There is good evidence that the deuteron ground state is primarily 1s
(n=1, ℓ=0). First, the lowest energy state in practically all the
model potentials is an s state. Secondly, the magnetic moment of H^2
is approximately the sum of the proton and the neutron moments, indi-
cating that $\underline{s}_n \parallel \underline{s}_p$ and no orbital motion of the proton relative to
the neutron; this is also consistent with the total angular momentum
of the ground state, I=1.

We therefore consider only the ℓ=0 radial wave equation,

$$-\frac{\hbar^2}{m}\frac{d^2 u(r)}{dr^2} + V(r)u(r) = E\, u(r)$$
$$\hookrightarrow -E_B$$

where u(r) = rR(r) and m is the neutron (or proton) mass.

Notice this equation is mathematically equivalent to the 1-D wave
equation except for the B.C., u(0) = 0.

Solutions:

$$u(r) = a\, e^{iKr} + b\, e^{-iKr} \qquad r<r_o, \quad K = [m(V_o-E_B)]^{1/2}/\hbar$$

$$u(r) = a'\, e^{K_\nu r} + b'\, e^{-K_\nu r} \qquad r>r_o, \quad K_\nu = \sqrt{mE_B}/\hbar$$

Applying the boundary conditions

    (i)   u(0) = 0, so R(0) is finite

    (ii) u(r$\rightarrow\infty$) = 0 since we are dealing with a bound state

    (iii) continuity of u and first-order derivative at r=r$_o$
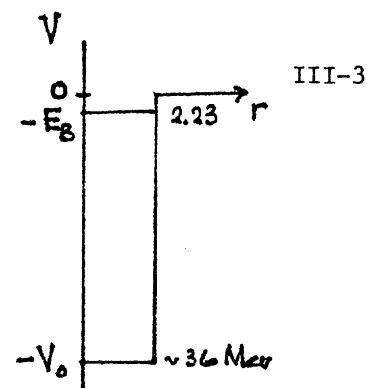
one obtains

$$u(r) = c \sin Kr \qquad r<r_o$$
$$a'=0, \quad \therefore\; u(r)=b'\, e^{-K_\nu r} \qquad r>r_o$$

$$K \cot Kr_o = -K_\nu \qquad \text{this is a relation between } V_o, E_B \text{ and } r_o$$

$$\text{or } \tan Kr_o = -\frac{K}{K_\nu} = -\left(\frac{V_o-E_B}{E_B}\right)^{1/2}$$

Suppose $V_o \gg E_B$, then RHS is large and $Kr_o \sim \pi/2$ (we will see below that
$V_o \sim 36$ Mev as compared to $E_B \sim 2.23$ Mev).

Then $\quad K \sim \dfrac{\sqrt{m V_o}}{\hbar} \sim \dfrac{\pi}{2 r_o}$

or $\quad V_o r_o^2 \sim \left(\dfrac{\pi}{2}\right)^2 \dfrac{\hbar^2}{m} \sim 1 \text{ Mev-barn}$

We see from this result that a knowledge of E_B allows us to only determine the product $V_o r_o^2$. From n-p scattering (discussed below) we will find $r_o \sim 2F$, thus we obtain $V_o \sim 36$ Mev.

Since the interior wave function, $\sin Kr$, must match with the exterior wave function, $\exp(-Kr)$, the quantity $K r_o$ must be slightly greater than $\pi/2$ (more accurate estimate gives $116°$ instead of $90°$), so if we write

$$K = 2\pi/\lambda \sim \pi/2r_o$$

then the 'effective wavelength' λ is approximately $4 r_o$ which suggests that much of the wave function is not in the interior region.

