

22.616 Plasma Transport Theory

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Problem # 2 Solutions

1. Equilibrium: Find in text book:

$$\nu_s^{ab}(v) = \widehat{\nu}_{ab} \frac{2T_a}{T_b} \left(1 + \frac{m_b}{m_a}\right) \frac{G(X_b)}{X_a}$$

$$\nu_{||}^{ab}(v) = 2 \widehat{\nu}_{ab} \frac{G(X_b)}{X_a^3}$$

where $X_a \equiv \frac{v}{v_{Ta}}$, $X_b \equiv \frac{v}{v_{Tb}}$, $v_{Ta} \equiv \sqrt{\frac{2T_a}{m_a}}$

For like particles, $a=b$, let $X = \frac{v}{v_T}$, $v_T \equiv v_a v_{Ta} = v_{Tb}$

Then $T_a = T_b \equiv T$, we can easily find

$$\frac{\nu_s^{ab}(v)}{\nu_{||}^{ab}(v)} = \left(1 + \frac{m_{ab}}{m_{ba}}\right) X_a^2 = 2 X^2 = 2 \frac{v^2}{v_T^2}$$

i.e.,
$$\frac{\nu_s^{ab}(v)}{v \nu_{||}^{ab}(v)} = 2 \frac{v}{v_T^2}$$

From Eq. (3.40), we find the velocity magnitude part of the collisional operator:

$$\begin{aligned} \mathcal{C} &= \frac{1}{v^2} \frac{\partial}{\partial v} \left(v^3 \frac{m_{ab}}{m_a + m_b} \nu_s^{ab} f_a + \frac{1}{2} \nu_{||}^{ab} v \frac{\partial f_a}{\partial v} \right) \\ &= \frac{1}{2} \frac{1}{v^2} \frac{\partial}{\partial v} \left(v^4 \nu_{||}^{ab} \left(\frac{\nu_s^{ab}}{v \nu_{||}^{ab}} f_a + \frac{\partial f_a}{\partial v} \right) \right) \\ &= \frac{1}{2v^2} \frac{\partial}{\partial v} \left[v^4 \nu_{||}^{ab} \left(2 \frac{v}{v_T^2} f + \frac{\partial f}{\partial v} \right) \right] \quad (f \equiv f_a) \end{aligned}$$

Therefore $\mathcal{D}_{11}(v) = \mathcal{D}_{11}^{ab}(v)$

$$= 2 \hat{\mathcal{D}}_{aa} \frac{G(x_a)}{\lambda_a^3} = 2 \hat{\mathcal{D}}_{aa} v_T^2 \frac{G(x)}{v^3}$$

When $G \rightarrow 0$

$$v^4 \mathcal{D}_{11}(v) \left(2 \frac{v}{v_T^2} f + \frac{\partial f}{\partial v} \right) = \text{const}$$

Notice $v^4 \mathcal{D}_{11}(v) \propto G(x) v$ & $G(0) = 0$

So when $v \rightarrow 0$, $v^4 \mathcal{D}_{11}(v) = 0$, then $\text{const} = 0$, i.e.

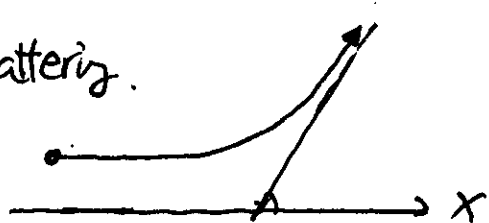
$$2 \frac{v}{v_T^2} f + \frac{\partial f}{\partial v} = 0$$

Solve it we get: $f = \frac{1}{\sqrt{\pi} v_T^2} e^{-\frac{v^2}{v_T^2}} = f_M$

2. Fokker-Planck equation accuracy

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial v} \cdot A f + \frac{\partial^2}{\partial v^2} \cdot B f \equiv \underline{D} f + \frac{\partial^2}{\partial v^2} \cdot \underline{I} f$$

Use the notation in the textbook; In an orthogonal coordinate system (x, y, z) , with x in the direct of the incident velocity of test particle, for small angle scattering.



