiparallel, magnetic fields cancel so that the field energy is minimized. With no applied field, atomic currents are oriented randomly, and there is no macroscopic field. The situation is analogous to that of molecules in a dielectric. Material that can be described by this classical viewpoint is called *paramagnetic*, which means "along or parallel to the field." Reference to Figure 5.9b shows that when a magnetic field is applied to a paramagnetic material, the atomic currents line up so that the field inside the loop is in the same direction as the applied field while the return flux is in opposition. As in dielectrics, the fractional alignment is small since the change in magnetostatic energy is much less than the average thermal energy of an atom. Figure 5.9c shows what the net magnetic field looks like when magnetic moments of atoms in a dense medium are aligned; there is an increase of magnetic flux inside the material above the applied field. Negative flux returns around the outside of the material. A view of the atomic and real currents normal to the applied field clarifies the process (Fig. 5.9d). Alignment of magnetic moments does not produce a net atomic current inside the material, but results in a surface current in the same direction as the applied field. The surface current is the magnetic analogy of the surface charge of dielectrics.

The field inside a paramagnetic material is approximately proportional to the applied field, or

$$\boldsymbol{B} = (\mu/\mu_o) \ \boldsymbol{B}_1 = \boldsymbol{B}_1 + \boldsymbol{B}_2. \tag{5.10}$$

The quantity  $\mu/\mu_o$  is the *relative permeability*. The magnetic field intensity **H** is a vector quantity proportional to the magnetic field minus the contribution of atomic material currents, or

$$\boldsymbol{H} = \boldsymbol{B}_{1}/\boldsymbol{\mu}_{o}.$$



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Figure 5.9 Behavior of magnetic materials. (a) Interactions between atoms with magnetic moments; force exerted on one current loop (B) by another (A). (b) Response of atoms in a paramagnetic material to an applied magnetic field. (c) Macroscopic magnetic fields produced by alignment of atomic currents in a material. (d) Macroscopic surface current in a material resulting from alignment of atoms with magnetic moments.

The field intensity is related to the magnetic field inside a magnetic material by

(c)

$$\boldsymbol{H} = \boldsymbol{B}/\boldsymbol{\mu}. \tag{5.11}$$

(d)

Magnetic fields obey the principle of superposition. Equation (4.39) can be written

$$\int \boldsymbol{B}_1 \cdot \boldsymbol{dl} = \boldsymbol{\mu}_o \boldsymbol{I}_1.$$

for the applied fields. This can be expressed in terms of H by

$$\int \boldsymbol{H} \cdot \boldsymbol{dl} = \boldsymbol{I}_1. \tag{5.12}$$

Thus, the magnetic intensity is determined only by free currents and has the dimensions amperes per meter in the mks system.

The relative permeability in typical paramagnetic materials is only about a factor of  $10^{-6}$  above unity. Paramagnetic effects are not important in accelerator applications. Ferromagnetic materials, on the other hand, have  $\mu/\mu_o$  factors that can be as high as 10,000. This property gives them many important uses in magnets for charged particle acceleration and transport. Ferromagnetism is a quantum mechanical phenomenon with no classical analogy. In some materials (chiefly iron, nickel, and iron alloys), the minimum energy state consistent with the exclusion principle has atomic magnetic moments aligned parallel rather than antiparallel. The energy involved in this alignment is greater than thermal kinetic energies at room temperature. On a microscopic scale, all the magnetic moments in a ferromagnetic material are aligned in the minimum energy state.

Alignment does not extend to macroscopic scales. Macroscopic alignment of magnetic moments produces fields outside the material which require additional energy. Two opposing factors are balanced in ferromagnetic materials in the minimum energy state. On the microscopic scale, minimum energy is associated with atomic alignment, while on the macroscopic scale minimum energy is equivalent to maximum disorder. The situation is resolved by the formation of domains, small regions in which all magnetic moments are aligned. On a macroscopic scale, the domains are randomized (Fig. 5.10) so that there is no magnetic field outside the ferromagnetic material in its ordinary state. The domain size (the separatrix between the quantum mechanical and classical regimes) is about 10<sup>-5</sup> cm, or 1000 atoms wide.