We can estimate the relaxation constant (or decay length) in the exterior region, $r > r_o$

$$\dfrac{1}{K} = \dfrac{\hbar}{\sqrt{m E_B}} \sim 4.3 \text{ F}$$

This means that the two nucleons in H^2 spend a large fraction of their time at $r > r_o$, the classically forbidden reion of negative kinetic energy. We can calculate the root-mean-square radius of the deuteron wave function,
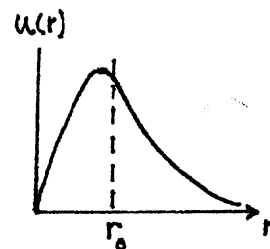
$$R_{rms}^2 \equiv \dfrac{\int d^3 r \; r^2 R^2(r)}{\int d^3 r \; R^2(r)} = \dfrac{\int_0^\infty r^2 dr \; r^2 R^2(r)}{\int_0^\infty r^2 dr \; R^2(r)}$$

If we put $R(r) \sim e^{-Kr}$ for all r which should result in an overestimate, one gets

$$R_{rms}^2 = \dfrac{\hbar}{\sqrt{2 m E_B}} = 3 \text{ F}$$

which can be compared with

$$\left(1.4 \times A^{1/3}\right)^2 \sim 3.11 \quad \text{or} \quad \left(1.2 \times A^{1/3}\right)^2 \sim 2.29 \qquad A = 2$$

B. Neutron-proton Scattering

One can also obtain information about the neutron-proton interaction by studying the scattering of neutrons by a hydrogen sample. To describe the scattering process we imagine the incident neutrons are represented by a plane wave of the form

$$\Psi_{in} = b\, e^{i(kz - \omega t)}$$

where $k = \sqrt{2\mu T}/\hbar$, μ is the reduced mass
of the incoming particle.

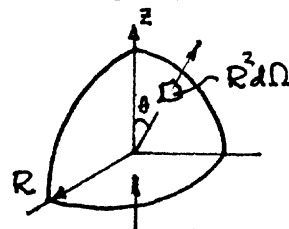T is the kinetic energy in center-of-mass coordinates (CMCS).

Incident flux $J_{in} = v\, \Psi_{in}^* \Psi_{in} = v |b|^2$, v is relative speed.
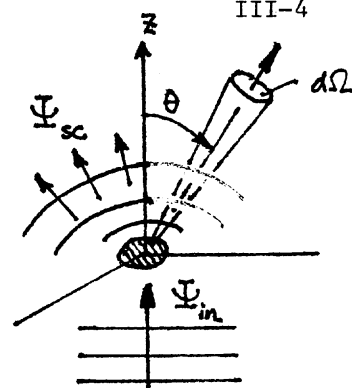
We look for the scattered wave in the form of a spherical outgoing
wave,

$$\Psi_{sc} = f(\theta)\, b\, \frac{e^{i(kr - \omega t)}}{r}$$

where f(θ) is called the scattering amplitude.

$$dN = \left(\Psi_{sc}^* \Psi_{sc} v \right) R^2 d\Omega$$

No. particles scattered
into area $R^2 d\Omega$ per
unit time.

$$\frac{d\sigma}{d\Omega} = \frac{dN/d\Omega}{J_{in}} = |f(\theta)|^2$$

angular differential
scattering cross
section

$$\sigma = \int d\Omega\, |f(\theta)|^2$$

total scattering
cross section

The quantity we want to calculate is f(θ). Stating the problem in
another way, we want to solve the wave equation

$$-\frac{\hbar^2}{2\mu} \nabla^2 \psi + V(r)\psi = T\psi$$

subject to the B.C.

$$\psi \xrightarrow[r \to \infty]{} b \left[e^{ikz} + f(\theta)\, \frac{e^{ikr}}{r} \right]$$

We proceed by considering a <u>partial wave analysis</u>:

Since V(r) is spherically symmetric it is most conveninet to
discuss the solution in spherical coordinates. We already know that
the general solution is of the form

$$R(r)\, P_\ell^{(m)}(\cos\theta)\, e^{im\varphi}$$

which, because the potential does not depend on φ, reduces to
$R_\ell(r) P_\ell(\cos\theta)$. Thus we expand

$$\psi(r,\theta) = \sum_{\ell=0}^{\infty} \psi_\ell(r) P_\ell(\cos\theta)$$

one can show that $e^{ikz} = e^{ikr\cos\theta} = \sum_{\ell} F_{\ell}(r) P_{\ell}(\cos\theta)$,

then $F_0(r) = \sin kr / kr$

[For arbitrary ℓ, one finds $F_{\ell}(r) \sim i^{\ell}(2\ell+1) \dfrac{\sin(kr - \ell\pi/2)}{kr}$].

Furthermore, we will write $f(\theta) = \sum_{\ell} f_{\ell} P_{\ell}(\cos\theta)$.

Then

$$\psi_0 \underset{r\to\infty}{\sim} b \cdot \left[\frac{\sin kr}{kr} + f_0 \frac{e^{ikr}}{r} \right]$$

$$= \frac{b}{2ikr} \left[(1 + 2ikf_0) e^{ikr} - e^{-ikr} \right] \equiv R_0(r) = \frac{u_0(r)}{r} \qquad (*)$$

$$\underset{\text{outgoing}}{\downarrow} \qquad \underset{\text{incoming}}{\downarrow}$$

For $r > r_0$, the wave equation is

$$-\frac{\hbar^2}{2\mu} \frac{d^2 u_0}{dr^2} = T u_0 \quad , \qquad k = \sqrt{2\mu T}/\hbar$$

$$u_0(r) = A \sin(kr + \delta_0) \quad \text{s-wave phase shift}$$

$$= \frac{A e^{-i\delta_0}}{2i} \left(e^{ikr} e^{2i\delta_0} - e^{-ikr} \right) \qquad (**)$$

Comparing (*) and (**) one obtains

$$1 + 2ikf_0 = e^{2i\delta_0}$$

$$f_0 = \frac{e^{2i\delta_0} - 1}{2ik} = \frac{e^{i\delta_0} \sin\delta_0}{k} \qquad (\dagger)$$

Thus,

$$\frac{d\sigma}{d\Omega} = \frac{\sin^2\delta_0}{k^2} = \lambdabar^2 \sin^2\delta_0 \quad , \qquad \sigma = 4\pi\lambdabar^2 \sin^2\delta_0$$

We will define the scattering length as $a \equiv \lim_{k\to 0} (-f_0)$.

Since f_0 can not $\to\infty$ as $k \to 0$ (otherwise $\sigma \to \infty$), ($\dagger$) shows that δ_0 must $\to 0$ as $k \to 0$, so

$$a = \lim_{k\to 0} \left[- e^{i\delta_0} \frac{\sin\delta_0}{k} \right] = - \frac{\delta_0}{k}$$

We can therefore obtain

$$\lim_{k\to 0} \frac{d\sigma}{d\Omega} = a^2$$

and
$$\lim_{k \to 0} \sigma = 4\pi a^2$$

$$\delta > 0 \quad (\text{attractive pot.})$$
$$\delta < 0 \quad (\text{repulsive})$$

Significance of a as an extrapolation distance:

Notice that

$$u_o = A \sin k(r-a) \quad , \quad r > r_o$$

$$\underset{kr_o \to 0}{\sim} \quad Ak(r-a)$$

positive a

negative a
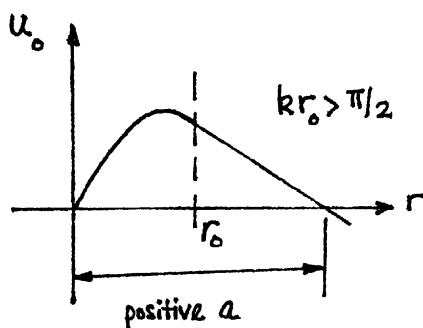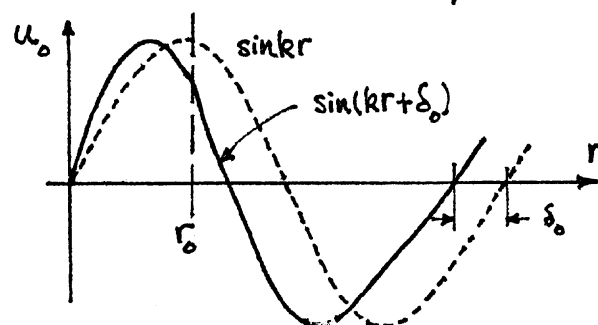
[this case corresponds to a potential which can give rise to a bound state]
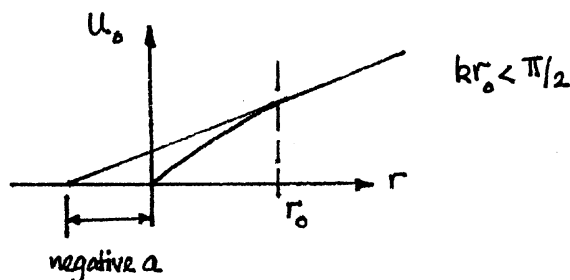
[this potential can give rise to a virtual state]

We now apply these results to n-p scattering. In this case we have the same equation as the deuteron calculation except that $E > 0$.

$$r < r_o \qquad u_o = C \sin K'r \quad , \qquad K' = [2\mu(V_o + T)]^{1/2}/\hbar$$

$$r > r_o \qquad u_o = A \sin(kr + \delta_o), \qquad k = \sqrt{2\mu T}/\hbar \quad , \quad \mu = \text{reduced mass}$$
$$= m/2$$

Matching B.C. at $r = r_o$,

$$K' \cot K'r_o = k \cot(kr_o + \delta_o)$$

this gives δ_o in terms of V_o, r_o, and T.

To simplify the calculation we will assume

(i) $a \gg r_o$ so that $kr_o \ll \delta_o$
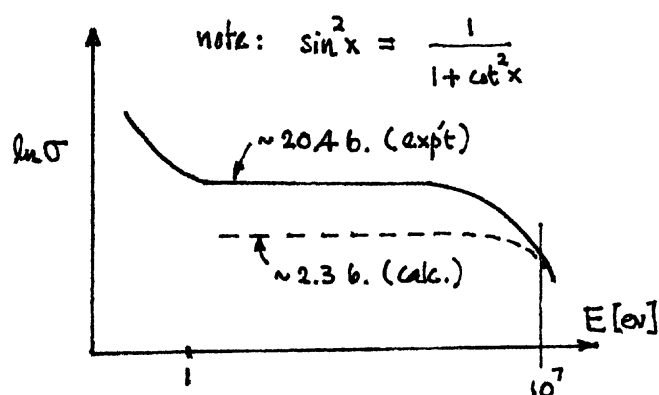
(ii) $T \ll V_o$ so $K' \sim K$

then

$$k \cot \delta_o \approx K \cot K r_o = -K, \quad \longleftarrow \text{ from the deuteron problem}, \quad K = \sqrt{mE_B}/\hbar$$

$$\frac{d\sigma}{d\Omega} = \frac{\sin^2 \delta_o}{k^2} = \frac{1}{k^2} \frac{1}{1 + K^2/k^2}$$

$$\frac{d\sigma}{d\Omega} \approx \frac{1}{k^2 + K^2} = \frac{\hbar^2}{m} \frac{1}{T + E_B}$$

$$\sigma \approx \frac{4\pi\hbar^2}{m} \frac{1}{T + E_B}$$

note: $\sin^2 x = \frac{1}{1 + \cot^2 x}$

~20.4 b. (exp't)

~2.3 b. (calc.)