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$$\Delta V_x = - \left(1 + \frac{m_a}{m_b}\right) \left(\frac{e_a e_b}{2\pi\epsilon_0 m_a}\right)^2 \frac{1}{2r^2 u^3}$$

$$\Delta V_y = \frac{e_a e_b}{2\pi\epsilon_0 m_a} \frac{\cos\phi}{ur}$$

$$\Delta V_z = \frac{e_a e_b}{2\pi\epsilon_0 m_a} \frac{\sin\phi}{ur}$$

with $u = |\mathbf{v} - \mathbf{v}'|$

First consider tensor \mathbb{I} .

$$\mathbb{I} T_{ijk} = \left\langle \frac{\Delta V_i \Delta V_j \Delta V_k}{\Delta t} \right\rangle \frac{1}{6}, \quad i, j, k = x, y, z$$

$$= \frac{1}{6} \int d^3v' dr d\phi f_b(v') u \Delta V_i \Delta V_j \Delta V_k$$

Since $\Delta V_x \propto \frac{1}{r^2}$, $\Delta V_y, \Delta V_z \propto \frac{1}{r}$. So ~~the~~ $\Delta V_i \Delta V_j \Delta V_k$ at ~~most~~ ^{least}

gives $\frac{1}{r^3}$, then the integral $\int_{r_{\min}}^{r_0} \frac{1}{r^3} r dr = \frac{1}{r_{\min}}$

Also, we notice $\int_0^{2\pi} d\phi \sin\phi = 0$, $\int_0^{2\pi} d\phi \cos\phi = 0$, $\int_0^{2\pi} d\phi \sin^3\phi = 0$, $\int_0^{2\pi} d\phi \cos^3\phi = 0$

$$= \int_0^{2\pi} d\phi \sin^2\phi \cos\phi = \int_0^{2\pi} d\phi \cos^2\phi \sin\phi = 0. \quad \text{Then only } \Delta V_x \Delta V_y^2$$

$\Delta V_x \Delta V_z^2, \Delta V_x^3$ give ~~non-vanishing~~ non-vanishing terms. These terms

at least give $\frac{1}{r^4}$, so there is no divergent term in Λ in

the tensor of T_{ijk} , For the same reasoning, this is true for even higher order terms than \underline{T} .

From this point of view, the Fokker-Planck equation has an inherent error of $\frac{1}{\ln \Lambda} \sim 3\%$ for Fusion Plasma.

3. Collision Operator Properties:

$$C_{ab}(f_a, f_b) = \frac{1}{2} \Gamma_{ab} \frac{\partial}{\partial \underline{v}} \cdot \int d^3v' \underline{U}(\underline{v} - \underline{v}') \cdot \left(\frac{\partial}{\partial \underline{v}} - \frac{m_a}{m_b} \frac{\partial}{\partial \underline{v}'} \right) f_a(\underline{v}) f_b(\underline{v}')$$

$$\text{with } \Gamma_{ab} \equiv \frac{4\pi Z_a^2 Z_b^2 e^4 \ln \Lambda}{m_a^2}$$

$$\text{Define } \Gamma \equiv \frac{4\pi Z_a^2 Z_b^2 e^4 \ln \Lambda}{m_a m_b}, \text{ we first prove the two-species}$$

case, then use the two-species result, write down the single-species result directly.

Two species - case

I⁰. particle conservation

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$$\int C_{ei} (f_e, f_i) d^3v$$

$$= \frac{1}{2} \Gamma e i \int d^3v \frac{\partial}{\partial v} \cdot \int d^3v' U(v-v') \cdot \left(\frac{\partial}{\partial v} - \frac{m_e}{m_i} \frac{\partial}{\partial v'} \right) f_e(v) f_i(v')$$

Gauss' law

$$= \frac{1}{2} \Gamma e i \oint_{\infty} d\vec{s} \cdot \int d^3v' U(v-v') \cdot \left(\frac{\partial}{\partial v} - \frac{m_e}{m_i} \frac{\partial}{\partial v'} \right) f_e(v) f_i(v')$$

Because $f_e(v), f_i(v'), \frac{\partial}{\partial v} f_e, \frac{\partial}{\partial v'} f_i$ vanishes at infinity

$$\underline{\int C_{ei} (f_e, f_i) d^3v = 0}$$

For the same reason $\int C_{ie} (f_i, f_e) d^3v = 0$

$$\int C_{ee} (f_e, f_e) d^3v = 0, \quad \int C_{ii} (f_i, f_i) d^3v = 0$$

2° Momentum Conservation.

