From 22.111 Lecture Notes by S. Yip, Chap VII

VII. Interaction of Neutrons with Matter*

Since a neutron has no charge it can easily enter into a nucleus and cause reaction. Neutrons interact mostly with the nucleus of an atom; except in the special case of magnetic scattering of thermal neutrons (which is of no interest in this course) one can neglect the electrons and think of atoms and nuclei interchangeably. Neutron reactions can take place at any energy, so one has to pay particular attention to the energy dependence of the interaction cross section. In the nuclear reactor neutrons with energies from 10^{-3} ev (1 mev) to 10^{7} ev (10 Mev) are of interest, this means an energy range of 10^{10}.

For a given energy region (thermal, epithermal, resonance, fast) not all the possible reactions are equally important. What is important depends on the problem at hand. Generally speaking the important types of interactions, in the order of increasing complexity from the standpoint of theory, are:

(n,n) - elastic scattering

- potential scattering: interaction at nuclear surface, billiard ball-like collision
- resonance scattering: compound nucleus formation

(n,γ) - radiative caputre

(n,n')- inelastic scattering

(n,p),(n,α), ... - charged particle emission

(n,f) - fission

If we were interested in fission reactors the reactions in the order of importance would be:

fission

capture (in fuel and other reactor materials)

scattering (elastic and inelastic)

fission product decay by β-emission (delayed neutron, heat production)

In this section we will study mostly potential scattering. The other

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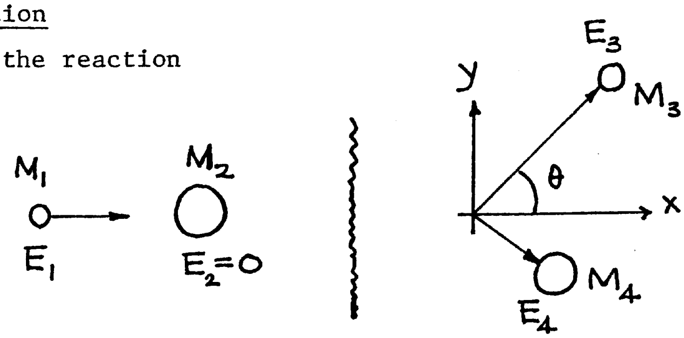
*This topic is treated clearly, although somewhat too briefly, in Meyerhof, Sec. 3.3. There is a very readable and quite complete semqualitative discussion in Lamarsh, Nuclear Reactor Theory (1966), Chap. 2. Detailed treatment of reaction energetics is given in Evans, Chap. 12.

reactions all involve compound nucleus formation; they will be discussed later in Sec. X.

A. The Q-Equation

Consider the reaction



We will derive an equation relating the outgoing energy E_3 to the angle θ using the conservation of energy (kinetic plus rest mass - ignore relativistic effects) and linear momentum.* For the case of stationary or low-temperature target, $E_2 = 0$,

$$(E_1 + M_1c^2) + M_2c^2 = (E_3 + M_3c^2) + (E_4 + M_4c^2)$$

Rewriting the momentum equation as

$$\begin{aligned} p_4^2 &= (\vec{p}_1 - \vec{p}_3)^2 \\ &= p_1^2 + p_3^2 - 2p_1p_3\cos\theta = 2M_4E_4 \end{aligned}$$

and recalling

$$\begin{aligned} Q &= (M_1 + M_2 - M_3 - M_4)c^2 \\ &= E_3 + E_4 - E_1 \end{aligned}$$

we obtain

$$Q = E_3\left(1 + \frac{M_3}{M_4}\right) - E_1\left(1 - \frac{M_1}{M_4}\right) - \frac{2}{M_4}\sqrt{M_1M_3E_1E_3}\cos\theta \qquad (*)$$

Q-Equation

Notice that the energies E_i and angle θ are in LCS while Q is independent of coordinate system (since Q can be expressed in terms of masses which

*Other quantities which are also conserved in nuclear reactions are: charge, mass number, statistics, angular momentum, and isobaric spin (cf. Evans, Chap. 12). However, these do not enter into the present discussion.

of course do not depend on coordinate system). Usually the incident energy E_1, the masses, the Q-value are all known, and one is interested in solving this equation for E_3 in terms of $\cos\theta$, or vice versa.