Ferromagnetic materials respond to applied magnetic fields by shifting domain boundaries to favor domains aligned with the field. In contrast to paramagnetic materials, the resulting high degree of atomic orientation produces large magnetic effects. Saturation (total alignment) can occur at attainable applied field strengths ( $\sim 2$  T). Although the magnetic field is a monotonic



Figure 5.10 Random orientation of domains in unmagnetized material.

function of the applied field, we cannot expect the response to be linear or reversible. Equation (5.11) is no longer valid. We can preserve the concept of the permeability by considering the response of ferromagnetic materials to small excursions in the applied field about an equilibrium value. The small signal  $\mu$  is defined by

$$\Delta \boldsymbol{B} = \boldsymbol{\mu}(\boldsymbol{H}) \ \Delta \boldsymbol{H}. \tag{5.13}$$

or

$$\mu(H) = (dB/dH)_{H}$$

## 5.4 STATIC HYSTERESIS CURVE FOR FERROMAGNETIC MATERIALS

In this section we shall look in more detail at the response of ferromagnetic materials to an applied field. In unmagnetized material, the directions of domains are randomized because energy is required to generate magnetic fields outside the material. If the external magnetic field energy is supplied by an outside source, magnetic moments may become orientated, resulting in large amplified flux inside the material. In other words, an applied field tips the energy balance in favor of macroscopic magnetic moment alignment.

A primary use of ferromagnetic materials in accelerators is to conduct magnetic flux between vacuum regions in which particles are transported. We shall discuss relationships between fields inside and outside ferromagnetic materials when we treat magnetic circuits in Section 5.7. In this section we limit the discussion to fields confined inside ferromagnetic materials. Figure 5.11 illustrates such a case; a ferromagnetic torus is enclosed in a tight uniform magnet wire winding. We want to measure the net toroidal magnetic field inside the material, *B*, as a function of the applied field  $B_1$ , or the field intensity *H*. The current in the winding is varied slowly so that applied field permeates the material uniformly. The current in the winding is related to  $B_1$  through Eq. (4.42). By Eq. (5.1 1), H = NI/L, hence the designation of *H* in ampere-turns per meter. The magnetic field inside the material could be measured by a probe inserted in a thin gap. A more practical method is illustrated in Figure 5.11. The voltage from a loop around the torus in integrated electronically. According to Section 3.5, the magnetic field enclosed by the loop of area *A* can be determined from  $B = \int V dt/A$ .

With zero current in the windings, a previously unmagnetized core has randomly orientated domains and has no macroscopic magnetization (H = 0, B = 0). Domains become aligned as the applied field is raised. Both H and B increase, as shown in Figure 5.12. The field in the material (the sum of the applied and atomic contributions) may be over 1000 times that of the applied field alone; thus, the small signal  $\mu$  is high. At some value of applied field, all the domains are aligned. This is called *saturation*. Beyond this point, there is no increase in the contribution of



Figure 5.11 Circuit to measure response of atomic currents in a ferromagnetic material.

material currents to the field with increasing applied field; therefore, the small signal  $\mu$  drops to  $\mu_0$ . The portion of the *B*-*H* curve from (*H* = 0, *B* = 0) to saturation (the dashed line in Fig. 5.12) is called the *virgin magnetization curve*. Unless the material is completely demagnetized, it will not be repeated again.

The next step in the hypothetical measurement is to reduce the applied field. If the magnetization process were reversible, the B-H curve would follow the virgin magnetization



Figure 5.12 Hysteresis curve.

curve back to the origin. This does not happen since it takes energy to shift domain boundaries. When *H* is returned to zero, the domains are energetically able to retain much of their alignment so that the torus remains magnetized. This occurs because all field lines are contained in the torus and no energy need be expended to produce external fields. The magnetization would be reduced if there were an air gap in the core (see Section 5.8). If a reverse current is applied to the driving circuit, the magnetic field will return to zero and eventually be driven to reverse saturation. The *B-H* curve of a magnetic material exhibits *hysteresis* or nonreversibility. The term derives from a Greek word meaning shortcoming or lagging. The B-H curve described is a particular case, the *saturation hysteresis curve*. There is a nested family of hysteresis curves converging to the origin, depending on the magnitude of current in the driving circuit.

