22.616. Plasma Transport theory

Problem #3 Solutions

1. Momentum Equation Structure

Vlasov Equation

$$\frac{\partial f}{\partial t} + \underline{v}\cdot\nabla f + \frac{q}{m}\left(\underline{E} + \frac{1}{c}\underline{v}\times\underline{B}\right)\cdot\frac{\partial}{\partial \underline{v}} f = 0 \qquad \text{---(1)}$$

Assume

$$f \simeq f^M = \frac{n}{(2\pi T/m)^{3/2}} e^{-\frac{m(\underline{v}-\underline{V})^2}{2T}} \qquad \text{---(2)}$$

Let $\boxed{\underline{v}' = \underline{v} - \underline{V},}$ then $\underline{v} = \underline{v}' + \underline{V}$ --(3)

$$f^M = \frac{n}{(2\pi T/m)^{3/2}} \exp\left(-\frac{m v'^2}{2T}\right) \qquad \text{---(4)}$$

1° Take the density moment of Eq(1).

$$\frac{\partial}{\partial t}\int d^3v f + \nabla\cdot\int d^3v\, \underline{v} f + \int d^3v \frac{q}{m}\left(\underline{E} + \frac{1}{c}\underline{v}\times\underline{B}\right)\cdot\frac{\partial}{\partial \underline{v}} f = 0$$

Where

$$\int d^3v f = \int d^3v (f - f_M) + \int d^3v f_M$$

$$= \int d^3v (f - f_M) + n \simeq n \qquad \text{---(5)}$$

$$\int d^3v\, \underline{v} f = \int d^3v' (\underline{v}' + \underline{V}) f$$

$$= \int d^3v' \underline{v}' f + \int d^3v' f\, \underline{V} \simeq n\underline{V} \qquad \text{--- (6)}$$

Assume $\int d^3v' \underline{v}' f = \int d^3v' \underline{v}' (f - f^M) = 0 \qquad \text{--- (7)}$

~~Take the~~

$$\frac{q}{m} \int (\underline{E} + \frac{1}{c} \underline{v} \times \underline{B}) \cdot \frac{\partial}{\partial \underline{v}} f \, d^3v$$

Integrate by parts $= -\frac{q}{m} \int f \nabla_{\underline{v}} \cdot (\underline{E} + \frac{1}{c} \underline{v} \times \underline{B})\, d^3v$

$\underline{E} = \underline{E}(\underline{x}, t)$ so $\nabla_{\underline{v}} \cdot \underline{E} = 0$

$$\nabla_{\underline{v}} \cdot (\underline{v} \times \underline{B}) = \frac{\partial}{\partial v_i} (\varepsilon_{ijk} V_j B_k) = \varepsilon_{ijk} B_k \delta_{ij} = \varepsilon_{iik} B_k = 0$$

Therefore $\frac{q}{m} \int (\underline{E} + \frac{1}{c} \underline{v} \times \underline{B}) \cdot \frac{\partial}{\partial \underline{v}} f \, d^3v = 0 \qquad \text{--- (8)}$

Then we have

$$\boxed{\frac{\partial n}{\partial t} + \nabla \cdot (n \underline{V}) = 0} \qquad \text{--- (9)}$$

2° Take the momentum moment of Eq (1)

$$\underbrace{\frac{\partial}{\partial t}\int d^3v\, m\underline{v} f}_{①} + \nabla\cdot\underbrace{\int d^3v\, m\underline{v}\,\underline{v} f}_{②} + \underbrace{\int d^3v\, \underline{v}\, q\left(\underline{E} + \frac{1}{c}\underline{v}\times\underline{B}\right)\cdot\frac{\partial}{\partial \underline{v}} f}_{③} = 0$$

$$① = \int d^3v\, m\underline{v} f = nm\underline{V} \quad (\text{proved at eq (6)})$$

$$② = m\int d^3v' (\underline{v}'+\underline{V})(\underline{v}'+\underline{V}) f$$

$$= m\int d^3v' (\underline{v}'\underline{v}' + \underline{v}'\underline{V} + \underline{V}\underline{v}' + \underline{V}\,\underline{V}) f$$

$$= m\int d^3v'\, \underline{v}'\underline{v}' f + mn\underline{V}\,\underline{V}$$

$$= m\int d^3v'\, \underline{v}'\underline{v}'(f - f^M) + m\int d^3v'\, \underline{v}'\underline{v}' f^M + nm\underline{V}\,\underline{V}$$

$$= \underline{\underline{\Pi}} + nT\underline{\underline{I}} + nm\underline{V}\,\underline{V} \qquad \text{---- (11)}$$

$$③ = -\int d^3v\, q\left(\underline{E} + \frac{1}{c}\underline{v}\times\underline{B}\right)\cdot\underline{\underline{I}} f$$

$$= -qn\underline{E} - \frac{q}{c}\left(\int d^3v\, \underline{v} f\right)\times\underline{B}$$

$$= -qn\underline{E} - \frac{q}{c} n\underline{V}\times\underline{B} \qquad \text{---- (12)}$$