 ~~$\int C_{ei}$~~ The momentum increase[†] of electron due to colliding

with ions

$$\int C_{ei} (f_e, f_i) m_e v d^3v$$

$$= \frac{1}{2} \Gamma e i \int d^3v m_e v \frac{\partial}{\partial v} \cdot \int d^3v' U(v-v') \cdot \left(\frac{\partial}{\partial v} - \frac{m_e}{m_i} \frac{\partial}{\partial v'} \right) f_e(v) f_i(v')$$

Integrate by parts

$$= -\frac{m_e}{2} \Gamma e i \int d^3v d^3v' U(v-v') \cdot \left(\frac{\partial}{\partial v} - \frac{m_e}{m_i} \frac{\partial}{\partial v'} \right) f_e(v) f_i(v')$$

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$$= \frac{\Gamma}{2} \int d^3v d^3v' \underline{U}(\underline{v}-\underline{v}') \cdot (m_e \frac{\partial}{\partial \underline{v}'} - m_i \frac{\partial}{\partial \underline{v}}) f_e(\underline{v}) f_i(\underline{v}')$$

Similarly

$$\int C_{ie} (f_i, f_e) m_e \underline{v} d^3v$$

$$= \frac{\Gamma}{2} \int d^3v d^3v' \underline{U}(\underline{v}-\underline{v}') \cdot (m_i \frac{\partial}{\partial \underline{v}'} - m_e \frac{\partial}{\partial \underline{v}}) f_i(\underline{v}) f_e(\underline{v}')$$

Change of variables, $\underline{v} \Leftrightarrow \underline{v}'$, Note $\underline{U}(\underline{v}-\underline{v}') = \underline{U}(\underline{v}'-\underline{v})$

$$\int C_{ie} (f_i, f_e) m_e \underline{v} d^3v$$

$$= - \frac{\Gamma}{2} \int d^3v d^3v' \underline{U}(\underline{v}-\underline{v}') \cdot (m_e \frac{\partial}{\partial \underline{v}'} - m_i \frac{\partial}{\partial \underline{v}}) f_e(\underline{v}) f_i(\underline{v}')$$

Therefore

$$\int d^3v (C_{ei} (f_e, f_i) m_e \underline{v} + C_{ie} (f_i, f_e) m_i \underline{v}) = 0$$

i.e. ~~The net transfer of momentum between electrons & ions are~~

~~Zero, momentum is conserved.~~

i.e. All the momentum electron (lost (gain)) is equal to the momentum ions gain (lost), Momentum is conserved.

3. Energy Conservation

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$$\int C_{ei} (f_e, f_i) \frac{m_e v^2}{2} d^3v$$

$$= \frac{1}{4} \Gamma \rho_{ei} \int d^3v m_e v^2 \frac{\partial}{\partial v} \cdot \int d^3v' \underline{U}(\underline{v}-\underline{v}') \cdot \left(\frac{\partial}{\partial v} - \frac{m_e}{m_i} \frac{\partial}{\partial v'} \right) f_e(\underline{v}) f_i(\underline{v}')$$

$$= \frac{1}{4} \Gamma \int d^3v v^2 \frac{\partial}{\partial v} \cdot \int d^3v' \underline{U}(\underline{v}-\underline{v}') \cdot \left(m_i \frac{\partial}{\partial v} - m_e \frac{\partial}{\partial v'} \right) f_e(\underline{v}) f_i(\underline{v}')$$

integrate by parts

$$= \frac{1}{2} \Gamma \int d^3v d^3v' \underline{v} \cdot \underline{U}(\underline{v}-\underline{v}') \cdot \left(m_e \frac{\partial}{\partial v'} - m_i \frac{\partial}{\partial v} \right) f_e(\underline{v}) f_i(\underline{v}')$$

Similarly

$$\int C_{ie} (f_i, f_e) \frac{m_i v^2}{2} d^3v$$

$$= \frac{1}{2} \Gamma \int d^3v d^3v' \underline{v} \cdot \underline{U}(\underline{v}-\underline{v}') \cdot \left(m_i \frac{\partial}{\partial v'} - m_e \frac{\partial}{\partial v} \right) f_i(\underline{v}) f_e(\underline{v}')$$

change of variables $\underline{v} \leftrightarrow \underline{v}'$

$$= \frac{1}{2} \Gamma \int d^3v d^3v' \underline{v}' \cdot \underline{U}(\underline{v}-\underline{v}') \cdot \left(m_i \frac{\partial}{\partial v} - m_e \frac{\partial}{\partial v'} \right) f_e(\underline{v}) f_i(\underline{v}')$$