Magnet engineering is based on some straightforward concepts and a large body of terminology. A clear understanding of the definitions of magnetic properties is essential to utilize data on magnetic materials. To facilitate this, important terms will be singled out in this section and in Sections 5.7 and 5.8. Terms related to the hysteresis curve are illustrated in Figure 5.12. Most data on magnetic materials and permanent magnets is given in cgs units, so it is important to know the transformation of quantities between mks and cgs.

**H** Magnetic Intensity. Also called magnetizing force. The cgs unit is oersteds (Oe), where I A-turn/m = 0.01256 Oe.

**B** Magnetic Induction Field. Also called magnetic field, magnetic induction, magnetic flux density. The cgs unit is the gauss (G), where 1 tesla  $(T) = 10^4$  gauss.

**B**<sub>s</sub> **Saturation Induction**. The magnetic field in a ferromagnetic material when all domains are aligned.

 $\mathbf{H}_{s}$  The magnetizing force necessary to drive the material to saturation.

 $H_c$  Coercive Force. The magnetic intensity necessary to reduce the magnetization of a previously saturated material to zero.

 $\mathbf{B}_{\mathbf{r}}$  **Remanence Flux**. Also called residual induction. The value of the magnetic field on the saturation hysteresis curve when the driving current is zero. It is assumed that all magnetic flux is contained in the material.

**Soft Magnetic Material**. Ferromagnetic materials which require a small magnetizing force to be driven to saturation, typically 10 Oe. The area enclosed by the saturation hysteresis curve is relatively small, and we shall see that this is equivalent to small energy input to magnetize or demagnetize the material. Soft magnetic materials are used to conduct field lines and as isolators in induction accelerators.

**Hard Magnetic Materials**. Ferromagnetic materials which require considerable energy to reorient the domains. The coercive force can be as high as 8000 Oe. The large amount of energy stored in hard magnetic materials during magnetization means that more energy is available to produce fields external to the material. Hard magnetic materials are used for permanent magnets.



Figure 5.13 Saturation hysteresis curves. (a) Low-carbon steel. (b) Steel with 3.25% silicon. (Courtesy M. Wilson, National Bureau of Standards.)

Figure 5.13 shows hysteresis curves for carbon steel (a material used for magnet poles and return flux yokes) and silicon steel (used for pulsed transformer cores).

## **5.5 MAGNETIC POLES**

Figure 5.14 shows a boundary between a magnetic material with permeability  $\mu$  and vacuum with  $\mu_{o}$ . A thin volume element encloses the boundary. The equation  $\nabla \cdot \mathbf{B} = 0$  implies that the integral of the normal component of **B** on the surfaces of the volume is zero (Fig. 5.14a). The main contributions to the integral are from the upper and lower faces, so that

$$B_{\perp\alpha} = B_{\perp\beta}. \tag{5.14}$$

Noting that there is no free current enclosed, Eq. (5.12) can be applied around the periphery of the volume (Fig. 5.14b). The main contributions to the circuital integral are on the faces.

$$\int (\boldsymbol{H}_{\alpha} - \boldsymbol{H}_{\beta}) \cdot d\boldsymbol{l} =$$
 (5.15)

or

$$B_{\parallel \alpha}/\mu_o = B_{\parallel \beta}/\mu. \tag{5.16}$$

For ferromagnetic materials  $(\mu \gg \mu_o)$ , the parallel component of magnetic field outside the ferromagnetic material is much smaller than the parallel component inside the material. Thus, magnetic field lines just outside a ferromagnetic material are almost normal to the surface. This simple boundary condition means that ferromagnetic materials define surfaces of constant



Figure 5.14 Boundary conditions for magnetic field components at a boundary between vacuum and a ferromagnetic material. (a) Geometry to find the relationship of normal field components. (b) Geometry to find the relationship of parallel field components.