In general there are two solutions to the Q-equation, $\pm\sqrt{E_3}$. The physical solution is the one that is real and positive. However, there are situations where both solutions are real and positive (cf. Evans, pp. 413-415).

Consider two special cases of the Q-equation

(i) elastic scattering: $Q = 0$, $M_1 = M_3 = M_n = m$, $M_2 = M_4 = M$

$$E_3\left(1+\frac{m}{M}\right) - E_1\left(1-\frac{m}{M}\right) - \frac{2m}{M}\sqrt{E_1E_3}\cos\theta = 0 \qquad (\dagger)$$

We see that $E_3 \neq E_1$, or neutron loses energy even in an elastic scattering collison (elastic here mean no excitation of the nucleus); indeed $E_4 = E_1 - E_3$ is just the recoil energy of the target nucleus. Eq. ($\dagger$) is the starting point for the analysis of neutron slowing down in a moderator.

(ii) inelastic scattering: $Q = -E^*$, $M_1 = M_3 = m$, $M_2 = M$, $M_4 = M^*$
In this case E^* is the excitation energy imparted to the target nucleus, with $M^* = M + E^*/c^2$. If we take $\theta = 0$ and $E_3 \simeq 0$, then we have

$$E_1 = E^*\left(\frac{M_4}{M_4 - M_1}\right) \simeq E^*\left(\frac{M+m}{M}\right)$$

This is a good approximation to the threshold energy, the minimum possible value of E_1 for which the reaction can take place. Notice that the threshold energy is slightly larger than the excitation energy; this is to be expected since some of the incoming energy has to go into the energy of the center-of-mass which is not available for the reaction. We will come back to this point in Sec. X.

B. Elastic Scattering

We consider Eq. ($\dagger$) as a quadratic equation for $\sqrt{E_3}$,

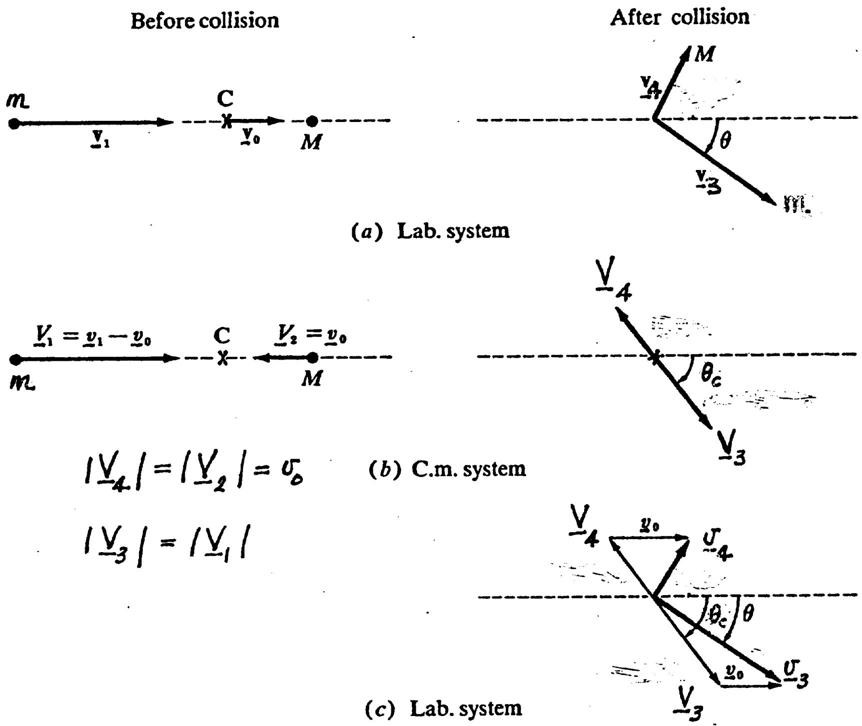
$$E_3 - \frac{2m}{M+m}\cos\theta\sqrt{E_1E_3} - \frac{M-m}{M+m}E_1 = 0$$

