Fall Term 2003
Plasma Transport Theory, 22.616
Problem Set #2
Prof. Molvig

Passed Out: Sept. 18, 2003 DUE: Sept. 25, 2003

Reading: Chapters 2 & 3 of Sigmar & Helander

1. **Equilibration:** Section 3.3 in the book considers collisions of test particles with a Maxwellian field particle distribution. The result in eq. (3.40) of the book involves collision frequencies, $\nu_s^{ab}(v)$ and, $\nu_\parallel^{ab}(v)$ and it is *not* obvious that a Maxwellian will result for the test particles in equilibrium. Consider identical field and test particles, so that, $m_a = m_b$. Show that actually,

$$\frac{\nu_s^{ab}(v)}{v\nu_\parallel^{ab}(v)} = 2\frac{v}{v_T^2}$$

   You may find equations, 3.45-3.48 helpful for this. Now you can write the velocity magnitude part of the operator as,

$$\mathcal{C}_v \equiv \frac{1}{2v^2}\frac{\partial}{\partial v}v^4\nu_\parallel(v)\left(2\frac{v}{v_T^2}f + \frac{\partial f}{\partial v}\right)$$

   This is now analagous to the 1D example we looked at in lecture, except for the magnetude of velocity, v, in a 3D velocity space. Show that for, $\mathcal{C}_v \to 0$, the distribution goes to a Maxwellian, $f \to f_M$.

2. **Fokker-Planck equation accuracy:** Considering the Fokker-Planck equation as a Taylor series expansion, we *could* continue to higher order as follows,

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial \mathbf{v}}\cdot\mathbf{A}f + \frac{\partial^2}{\partial\mathbf{v}\partial\mathbf{v}}:\mathbf{D}f + \frac{\partial^3}{\partial\mathbf{v}\partial\mathbf{v}\partial\mathbf{v}}\vdots\mathbf{T}f$$

   where, $\mathbf{T}$, is some rank 3 tensor. Make a simple scaling argument on the coefficients (assuming the small angle expansion) to show that the terms in $\mathbf{T}$ (and higher order terms) are order unity compared to the divergent, $\sim \ln\Lambda$, terms retained in the Fokker-Planck equation. Estimate from this the inherenet error in the Fokker-Planck operator. You may find some helpful arguments in the book for this problem.

3. **Collision Operator Properties:** Prove conservation of mass, momentum, and energy first for the single species collision operator, and then for a 2 species system consisting of electrons (subscript, e), and a single species of ions (subscript, i).

4. **H-Theorem:** Prove the H-theorem as follows:

   Show that the rate of change of entropy is given by,

$$\frac{dS}{dt} = -\frac{d}{dt}\int d^3v f\ln f = -\int d^3v \ln f\mathcal{C}(f,f)$$

By appropriate manipulations (integration by parts, reversing dummy variables, etc.) work this into the expression,

$$\frac{dS}{dt} = \frac{1}{2}\Gamma \int d^3v d^3v' f(\mathbf{v}) f(\mathbf{v}') \left(\frac{\partial}{\partial \mathbf{v}} \ln f - \frac{\partial}{\partial \mathbf{v}'} \ln f' \right) \cdot \mathbf{U} \cdot \left(\frac{\partial}{\partial \mathbf{v}} \ln f - \frac{\partial}{\partial \mathbf{v}'} \ln f' \right)$$

where, $f' = f(\mathbf{v}')$.

Show that, $\mathbf{c} \cdot \mathbf{U} \cdot \mathbf{c} = |\mathbf{u} \times \mathbf{c}|^2 / u^3 > 0$, for any vector, $\mathbf{c}$. It now follows that,

$$\frac{dS}{dt} \geq 0$$

Why?

$dS/dt = 0$ if and only if, $\mathbf{u} \times \mathbf{c} = 0$, and this must hold for all, $\mathbf{v}$ and $\mathbf{v}'$. Show then that this implies,

$$(\mathbf{v} - \mathbf{v}') \times \left(\frac{\partial}{\partial \mathbf{v}} \ln f - \frac{\partial}{\partial \mathbf{v}'} \ln f' \right) = 0$$

and that this implies that f must be Maxwellian, $f =$const.exp$\left(- (\mathbf{v} - \mathbf{V})^2 / v_T^2 \right)$. Here, $\mathbf{V}$, is some constant, fluid, velocity.

5. **Positivity:** Show that, $f > 0$, at $t = 0$, implies, $f > 0$, for all times.