magnetic potential  $U_m$ . Ferromagnetic surfaces can be used to generate magnetostatic field distributions in the same way that electrodes are used for electrostatic fields. Since both the electrostatic and magnetic potentials satisfy the Laplace equation, all electrostatic solutions can be applied to magnetic fields. Ferromagnetic surfaces used to shape magnetic field lines in a vacuum (or air) region are called *pole pieces*. By convention, magnetic field vectors point from the North to the South pole. Figure 5.15 shows an example of a numeric calculation of magnetic fields for a synchrotron magnet. Shaping of the pole piece near the gap provides the proper field gradient for beam focusing.

The boundary condition is not valid when the pole material becomes saturated. In high-field magnets, regions of high flux may become saturated before rest of the pole piece. This distorts the magnetic field pattern. The fields of partially saturated magnets are difficult to predict. Note that local saturation is avoided in the design of Figure 5.15 by proper shaping and avoidance of sharp edges. This allows the maximum magnetic field without distortion. In the limit of fields well above the saturation limit (» 2 T), the effective relative permeability decreases to unity and the field pattern approaches that of the exciting coil only.

The analogy between electrostatic and magnetostatic solutions leads to the magnetic quadrupole, illustrated in Figure 5.16. The pole pieces follow hyperbolic surfaces of constant magnetic potential [Eq. (4.26)]. In contrast to the electrostatic quadrupole, the *x*-*y* forces on a beam moving along the *z* axis are perpendicular to the magnetic field lines. It is usually more convenient to analyze a magnetic quadrupole in terms of *x*-*y* axes rotated 45° from those used for



Figure 5.15 Numerical calculation of magnetic field lines in a synchrotron magnet. (Adapted from J. S. Colonias, *Particle Accelerator Design*; *Computer Programs*, used by permission, Academic Press.)



Figure 5.16 Magnetic quadrupole lens (cross section).

the electric field version. In the coordinate system of Figure 5.16, the magnetic field components are

$$B_x = B_o y/a, \quad B_y = B_o x/a. \tag{5.17}$$

## **5.6 ENERGY DENSITY OF ELECTRIC AND MAGNETIC FIELDS**

As mentioned in Section 3.2, the field description summarizes the electromagnetic interactions between charged particles. Although exchange of energy in a system takes place between charged particles, it is often more convenient to imagine that energy resides in the fields themselves. In this section, we shall use two examples to demonstrate the correspondence between the electromagnetic energy of particles and the concept of *field energy density*.

The electric field energy density can be determined by considering the parallel plate capacitor (Fig. 5.6c). Initially, there is no voltage difference between the plates, and hence no stored charge or energy. Charge is moved slowly from one plate to another by a power supply. The supply must perform work to move charge against the increasing voltage. If the plates have a voltage V', the differential energy transfer from the power supply to the capacitor to move an amount of charge dQ' is  $\Delta U = V'dQ'$ . Modifying Eq. (3.9) for the presence of the dielectric, the electric field is related to the total charge moved between the plates by

$$E = \frac{Q/A}{\varepsilon}$$

where A is the area of the plates. In terms of incremental quantities during the charging cycle,  $dQ' = \varepsilon dE'A$ . The voltage is related to the field through V' = E'd. The total energy stored in the capacitor proceeding from zero electric field to E is

$$U = \int_{0}^{E} (E'd) (\varepsilon A dE') = (\varepsilon E^{2}/2) (Ad).$$
 (5.18)

The total energy can be expressed as the product of an energy density associated with the field lines times the volume occupied by field (Ad). The electrostatic energy density is thus

$$U(E) = \varepsilon E^{2}/2 = D \cdot E/2.$$
 (J/m<sup>3</sup>) (5.19)

The last form is the three-dimensional extension of the derivation.

