Fall Term 2003
Plasma Transport Theory, 22.616
Problem Set #1
Prof. Molvig

Passed Out: Sept. 11, 2003 DUE: Sept. 18, 2003

1. **Fusion Transport estimates:** Consider a tokamak fusion plasma with parameters,

$$\begin{aligned} T &\simeq 20\ KeV \\ n &= 10^{20}\ m^{-3} \\ B &= 5\ T \\ B_p &= \frac{\epsilon}{q} B \simeq 1\ T \\ a &= 3\ m \\ R &= 9\ m \end{aligned}$$

Here, B_p, is the poloidal magnetic field. Estimate the particle and energy confinement times, $\tau_p = a^2/D$, and, $\tau_E = a^2/\chi$, where diffusion and thermal diffusivity coefficients are to be computed with the *poloidal gyroradius.* Write down the formulas for D and χ first, in terms of appropriate collision frequencies. State your assumptions. Now make an estimate of the neoclassical pinch time scale, assuming the pinch velocity to be,

$$V_p = \frac{E_T}{B_p}$$

and you may take the toroidal loop voltage to be, $V_T = 0.02\ Volts$. Compare the pinch and particle diffusion time scales and state which effect is strongest.

2. **Diffusion equation solution and properties:** Show that the function,

$$P(x,t) = \frac{1}{\sqrt{4\pi D t}} \exp\left(-\frac{x^2}{4Dt}\right)$$

solves the diffusion equation,

$$\frac{\partial P}{\partial t} = D \frac{\partial^2 P}{\partial x^2}$$

with intial condition, $P(x,0) = \delta(x)$, with, δ, being the Dirac delta function. As such this P may be interpreted as a probability density for a particle launched from the origin on a random walk. Verify the other properties of P required by this interpretation,

$$\begin{aligned} \int_{-\infty}^{+\infty} dx P(x,t) &= 1 \\ \int_{-\infty}^{+\infty} dx x P(x,t) &= \langle x \rangle = 0 \\ \int_{-\infty}^{+\infty} dx x^2 P(x,t) &= \left\langle x^2 \right\rangle = 2Dt \end{aligned}$$

3. **Diffusion Equation Green's Function:** A particle launched from position, x', has the probability,

$$P(x,t) = \frac{1}{\sqrt{4\pi Dt}} \exp\left(-\frac{(x-x')^2}{4Dt}\right)$$

This function can be used as a *Green's function*, $G(x,x',t)$, for the intial value problem (on the full space, $-\infty \leq x \leq +\infty$), since it solves the diffusion equation and goes to, $G(x,x',t) \to \delta(x-x')$, as $t \to 0$.

Show that for any initial temperature distribution, $T_0(x) = T(x,0)$, the integral,

$$T(x,t) = \int dx' G(x,x',t) T_0(x')$$

satisfies the diffusion equation for temperature, with boundary condtion, $T(|x| \to \infty, t) \to 0$.

4. **Metallic Heat Conduction:** This is a 1D model problem for putting the end of one long metallic bar into a big water tank with a constant temperature T_H. Assume the bar is at zero temperature before it reaches the tank. Consider this as a 1-D heat transport problem. Please solve the heat diffusion equation for $T(x,t)$ and plot the resultant $T(x,t)$ at a series of time moments t. Assume the heat diffusivity of the bar is χ_0. $T(x,t)$ satisfies

$$\frac{\partial T}{\partial t} = \chi_0 \frac{\partial^2 T}{\partial x^2}, 0 < x < +\infty, 0 < t < +\infty$$

$$T(x,0) = 0, 0 < x < +\infty$$

$$T(0,t) = T_H, 0 < t < +\infty$$

(Hint: You may want to use the *Green's function* result of problem 3 . Try to extend the initial condition $T(x,0)$ to the whole range of spatial space, i.e., $-\infty < x < +\infty$.)

5. *Monte Carlo* solution to diffusion equation and demonstration of the *Central Limit Theorem*. This problem gives you a numerical scheme that can be generalized to solve quite complex transport problems and in the process gives you a example of how the Central Limit Theorem works. It is named after the famous gambling country along the Riveira!

For reference, with somewhat crude phrasing, the Central Limit Theorem states the following:

A random variable that is the sum of a large number of *independent* random variables obeys a Gaussian probability distribution *regardless of the probability distribution of the individual random variables* in the sum. Formally, when,

$$R = \sum_{n=0}^{N} R_n$$

then, $P(R) \to$ Gaussian, for any, $P(R_n)$, as $N \to \infty$.

We can use this property to make a random walk scheme that is efficient numerically (does not require the random steps, R_n, be picked from a Gaussian distribution) and gives the Gaussian for large numbers of particles. Here is a simple scheme: Let R_n be picked by computer generating a random number between $-1/2$ and $+1/2$, and then take a spatial step according to,

$$\delta x_n = sR_n$$

with, s, a scale factor to make the variance of this correct for the diffusion process. This means, with,

$$X \equiv \sum_{n=0}^{N} \delta x_n$$

$$\left\langle X^2 \right\rangle \equiv N \left\langle \delta x_n^2 \right\rangle = 2N$$

for a diffusion process with diffusion coefficient, $D = 1$. Therefor the scale factor, s, is computed such that,

$$\left\langle \delta x_n^2 \right\rangle = s^2 \left\langle R_n^2 \right\rangle = 2$$

Calculate the variance of $\langle R_n^2 \rangle$, of R_n and to show that,

$$s = \sqrt{24}$$

Yong Xiao has made up a nice little Matlab code (http://web.mit.edu/hardes/www/prob1Fall03/) so you can watch the diffusion and measure diffusion constant. Change the scale factor to see whether the measured diffusion constant is consistant with what you predict from theory.