Therefore we have

$$\frac{\partial}{\partial t}(nm\underline{V}) + \nabla\cdot\left(\underline{\underline{\Pi}} + nT\underline{\underline{I}} + nm\underline{V}\,\underline{V}\right) - qn\underline{E} - qn\frac{1}{c}\underline{V}\times\underline{B} = 0$$

i.e.

$$nm\frac{\partial \underline{V}}{\partial t} + m\underline{V}\frac{\partial n}{\partial t} + \nabla\cdot\underline{\underline{\Pi}} + \nabla p + nm\underline{V}\cdot\nabla\underline{V} + m\underline{V}\,\nabla\cdot(n\underline{V})$$

$$= qn\left(\underline{E} + \frac{1}{c}\underline{V}\times\underline{B}\right)$$

Plug in the density moment equation (9). we have

$$\boxed{\begin{aligned} & nm\left(\frac{\partial \underline{V}}{\partial t} + \underline{V}\cdot\nabla\underline{V}\right) = -\nabla p + qn\left(\underline{E} + \frac{1}{c}\underline{V}\times\underline{B}\right) - \nabla\cdot\underline{\underline{\Pi}} \qquad \text{---(13)} \\ & \text{with} \\ & \qquad \underline{\underline{\Pi}} \equiv m\int d^3v\,\underline{v}'\underline{v}'\,(f - f^M) \quad (\text{Stress tensor}) \qquad \text{---(14)} \end{aligned}}$$

3°. Take the Energy moment of Eq(1), we get

$$\frac{\partial}{\partial t}\overbrace{\int d^3v\,\frac{1}{2}mv^2 f}^{1} + \nabla\cdot\Big(\overbrace{\int d^3v\,\frac{1}{2}mv^2\,\underline{v}\,f}^{2}\Big) + \overbrace{\Big(\int d^3v\,\frac{q}{m}\left(\underline{E} + \frac{1}{c}\underline{v}\times\underline{B}\right)\cdot\frac{\partial f}{\partial \underline{v}} \times \frac{1}{2}mv^2\Big)}^{3} = 0$$

(5)

$$① = \frac{1}{2}m\int d^3v'(\underline{v}'+\underline{V})\cdot(\underline{v}'+\underline{V})f$$

$$= \frac{1}{2}m\int d^3v'(v'^2+V^2+2\underline{v}'\cdot\underline{V})f$$

$$= \frac{1}{2}m\int d^3v'(v'^2+V^2)f$$

$$\simeq \frac{1}{2}m\int d^3v'\,v'^2 f^M + \frac{nm}{2}V^2 + \frac{1}{2}m\int d^3v'\,v'^2(f-f^M)$$

$$= \frac{3}{2}nT + \frac{1}{2}nmV^2 \qquad \text{--- (15)}$$

(Assume $\frac{1}{2}m\int d^3v'v'^2(f-f^M) = 0$)

②

Define

$$\underline{Q} \equiv \int d^3v\,\frac{1}{2}mv^2\underline{v}f$$

$$= \frac{m}{2}\int d^3v'(\underline{v}'+\underline{V})^2(\underline{v}'+\underline{V})f$$

$$= \frac{m}{2}\int d^3v'(v'^2\underline{v}'+v'^2\underline{V}+V^2\underline{v}'+V^2\underline{V}+2\underline{v}'\cdot\underline{V}\underline{v}'+2\underline{v}'\cdot\underline{V}\underline{V})f$$

$$= \frac{m}{2}\int d^3v'(v'^2\underline{v}'+v'^2\underline{V}+V^2\underline{V}+2\underline{v}'\cdot\underline{V}\underline{v}')f \qquad \text{--- (16)}$$

let $\underline{q} \equiv \int d^3v'\frac{m}{2}v'^2\underline{v}'f$

As we defined in Eq (14)

$$\underline{\underline{\Pi}} = \int d^3v\, m\underline{v}'\underline{v}' (f - f^M)$$

$$= m\int d^3v' \underline{v}'\underline{v}' f - nT\underline{\underline{I}} \qquad \text{---(17)}$$