$$\sqrt{E_3} = \frac{m}{M+m}\cos\theta\sqrt{E_1} \pm \frac{\sqrt{E_1}}{M+m}\left[m^2\cos^2\theta + M^2 - m^2\right]^{1/2}$$ relates E_3 to θ

$$\text{or} \quad \sqrt{E_3} = \frac{m}{M+m}\sqrt{E_1}\left\{\cos\theta + \left[\left(\frac{M}{m}\right)^2 - \sin^2\theta\right]^{1/2}\right\} \qquad (\dagger\dagger)$$

where we have chosen the upper sign for the physical solution. Notice that in the case of forward scattering, $\theta = 0$, we have $E_3 = E_1$, and for backward scattering, $\theta = \pi$, and $E_3 = [(M-m)^2/(M+m)^2]E_1$. These two cases correspond to no energy loss and maximum energy loss.

Another way to analyze elastic scattering is to consider the center-of-mass coordinate system (velocities are denoted by capital letters and angle by subscript c).



(a) Lab. system

(b) C.m. system

$|V_4| = |V_2| = v_0$

$|V_3| = |V_1|$

(c) Lab. system

From diagram (c) we can deduce a number of useful relations. Perhaps the most important one is

$$\frac{1}{2}mv_3^2 = \frac{1}{2}m(\underline{V}_3 + \underline{v}_0)^2$$

$$= \frac{1}{2}m(V_3^2 + v_0^2 + 2V_3v_0\cos\theta_c)$$

or
$$E_3 = \frac{1}{2} E_1 \left[(1+\alpha) + (1-\alpha)\cos\theta_c \right] \qquad (\Delta)$$

where
$$\alpha \equiv \left(\frac{M-m}{M+m}\right)^2 = \left(\frac{A-1}{A+1}\right)^2, \quad A \equiv \frac{M}{m}$$

This result is actually identical to the square of Eq. (††),
$$E_3 = \frac{1}{(1+A)^2} E_1 \left\{ \cos^2\theta + A^2 - \sin^2\theta + 2\cos\theta\left[A^2 - \sin^2\theta\right]^{1/2} \right\}$$

which contains the LCS scattering angle θ instead of the CMCS scattering angle θ_c. To demonstrate the equivalence one needs a relation between θ and θ_c. This can be obtained from diagram (c).

$$\cos\theta = (v_0 + V_3\cos\theta_c)/v_3$$
$$= \frac{1 + A\cos\theta_c}{\sqrt{A^2+1+2A\cos\theta_c}}$$

Eq. (Δ) shows that there is a one-to-one relation between the scattering angle θ_c and the outgoing neutron energy. We will use this relation to derive the energy distribution for elastically scattered neutrons.

C. Energy and Angular Distributions

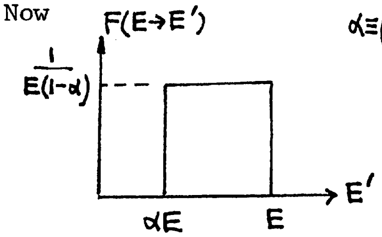
For $E_1 \lesssim 0.1$ Mev we need only consider s-wave scattering, and as we have discussed in Sec. III s-wave scattering is isotropic, or spherically symmetric, in CMCS. This means that if $P(\underline{\Omega}_c)\,d\Omega_c$ is the probability of scattering into solid angle $d\Omega_c$ about $\underline{\Omega}_c$ where $\underline{\Omega}_c$ is the unit vector along $\underline{V}_3$, then
$$P(\underline{\Omega}_c)\,d\Omega_c = \frac{d\Omega_c}{4\pi}$$