Then $\int \left(C_{ie} (f_i, f_e) \frac{m_i v^2}{2} + C_{ei} (f_e, f_i) \frac{m_e v^2}{2} \right) d^3v$

$$= \frac{\Gamma}{2} \int d^3v d^3v' (\underline{v}-\underline{v}') \cdot \underline{U}(\underline{v}-\underline{v}') \cdot \left(m_e \frac{\partial}{\partial v'} - m_i \frac{\partial}{\partial v} \right) f_e(\underline{v}) f_i(\underline{v}')$$

Because $(\underline{v}-\underline{v}') \cdot \underline{v} (\underline{v}-\underline{v}') = 0$, So

$$\int d^3v \left(C_{ei}(f_e, f_i) \frac{m_e v^2}{2} + C_{ie}(f_i, f_e) \frac{m_i v^2}{2} \right) = 0$$

This means, ~~the~~ energy is conserved.

Single species case:

1° Particle conservation: proved in the two-species case.

2° Momentum Conservation: use the result from two-species case

$$\int C_{aa}(f_a, f_a) m_a \underline{v} d^3v$$

$$= -m_a \frac{\Gamma}{2} \int \underline{v} (\underline{v}-\underline{v}') \cdot \left(\frac{\partial}{\partial \underline{v}} - \frac{\partial}{\partial \underline{v}'} \right) f_a(\underline{v}) f_a(\underline{v}') d^3v' d^3v$$

$$\underline{v} \Leftrightarrow \underline{v}'$$

$$= -m_a \frac{\Gamma}{2} \int \underline{v} (\underline{v}-\underline{v}') \cdot \left(\frac{\partial}{\partial \underline{v}'} - \frac{\partial}{\partial \underline{v}} \right) f_a(\underline{v}) f_a(\underline{v}') d^3v' d^3v$$

$$= - \int C_{aa}(f_a, f_a) m_a \underline{v} d^3v$$

$$\text{So } \int C_{aa}(f_a, f_a) m_a \underline{v} d^3v = 0$$

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3°. Energy Conservation: use the intermediate result from two-species case

$$\int C_{aa}(f_a, f_a) \frac{1}{2} m_a v^2 d^3v$$

$$= \frac{m_a}{2} \Gamma \int d^3v d^3v' \underline{v} \cdot \underline{U}(\underline{v} - \underline{v}') \cdot \left(\frac{\partial}{\partial \underline{v}'} - \frac{\partial}{\partial \underline{v}} \right) f_a(\underline{v}) f_a(\underline{v}') \quad (1)$$

$$\underline{v} \leftrightarrow \underline{v}'$$

$$= \frac{m_a}{2} \Gamma \int d^3v d^3v' \underline{v}' \cdot \underline{U}(\underline{v} - \underline{v}') \cdot \left(\frac{\partial}{\partial \underline{v}} - \frac{\partial}{\partial \underline{v}'} \right) f_a(\underline{v}) f_a(\underline{v}') \quad (2)$$

$$\frac{(1)+(2)}{2}$$

$$= \frac{m_a}{2} \Gamma \int d^3v d^3v' (\underline{v} - \underline{v}') \cdot \underline{U}(\underline{v} - \underline{v}') \cdot \left(\frac{\partial}{\partial \underline{v}'} - \frac{\partial}{\partial \underline{v}} \right) f_a(\underline{v}) f_a(\underline{v}')$$

$$= 0$$

4. H-Theorem: (single species)

$$S \equiv - \int f \ln f$$

$$\text{Then } \frac{dS}{dt} = - \frac{d}{dt} \int d^3v f \ln f$$

$$= - \int d^3v \left(\frac{\partial f}{\partial t} \ln f + \frac{\partial f}{\partial t} \right)$$

$$= - \int d^3v C(f, f) \ln f + - \int d^3v C(f, f)$$

$$= - \int d^3v \ln f C(f, f)$$

|| particle conserving
0

Plug into the Landau Collisional Operator

$$\frac{ds}{dt} = -\frac{1}{2} \Gamma \int d^3v \ln f \frac{\partial}{\partial v} \cdot \int d^3v' \underline{U}(\underline{v}-\underline{v}') \cdot \left(\frac{\partial}{\partial v} - \frac{\partial}{\partial v'} \right) f f'$$