Since $T < E_B (2.23 \text{ Mev})$, $\sigma \sim 2.3$ barns

This does not agree with the experimental value which is ~20.4 barns.

Wigner (1933) has suggested that n-p scattering depends on whether the neutron and proton spins are parallel (triplet state, I=1) or anti-parallel (singlet state, I=0); for these two cases the potential (and $\therefore \delta_o$) is different. Following this idea one can write

$$\frac{d\sigma}{d\Omega} = \frac{1}{k^2} \left[\frac{1}{4} \sin^2 \delta_{os} + \frac{3}{4} \sin^2 \delta_{ot} \right]$$

singlet   triplet

We already know that the triplet state gives rise to the ground state of the deuteron at $E = -E_B$. If the singlet state produces a state of energy $E=E^*$ (E^* could be positive or negative), then

$$\sigma \approx \frac{\pi\hbar^2}{m} \left(\frac{3}{T + E_B} + \frac{1}{T + |E^*|} \right)$$

With $|E^*| \approx 70$ kev, σ is now in good agreement with experiment. Since $|E^*| \ll V_o$, just like $E_B \ll V_o$ in the triplet state, the singlet wave function inside the nuclear potential is also $\sim \frac{1}{4}$ of a sine wave,

$$K r_o \approx \frac{\pi}{2}, \quad \text{or}$$

$$V_{os} r_{os}^2 \approx V_{ot} r_{ot}^2 \sim 1.0 \text{ Mev-barn}$$

However, using information from neutron scattering from parahydrogen (I=0) one finds a scattering length $a_s = -24$ F, thus indicating that the singlet interaction gives rise to a virtual state ($a_s < 0$).

In summary, one finds from such considerations the following results [cf. Preston, _Physics of the Nucleus_]

Interaction	Scattering lengths a[F]	r_o[F]	V_o[Mev]
n-p(triplet)	5.4	2	36
n-p(singlet)	-23.7 (n-p)	~ 2.5	18
	-17 (p-p)		

Final remark:

Experimentally it is known that the total angular momentum (nuclear spin) of the deuteron ground state is I=1, where

$$\underline{I} = \underline{L} + \underline{S}$$

where $\underline{L}$ is the orbital angular momentum and $\underline{S}$ the intrinsic spin. It is also known that the ground state is mostly 1s (ℓ=0), therefore for this state S=1 (n and p spins parallel). Now we have seen from the discussions of part A that the ground state is barely bound (E_B=2.23 Mev), so all the higher energy states are not bound states. Example, the 1s state with S=0 (n and p spins antiparallel), is a virtual state; it is unbound by ~60 kev. The significance of this is that nuclear interaction is different for different S states, i.e., nuclear forces are spin-dependent.

From 22.111 Lecture Notes by S. Yip, Chap. III.

A. Bound State of the Deuteron

Experimentally it is known that the binding energy E_B of the deuteron is 2.23 Mev. The deuteron is the only stable bound system of two nucleons. We will see later that the di-neutron and di-proton are not stable. The energy E_B is known from the γ energy in the reaction

$$n + H^1 \longrightarrow H^2 + \gamma \ (2.23 \text{ Mev})$$
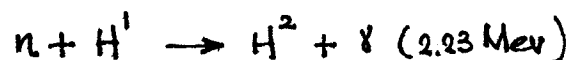
The inverse reaction of using electrons of known energy to produce external bremsstrahlung for (γ,n) reaction on H^2 also has been used. Besides the ground state no stable excited states of H^2 have been found. (There is a virtual state at ~ 2.30 Mev.)

Suppose we assume a square well potential (in 3-D),

$$V(r) = \begin{cases} -V_0 & r < r_0 \\ 0 & r > r_0 \end{cases}$$

Then we ask what is the level structure and what values should V_0 and r_0 take in order to be consistent with a bound state at energy E_B = 2.23 Mev?

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*For a general discussion of nuclear forces, see Chapter 6 of Meyerhof.