Let $G(\theta_c)\,d\theta_c$ be the probability that the scattering angle lies between θ_c and $\theta_c + d\theta_c$,
$$G(\theta_c)\,d\theta_c = \int_{\varphi=0}^{2\pi} d\varphi\, P(\underline{\Omega}_c)\sin\theta_c\, d\theta_c$$
$$= \frac{1}{2}\sin\theta_c\, d\theta_c$$

Since there is a one-to-one relation between $E_3 \equiv E'$ and θ_c, we can write

$$F(E\to E')\,dE' = G(\theta_c)\,d\theta_c$$

where $F(E\to E')dE'$ is the probability that given a neutron scattered at energy E it will have outgoing energy in dE' about E'. Now



$\alpha \equiv \left(\frac{M-m}{M+m}\right.$

$$F(E\to E') = G(\theta_c)\left|\frac{d\theta_c}{dE'}\right|$$

or

$$F(E\to E') = \begin{cases} \dfrac{1}{E(1-\alpha)} & \alpha E \leq E' \leq E \\ 0 & \text{otherwise} \end{cases}$$

Notice that $F(E\to E')$ is a distribution, or density: its dimension is 1/energy since it is a probability per unit outgoing energy. Also F is properly normalized in that its energy integral is unity as required by number conservation.

Knowing the probability distribution function $F(E\to E')$ one can construct the energy differential cross section

$$\frac{d\sigma_s(E\to E')}{dE'} \equiv \sigma_s(E\to E') = \sigma_s(E)\,F(E\to E')$$

which is also known as the scattering kernel (sometimes the scattering kernel is defined as the macroscopic cross section $N\sigma_s(E\to E') \equiv \Sigma_s(E\to E')$ with N the number of target nuclei per cm^3). Here $\sigma_s(E)$ is the total scattering cross section defined on p. III-4. For the discussion of neutron moderation such as that in a reactor or in a shielding material it is sufficient to consider only s-wave scattering, then $\sigma_s(E) = 4\pi a^2$, where a is the scattering length which is energy-independent. Our discussion thus far is valid for neutrons above thermal energies. For thermal neutrons one cannot assume the target nucleus is at rest (as we did in deriving the Q-equation). In this case the calculations become much more complicated as will be seen below.

The distribution $F(E\to E')$ is useful for calculating various energy averaged quantites. For example, the average energy loss per collision at energy E is

$$\int_{\alpha E}^{E} dE'(E-E')\,F(E\to E') = \frac{E}{2}(1-\alpha)$$

For hydrogen the average energy loss is therefore half the initial energy, whereas for a heavy nucleus it is $\sim 2E/A$.

We have used the fact that s-wave elastic scattering is spherically symmetric in CMCS. This means the angular differential corss section (in CMCS) is

$$\frac{d\sigma_s(E,\theta_c)}{d\Omega_c} \equiv \sigma_s'(\theta_c) = \frac{\sigma_s(E)}{4\pi}$$

One can ask what is the angular distribution of the scattered neutrons in LCS? To answer this question we can relate the angular distribution in CMCS to the angular distribution in LCS,

$$\sigma_s(\theta)\, d\Omega = \sigma_s'(\theta_c)\, d\Omega_c \quad \text{, or} \qquad \sigma_s(\theta) = \sigma_s'(\theta_c)\, \frac{\sin\theta_c}{\sin\theta}\, \frac{d\theta_c}{d\theta}$$

From the relation between $\cos\theta$ and $\cos\theta_c$ n p. VII-5 we obtain $d(\cos\theta_c)/d(\cos\theta) = \sin\theta_c\, d\theta_c / \sin\theta\, d\theta$, thus

$$\sigma_s(\theta) = \frac{\sigma_s(E)}{4\pi}\, \frac{(\gamma^2 + 2\gamma\cos\theta_c + 1)^{3/2}}{1 + \gamma\cos\theta_c}$$

with $\gamma = 1/A$.