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The magnetic field energy density can be calculated by considering the toroidal core circuit of Figure 5.11. Because the core is ferromagnetic, we will not assume a linear variation and well-defined  $\mu$ . The coil has *N* turns around a circumference *C*. A power supply slowly increases the current in the coil. To do this, it must counteract the inductive voltage *V'*. The energy transferred from the power supply to the circuit is  $\int V'Idt$ . The inductive voltage is V' = NA(dB'/dt), where *B'* is the sum of contributions to the field from *I* and the atomic current of the material. The applied field is related to the circuit current by  $B_1 = \mu_o NI/C$ . Summarizing the above considerations,

$$U = \int_{0}^{B} NA \ \frac{dB'}{dt} \ \frac{B_{1}C}{N\mu_{o}} = AC \ \int_{0}^{B} H \ dB'.$$
(5.20)

Recognizing that the volume occupied by fields is AC, the magnetic field energy density is

$$U(B) = \int_{0}^{B} H \, dB' \, (J/m^{3}). \tag{5.21}$$

If the relationship between *B* and *H* is known, the energy density of the final state can be evaluated. For instance, with a linear variation (constant  $\mu$ )

$$U(B) = B^{2}/2\mu = H \cdot B/2 \quad (J/m^{3}).$$
(5.22)

The magnetic field energy density in vacuum is  $B^2/2\mu^{\circ}$ .

The magnetic field energy density can be determined for nonlinear materials given the appropriate B-H curve. For instance, consider magnetizing a ferromagnetic toroidal core following the saturation hysteresis curve (Fig. 5.17a). Assume initially that the material is biased



Figure 5.17 Energy required to magnetize and demagnetize a ferromagnetic material. (a) Saturation hysteresis curve. (b) Quantities to calculate energy changes moving along saturation hysteresis curve. (c) Energy supplied by circuit (cross-hatched area) and energy returned to circuit (shaded portion).

to  $-B_r$  and a positive driving current is applied to bring it to  $+B_s$  (Fig. 5.17b). Both *H* and *dB* are positive, so work must be performed by the power source. The energy transfer is given by the shaded area of the graph (Fig. 5.17c) multiplied by the volume of the core. If the power supply is turned off, the core returns to  $+B_r$ . During this part of the cycle, *H* is positive but *dB* is negative, so that some energy is returned to the supply as an induced voltage. This energy is denoted by the darkly shaded portion of the graph; the net energy transfer in a half-cycle is the lightly shaded remainder. A similar process occurs for negative H. In most induction accelerators, a full cycle is traversed around the saturation hysterests curve for each beam pulse. An amount of energy equal to the area circumscribed by the hysteresis curve is lost to the core in each full cycle. This energy is expended in the irreversible process of domain reorientation. It ultimately appears in the core in the form of heat. In applications with continued and rapid recycling, it is clearly advantageous to use a ferromagnetic material with the smallest hysteresis curve area (soft material).

## **5.7 MAGNETIC CIRCUITS**

Magnetic fields used for charged particle transport are usually localized to small regions. This is the case for a bending magnet where the beam traverses a narrow vacuum region of parallel field lines. The condition of zero divergence implies that the field lines must curve around outside the transport region to return to the gap. Thus, most of the volume occupied by magnetic field serves no purpose for the application. If the surrounding region is vacuum or air ( $\mu/\mu_0 = 1$ ), most of the power input to the magnet is consumed to support the return flux. A more practical geometry for a bending magnet uses ferromagnetic material in the return flux region. We shall see that in this

case magnetic flux outside the gap is supported by the atomic currents in the material, with very little power required from the external supply.

The bending magnet is a circuit in the sense that the magnetic field lines circulate. The *magnetic circuit* has many analogies with electric circuits in which electrons circulate. The excitation windings provide the motive force (voltage), the vacuum gap is the load (resistance), and the ferromagnetic material completes the circuit (conducting wire). The magnetic circuit with iron is useful primarily when there is a small, well-defined gap. In applications requiring large-volume extended fields (such as hot plasma confinement devices for fusion reactors), there is little benefit in using ferromagnetic materials. Furthermore, ferromagnetic materials lose their advantages above their saturation field (typically 2 T). High-field applications (~6 T) have become practical through the use of superconductors. There is no energy penalty in supporting return flux lines in vacuum since the excitation windings draw no power.