Integrate by parts

$$= -\frac{1}{2} \Gamma \int d^3v \ln f \frac{\partial}{\partial v} \cdot \int d^3v' \underline{U}$$

$$= \frac{1}{2} \Gamma \int d^3v d^3v' \left(\frac{\partial}{\partial v} \ln f \right) \cdot \underline{U}(\underline{v}-\underline{v}') \cdot \left(\frac{\partial}{\partial v} - \frac{\partial}{\partial v'} \right) f f' \quad (1)$$

$$\underline{v} \Leftrightarrow \underline{v}'$$

$$= \frac{1}{2} \Gamma \int d^3v d^3v' \left(\frac{\partial}{\partial v'} \ln f' \right) \cdot \underline{U}(\underline{v}-\underline{v}') \cdot \left(\frac{\partial}{\partial v'} - \frac{\partial}{\partial v} \right) f f' \quad (2)$$

$\frac{(1)+(2)}{2}$

$$= \frac{1}{4} \Gamma \int d^3v d^3v' \left(\frac{\partial}{\partial v} \ln f - \frac{\partial}{\partial v'} \ln f' \right) \cdot \underline{U}(\underline{v}-\underline{v}') \cdot \left(\frac{\partial}{\partial v} - \frac{\partial}{\partial v'} \right) f f'$$

$$= \frac{1}{4} \Gamma \int d^3v d^3v' f f' \left(\frac{\partial}{\partial v} \ln f - \frac{\partial}{\partial v'} \ln f' \right) \cdot \underline{U}(\underline{v}-\underline{v}') \cdot \left(\frac{\partial}{\partial v} \ln f - \frac{\partial}{\partial v'} \ln f' \right)$$

where

$$\underline{U}(\underline{u}) = \frac{u^2 \underline{I} - \underline{u} \underline{u}}{u^3}$$

$$\text{So } \underline{c} \cdot \underline{U} \underline{c} = \frac{u^2 c^2 - (\underline{c} \cdot \underline{u})^2}{u^3}$$

$$\text{But } \frac{|\underline{u} \times \underline{c}|^2}{u^3} = \frac{1}{u^3} (\underline{u} \times \underline{c}) \cdot (\underline{u} \times \underline{c}) = \frac{1}{u^3} (\underline{c} \times (\underline{u} \times \underline{c})) \cdot \underline{u}$$

$$= \frac{1}{u^3} (\underline{u} c^2 - \underline{c} \underline{c} \cdot \underline{u}) \cdot \underline{u} = \frac{1}{u^3} (u^2 c^2 - (\underline{c} \cdot \underline{u})^2) = \underline{c} \cdot \underline{U} \underline{c}$$

So we have $\underline{c} \cdot \underline{u} \cdot \underline{c} = \frac{|\underline{u} \times \underline{c}|^2}{u^3} \geq 0$

Here $\underline{c} = \frac{\partial}{\partial \underline{v}} \ln f - \frac{\partial}{\partial \underline{v}'} \ln f'$

Because of its positivity (we'll prove later in Problem 5), $f \geq 0$
 $f' \geq 0$.

$$\frac{ds}{dt} = \frac{1}{2} \int d^3v d^3v' f(\underline{v}) f(\underline{v}') \frac{|\underline{u} \times \underline{c}|^2}{u^3} \geq 0.$$

When $\frac{ds}{dt} = 0$, then entropy reaches its maximum.

we require $\underline{u} \times \underline{c} = 0$, i.e. $\underline{u} \parallel \underline{c}$. So

$$\frac{\partial}{\partial \underline{v}} \ln f - \frac{\partial}{\partial \underline{v}'} \ln f = C_0 (\underline{v} - \underline{v}')$$

$\Rightarrow \frac{\partial}{\partial \underline{v}} \ln f = C_0 (\underline{v} - \underline{V})$, where C_0 & \underline{V} are arbitrary constants.