We can use the angular distribution to calculate the average value of $\mu = \cos\theta$.

$$\bar{\mu} = \frac{\int_{-1}^{1} d\mu\, \mu\, \sigma_s(\mu)}{\int_{-1}^{1} d\mu\, \sigma_s(\mu)} = \frac{\int_{-1}^{1} d\mu_c\, \mu\, \sigma_s'(\mu_c)}{\int_{-1}^{1} d\mu_c\, \sigma_s'(\mu_c)} = \frac{2}{3A}$$

This shows that in LCS the scattering distribution is more peaked in the forward direction. The deviation from spherical symmetry is stronger the lighter is the target nucleus. For heavy nuclei the angular distribution in LCS is essentially also spherically symmetric.

D. Effects of Anistropic Scattering and Thermal Motions

In arriving at the above result for $F(E \rightarrow E')$ we have made use of three assumptions, namely,

(i) elastic scattering

(ii) target nucleus at rest

(iii) isotropic scattering in CMCS

We will discuss briefly how each assumption can be relaxed.

Inelastic scattering can occur when the incident neutron energy is sufficiently large to excite the first nuclear level above the ground state. Inelastic scattering is a threshold reaction ($Q < 0$, endothermic), it can occur in heavy nuclei at $E \sim 0.05 - 0.1$ Mev or in medium nuclei at $\sim 0.1 - 0.2$ Mev. Typically $\sigma(n,n') \lesssim 1$ barn. Elastic scattering, on the other hand, is always present no matter what other reactions can take place. It is the primary means of slowing down neutrons, with $\sigma_s \sim 5 - 10$ barns or more ($\sigma_s(H) \sim 20$ barns).

When the incident neutron energy is comparable to the thermal energy of the target nuclei, the latter cannot be taken to be stationary any more. To take into account the thermal motions of the target one needs to know what is the physical state of the target. If the target is a crystal then the nuclei will be vibrating about their lattice positions. If the target were assumed to be a gas at some temperature T, then one finds the following behavior for $F(E \rightarrow E')$ (cf. Bell and Glasstone, Nuclear Reactor Theory, p. 336).

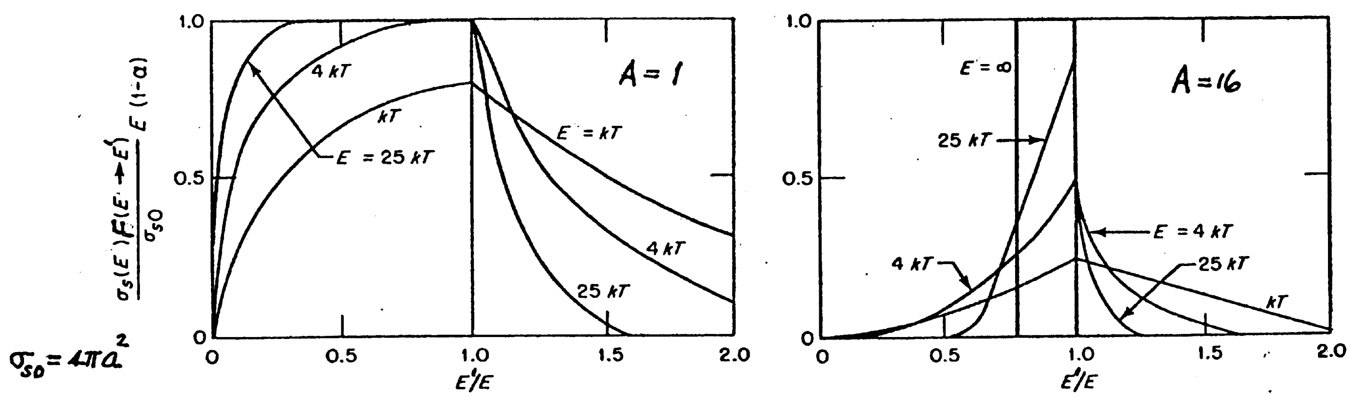
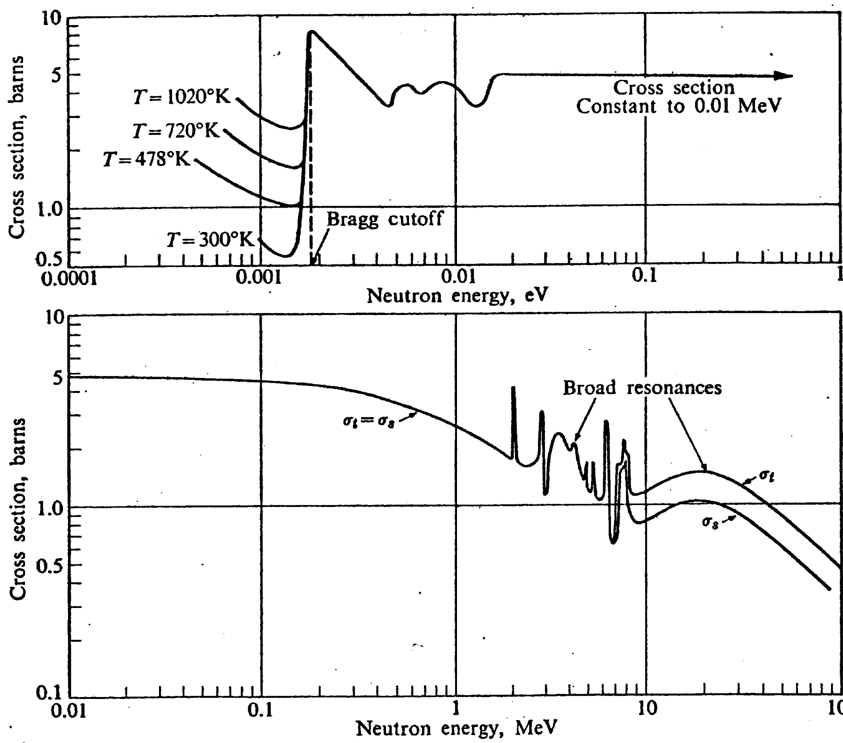