Figure 5.18 illustrates the advantage of including ferromagnetic material in ordinary magnetic circuits. Assume that both the air core and iron core geometries produce the same field,  $B_g$ , in equal gaps. In order to compare the circuits directly, windings are included in the air core circuit so that the return flux is contained in the same toroidal volume. The magnetic flux in any cross section is a constant and is the same for both circuits. The gap has cross-sectional area  $A_g$ , and the core (or return flux coil) has area  $A_c$ . The length of the gap is g, while the length of the core (coil) is l. The excitation coils have an ampere turn product given by the number of windings multiplied by the current input to the windings, NI. The wires that carry the current have resistivity in an ordinary magnet; the power necessary to support the field is proportional to N (the length of the wire) and to  $I^2$ . It is desirable to make NI as small as possible.

The ampere turn products for the two circuits of Figure 5.18 can be related to the magnetic field in the circuit through Eq. (5.12):

$$\int (\boldsymbol{B}/\boldsymbol{\mu}) \cdot \boldsymbol{dl} = NI. \tag{5.23}$$

The constant circuit flux is given by

$$\Psi = B_g A_g = B_c A_c. \tag{5.24}$$

For the air core circuit, Eq. (5.23) becomes

$$B_{g} (g/\mu_{o}) + B_{c} (l/\mu_{o}) = NI,$$

or

$$\Psi \left(g/A_g \mu_o + l/A_c \mu_o\right) = NI.$$
(5.25)



Modification of Electric and Magnetic Fields by Materials

Figure 5.18 Comparison of energy required to generate a gap field by (a) an iron core magnet and (b) an air core magnet.

Similarly, the following equation describes the ferromagnetic circuit:

$$\Psi \left(g/A_{g}\mu_{o} + l/A_{c}\mu\right) = NI.$$
(5.25)

Comparing Eqs. (5.25) and (5.26), the ampere turn product for equal flux is much smaller for the case with the ferromagnetic core when  $\mu \ll \mu_0$  and  $g \ll l$  (a small gap).

An iron core substantially reduces power requirements for a de magnet. An alternate view of the situation is that the excitation coil need support only the magnetic field in the gap since the second term in Eq. (5.26) is negligible. The return flux is supported by the atomic currents of the material. The excitation coils are located at the gap in Figure 5.18a to clarify this statement. In practice, the coils can be located anywhere in the circuit with about the same result. This follows

from the Laplace equation which implies that the field line configuration minimizes the net field energy of the system. The field energy is a minimum if the flux flows in the ferromagnetic material. The field lines are thus conducted through the core and cross the gap in a manner consistent with the boundary condition discussed in Section 5.5, relatively independent of the location of the exciting coils. Ferromagnetic materials also help in pulsed magnet circuits, such as those in the betatron. The *duty cycle* (time on/time off) of such magnets is usually low. Thus, the total field energy is of greater concern than the instantaneous power. Energy for pulsed fields is usually supplied from a switched capacitor bank. A ferromagnetic return flux core reduces the circuit energy and lowers the cost and size of the capacitor bank.

To summarize, ferromagnetic materials have the following applications in non-superconducting accelerator magnets:

The iron can be shaped near the gap to provide accurate magnetic field gradients.
 The exciting coil power (or net field energy) is reduced significantly compared to an air core circuit.

3. The iron conducts flux lines so that the exciting windings need not be located at the gap.

Equation (5.26) has the same form as that for an electric circuit consisting of a power source and resistive elements with the following substitutions.

**Magnetic Flux \Psi.** The analogy of current in an electric circuit. Although the magnitude of B may vary, the flux in any cross section is constant.

**Magnetomotive Force.** (Ampere turn product, NI.) The driving force for magnetic flux. Magnetomotive force corresponds to the voltage in an electric circuit.

**Reluctance**. Corresponds to the resistance. Equation 5.26 contains two reluctances in series,  $R_g = g/A_g\mu_o$  and  $R_c = l/A_c\mu$ . The higher the reluctance, the lower the flux for a given magnetomotive force. The reluctance of the iron return flux core is much smaller than that of the gap (the load), so it acts in the same way as a low-resistivity wire in an electric circuit.