Integrate above, we obtain

$$f = G e^{-\frac{C_0}{2} (\underline{v} - \underline{V})^2} \quad G \text{ another constant}$$

Let $\frac{C_0}{2} \equiv \frac{1}{v_T^2}$, $G = \frac{1}{\sqrt{2\pi} v_T^3}$, then

$$f = \frac{1}{\sqrt{2\pi} v_T^3} e^{-\frac{(\underline{v} - \underline{V})^2}{v_T^2}} \quad (\text{drift Maxwellian})$$

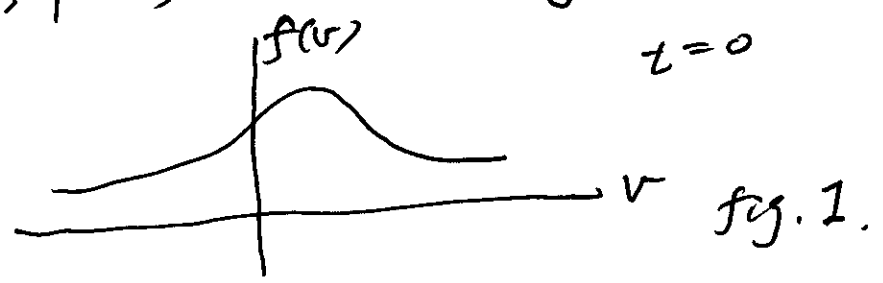
5. Positivity of f .

Consider a 1-D case. f satisfy the Fokker-Planck Equation

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial v} (A f) + \frac{1}{2} \frac{\partial^2}{\partial v^2} (D f)$$

$$= -\frac{\partial A}{\partial v} f - A \frac{\partial f}{\partial v} + \frac{1}{2} \frac{\partial^2 D}{\partial v^2} f + \frac{\partial D}{\partial v} \frac{\partial f}{\partial v} + \frac{1}{2} D \frac{\partial^2 f}{\partial v^2}$$

At $t=0$, $f > 0$, As shown in figure 1



Then f change continuously according to the Fokker-Planck Equation

Assume some time t_0 later, f reaches to zero for the first time at $v=v_0$, i.e. $f(v_0, t_0) = 0$, see fig 2.

Because of the continuity of f , $\frac{\partial f}{\partial v}(v_0, t_0)$ must be a minimum at t_0 . i.e.

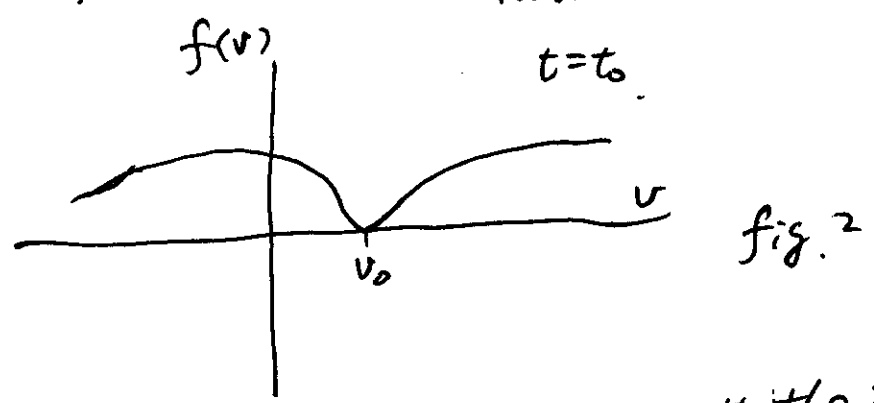
$$\frac{\partial f}{\partial v} \Big|_{v_0, t_0} = 0, \quad \frac{\partial^2 f}{\partial v^2} \Big|_{v_0, t_0} > 0$$

Therefore, According to Fokker-Planck Equation

$$\frac{\partial f}{\partial t} \Big|_{t_0, v_0} = \frac{1}{2} D \frac{\partial^2 f}{\partial v^2} \Big|_{t_0, v_0} > 0.$$

So At a little later $t_1 = t_0 + \Delta t$

$$f(t_0 + \Delta t, v_0) = \frac{1}{2} D \frac{\partial^2 f}{\partial v^2} \Big|_{v_0, t_0} \Delta t > 0, \Delta t \rightarrow 0.$$



i.e. The diffusion factor try to smooth the distribution curve.

f try to keep above zero.

So ~~we~~ if $f > 0$ at $t = 0$, f evolves according to Fokker-Planck

equation. ^{then} $f > 0$ for all times $t > 0$.