FIG. 7.5 ENERGY TRANSFER FUNCTION IN A MONATOMIC GAS WITH $A = 1$ and 16 (AFTER K. H. BECKURTS AND K. WIRTZ, REF. 14).

Notice that for $E \sim k_B T$ there is appreciable up-scattering (neutron gains energy) which is not possible with assumption (ii). As E becomes larger and larger compared to $k_B T$, up-scattering becomes less important. Thus the condition of target nucleus at rest really means that $E \gg k_B T$ (or $E_1 \gg E_2$ referring back to Sec. VII-A).

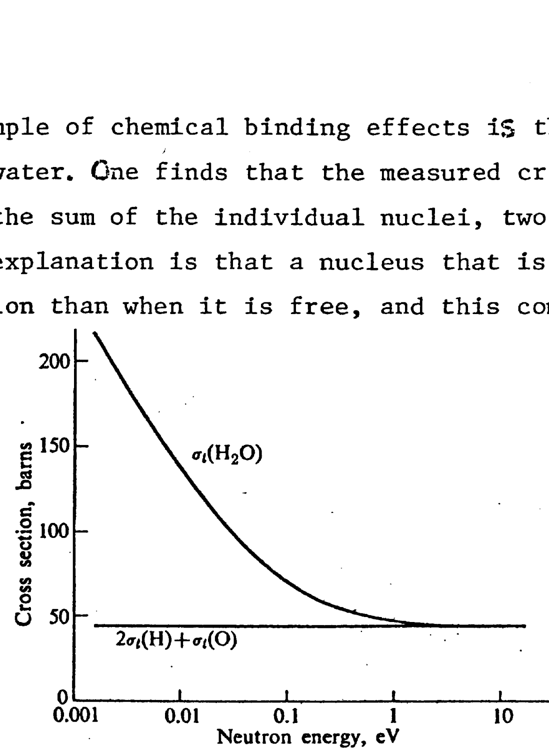
When thermal motions have to be taken into account, our previous result concerning $\sigma_s(E) = 4\pi a^2$ at low energies (i.e., s-wave scattering cross section is independent of energy) is also no longer valid. This is energy region of neutron thermalization and it covers the region from zero to about 0.1 - 0.5 ev. Thus our calculation of two-particle scattering in Sec. III holds for energies above ~ 1 ev. In other words, although $\sigma_s(E) = 4\pi a^2$ is a low-energy result in the sense $kr_o \ll 1$, r_o being the range of nuclear potential, it is at the same time a high-energy result in the sense that the neutron energy must be large compared to the kinetic energy of the target nucleus. We see an example of this in the case of graphite shown below. Notice the $\sigma_s = 4\pi a^2$ behavior from $\sim .02$ ev up to ~ 0.1 Mev. Below .02 ev one sees first Bragg diffraction (interference scattering) effects and then at still lower energies thermal motion effects (note the strong temperature dependence) characteristic of a crystalline material. Above 0.1 Mev we see first a gradual decrease of σ_s with increasing energy, a behavior we have already noted on p. III-7. Above 1 Mev one runs into resonances which can be explained in terms of compound nucleus formation. This then is completely out of the range of quantum mechanical calculations we have discussed thus far.