Permanence. The inverse of reluctance and the analogue of conductance.

The circuit analogy is useful for estimating operating parameters for complex magnets such as those found in electric motors. To illustrate a magnetic field calculation, consider the spectrometer magnet illustrated in Figure 5.19a. The components of reluctance already mentioned can be supplemented by additional paths representing fringing flux (magnetic field lines bulging out near the gap) and leakage flux (magnetic flux returning across the magnetic yoke without traversing the gap). These reluctances can be determined by a solution of the Laplace equation for the magnet. Reluctances are combined in series and in parallel, just as resistances. The equivalent circuit is illustrated in Figure 5.19b. The effect of leakage flux on the field in the gap can be





minimized by placing the excitation coils as close to the gap as possible.

A first-order estimate of driving coil parameters can be derived by neglecting the leakage and fringing contributions and assuming that the circuit reluctance resides predominantly in the gap. In this case it is sufficient to use Eq. (5.23), so that  $NI \approx B_g g/\mu_o$ . For example, production of a field of 1 T in a gap with a 0.02 in spacing requires 16-kA turns (160 turns of wire if a 100-A supply is available). For a given supply and excitation coil winding, the field magnitude is inversely proportional to the spacing of the magnet poles.

## **5.8 PERMANENT MAGNET CIRCUITS**

Permanent magnet circuits have the advantage that a dc magnetic field can be maintained with no power input. There are two drawbacks of permanent magnet circuits: (1) it is difficult to vary the field magnitude in the gap and (2) bulky magnets are needed to supply high fields over large areas. The latter problem has been alleviated by the development of rare-earth samarium cobalt magnets which have a maximum energy product three times that of conventional Alnico alloys. In other words, the same field configuration can be produced with a magnet of one-third the volume. Permanent magnet quadrupole lenses (Section 6.10) are an interesting option for focusing in accelerators (See K. Halbach, *Physical and Optical Properties of Rare Earth Cobalt Magnets*, Nucl. Instrum. Methods **187**, 109 (1981). In this section, we shall review some of the properties of permanent magnetic materials and first-order principles for designing magnetic circuits.



The second quadrant of a hysteresis curve for some common permanent magnet materials is shown in Figure 5.20. The plot is usually called the *demagnetization curve*. The most striking difference between Figure 5.20 and the hysteresis curve for soft iron (Fig. 5.13) is that the coercive force is about 100 times larger for the permanent magnets. In other words, it takes considerably more energy to align the domains and to demagnetize the material. Generation of magnetic fields in vacuum requires energy; a permanent magnet can produce fields because of the stored energy received during magnetization.

Figure 5.20 can be used to calculate the field produced in the gap of a magnetic circuit. The method used to find the operating point on the demagnetization curve is illustrated in Figure 5.21. In the first part of the figure, the permanent magnet is included in a zero reluctance circuit



Figure 5.21 Operating point of a permanent magnet. (a) Permanent magnet with a continuous iron flux conductor, zero magnetizing force. (b) Addition of an air gap with a coil to supply field energy. (c) Deactivation of the gap coil.

containing an ideal iron core ( $H_s = 0$ ) with no gap. There is no free current. The circuital integral of H is zero, so that  $H_m$  (the magnetic intensity in the permanent magnet) is zero. The magnetic field in the loop is equal to  $B_m$ , the remanence field of the permanent magnet. Next, assume that an air gap is introduced into the circuit, but excitation windings are placed around the gap with the proper current to produce a field equal to  $B_m$  (Fig. 5.21b). The current in the windings is in the same direction as the atomic currents in the ferromagnetic materials. Because the energy for the vacuum fields is supplied by an external source, the circuit still appears to have zero reluctance. The operating point of the permanent magnet remains at  $H_m = 0$ ,  $B_m = B_{rm}$ .