(from Larmash)

Fig. 2-9. The total and elastic scattering cross sections of C^{12}. [Based on BNL-325, Second Edition (1958).]

Another example of chemical binding effects is the scattering cross section of water. One finds that the measured cross section is much larger than the sum of the individual nuclei, two protons and one oxygen. The explanation is that a nucleus that is bound has a larger cross section than when it is free, and this comes about



(from Lamarsh)

Fig. 2–18. Measured total cross section of H_2O and the quantity $2\sigma_t(H) + \sigma_t(O)$.

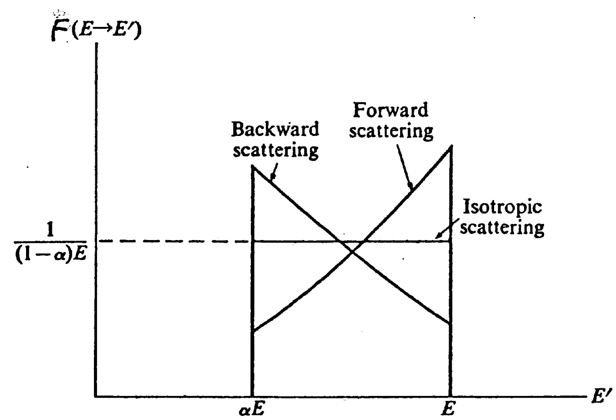
because the cross section is proportional to the square of the reduced mass,

$$\sigma_s \propto \mu^2 = \left(\frac{mM}{m+M}\right)^2 = \left(\frac{A}{A+1}\right)^2$$

Then
$$\sigma_{free} = \sigma_{bound}\left(\frac{A}{A+1}\right)^2$$

In the case of neutron-proton scattering, having the proton bound in a massive molecule (i.e., H_2O) makes the cross section about 4 times greater than when the proton is free. At low energies, like 0.01 ev, the neutron sees the proton as a bound particle, whereas at high energies, like 1 ev, the neutron sees the proton as a free particle. Hence the increase in the cross section as the energy is lowered.

We now consider relaxing assumption (iii). When the scattering is not isotropic in CMCS, the energy distribution will no longer be a constant. This will happen when we consider p-wave and higher-order orbital angular momentum contributions. In the case of p-wave scattering the distribution is proportional to $P_1(\cos\theta_c) = \cos\theta_c$, so it is more peaked in the forward scattering direction. Since the smaller scattering angles correspond to larger outgoing energies, or smaller energy losses, we therefore expect the distribution $F(E \rightarrow E')$ will be larger for E' close to E than for E' close to αE. The general shape of $F(E \rightarrow E')$ that one expects is indicated below. Notice the sharp cutoffs at $E' = E$ and αE are not affected when the scattering is anisotropic, since these occur because of assumption (ii).



(from Lamarsh)

Fig. 6–2. Elastic scattering distribution functions $F(E \rightarrow E')$ for forward scattering, backward scattering, and isotropic scattering, all in the center-of-mass system.