In the final state (Fig. 5.21c), the current in the excitation coils drops to zero. This is equivalent to the addition of a negative current to the existing current. The negative current demagnetizes the permanent magnet, or moves the operating point in Figure 5.20 to the left. Thus, an air gap in a permanent magnet circuit acts like an excitation winding with a current opposed to the atomic currents. There is no net applied current in the circuit of Figure 5.21c so that /H dl = 0. Neglecting the reluctance of the iron core, the operating point of the permanent magnet is determined by the gap properties through

$$H_m L_m = H_g L_g = B_g L_g / \mu_o.$$
(5.27)

where  $L_m$  is the length of the permanent magnet.

An important parameter characterizing the performance of a permanent magnet in a circuit is the energy product.

**Energy Product**. The product of magnet field times magnetic intensity at the operating point of a permanent magnet, or  $H_m B_m$ .

Equation (5.27) can be used to demonstrate the significance of the energy product. We again take the example of a simple circuit with a magnet, zero reluctance core, and air gap. Continuity of flux implies that  $B_gA_g = B_mA_m$ , where  $A_g$  and  $A_m$  are the cross-sectional areas of the gap and magnet respectively. This condition, combined with Eq. (5.27), yields

$$(B_g^2/2\mu_o) \ (A_g L_g) = (H_m B_m/2) \ (A_m L_m).$$
(5.28)

The first factor on the left is the magnetic field energy density in the gap, and the second term is the gap volume. On the right, the first factor is one-half the energy product and the second factor is the magnet volume. Thus, the magnet volume and the energy product determine the magnetic field energy in the gap.

Energy product is given in joules per cubic meter (mks units) or in megagauss oersteds (MG-Oe) in CGS units. The conversion is I MG-Oe = 7940 J/m<sup>3</sup>. Hyperbolic lines of constant  $B_m H_m$  are plotted in Figure 5.20. This is a graphic aid to help determine the energy product at different points of the demagnetization curve. A goal in designing a permanent magnet circuit is to produce the required gap field with the minimum volume magnet. This occurs at the point on the

demagnetization curve where  $B_m H_m$  is maximum. In Figure 5.20, parameters of the circuit should be chosen so that  $H_m$  is about 550 Oe.

Two examples will serve to illustrate methods of choosing permanent magnets. Both involve first-order design of a simple circuit with no leakage or fringing flux; the second-order design must invoke field calculations, tabulated gap properties or modeling experiments for an accurate prediction. To begin, suppose that we constrain the gap parameters ( $B_g = 8 \text{ kG}$ ,  $A_g = 10 \text{ cm}^2$  and  $L_g = 1 \text{ cm}$  and the type of magnetic material (Alnico 5). We must now determine the dimensions of the magnet that will produce the gap field. Using Eq. (5.27) with  $H_g = 8 \text{ kOe}$  and  $H_m = 600 \text{ Oe}$ , the length of the magnet must be 13.3 cm. The magnetic field is  $B_m = 8 \text{ kG}$ , so that the minimum magnet cross-sectional is 10 cm<sup>2</sup>.

In the second example, assume that the dimensions of the gap and magnet are constrained by the application. The goal is to determine what magnetic material will produce the highest gap flux and the value of this flux. For the simple circuit, the condition of constant flux can be combined with Eq. (5.27) to give

$$H_g L_g / B_g A_g = H_m L_m / B_m A_m = L_g / \mu_o A_g.$$

The expression on the right-hand side is the reluctance of the gap,  $R_g$ . We can then write

$$B_m/H_m = L_m/R_g A_m. (5.29)$$

With the stated conditions, the quantity  $B_m H_m$  must have a constant ratio. This motivates the definition of the permanence coefficient.

**Permanence Coefficient, or Load Line**. Equal to  $(B_m/H_m)$ . The permanence coefficient is a function only of the geometries of the magnetic load (system reluctance) and the magnet.

Fiducial points are usually included in demagnetization curves to lay out load lines. Given the load line, operating points on various permanent magnet materials can be determined. The highest gap flux will be produced by the material with the highest energy product at the intersection. The gap flux can then be determined from the magnet operating point.