Therefore

$$\underline{Q} = \underline{q} + \frac{m}{2}\int d^3v' v'^2 \underline{V} f + \frac{nm}{2} V^2 \underline{V} + (\underline{\underline{\Pi}} + nT\underline{\underline{I}})\cdot\underline{V}$$

$$= \underline{q} + \frac{m\underline{V}}{2}\int d^3v' v'^2 (f - f^M) + \frac{3}{2}nT\underline{V} + \frac{nmV^2}{2}\underline{V}$$

$$+ \underline{\underline{\Pi}}\cdot\underline{V} + nT\underline{V}$$

$$\Rightarrow \boxed{\underline{Q} = \underline{q} + \frac{5}{2}nT\underline{V} + \frac{nmV^2}{2}\underline{V} + \underline{\underline{\Pi}}\cdot\underline{V}} \qquad \text{---(18)}$$

$$③ = \frac{q}{m}\int d^3v\, \frac{1}{2}mv^2 \frac{\partial}{\partial \underline{v}}\cdot\left[(\underline{E} + \frac{1}{c}\underline{v}\times\underline{B}) f\right]$$

Integrate by parts $\Rightarrow$

$$③ = -\int d^3v\, q\underline{v}\cdot(\underline{E} + \frac{1}{c}\underline{v}\times\underline{B}) f$$

$$= -\int d^3v\, \frac{q}{m}\underline{v}\cdot \underline{E}\, f$$

$$= -\, q n \underline{V}\cdot \underline{E} \qquad \text{---(19)}$$

Therefore the energy moment equation turns out to be

$$\underline{\frac{\partial}{\partial t}\left(\frac{3}{2}nT + \frac{1}{2}mnV^2\right) + \nabla\cdot\underline{Q} = qn\underline{V}\cdot\underline{E}} \qquad \text{---(20)}$$

where

$$\nabla\cdot\underline{Q} = \nabla\cdot\left(q + \frac{5}{2}nT\underline{V} + \frac{nmV^2}{2}\underline{V} + \underline{\underline{\Pi}}\cdot\underline{V}\right)$$

$$= \nabla\cdot\left(q + \frac{5}{2}nT\underline{V}\right) + \underline{V}\cdot\nabla\left(\frac{1}{2}nmV^2\right) + \frac{1}{2}nmV^2\nabla\cdot\underline{V}$$

$$+ \underline{V}\cdot(\nabla\cdot\underline{\underline{\Pi}}) + \underline{\underline{\Pi}} : \nabla\underline{V}$$

$$= \nabla\cdot\left(q + \frac{5}{2}nT\underline{V}\right) + \frac{1}{2}nm\underline{V}\cdot\nabla V^2 + \frac{1}{2}mV^2\left(\underline{V}\cdot\nabla n + n\nabla\cdot\underline{V}\right)$$

$$+ \underline{V}\cdot(\nabla\cdot\underline{\underline{\Pi}}) + \underline{\underline{\Pi}} : \nabla\underline{V}$$

$$= \nabla\cdot\left(q + \frac{5}{2}nT\underline{V}\right) + \frac{1}{2}nm\underline{V}\cdot\nabla V^2 - \frac{1}{2}mV^2\frac{\partial n}{\partial t}$$

$$+ \underline{V}\cdot(\nabla\cdot\underline{\underline{\Pi}}) + \underline{\underline{\Pi}} : \nabla\underline{V} \qquad \text{---(21)}$$

Plug Eq (21) into Eq (20). We obtain

$$\frac{\partial}{\partial t}\left(\frac{3}{2}nT\right) + \nabla\cdot\left(\frac{5}{2}nT\underline{V} + \underline{q}\right) + \underline{V}\cdot\nabla\underline{\underline{\Pi}} + \underline{\underline{\Pi}}:\nabla\underline{V}$$

$$+ \frac{1}{2}nm\underline{V}\cdot\nabla V^2 - \frac{1}{2}mV^2\frac{\partial n}{\partial t} + \frac{\partial}{\partial t}\left(\frac{1}{2}nmV^2\right) = qn\underline{V}\cdot\underline{E}$$

i.e.

$$\frac{\partial}{\partial t}\left(\frac{3}{2}nT\right) + \nabla\cdot\left(\frac{5}{2}nT\underline{V} + \underline{q}\right) + \underline{V}\cdot\nabla\underline{\underline{\Pi}} + \underline{\underline{\Pi}}:\nabla\underline{V}$$

$$+ \frac{1}{2}nm\underline{V}\cdot\nabla V^2 + nm\underline{V}\cdot\frac{\partial \underline{V}}{\partial t} = qn\underline{V}\cdot\underline{E} \quad \text{---(22)}$$

Then evaluate $\underline{V}\cdot$ Eq.(13) (the Momentum moment Equation)

$$nm\left(\underline{V}\cdot\frac{\partial \underline{V}}{\partial t} + \underline{V}\cdot(\underline{V}\cdot\nabla\underline{V})\right) = -\underline{V}\cdot\nabla p + qn\underline{V}\cdot\underline{E} - \underline{V}\cdot(\nabla\cdot\underline{\underline{\Pi}})$$

$$\text{---(23)}$$

Notice

$$\underline{V}\cdot(\underline{V}\cdot\nabla\underline{V}) = V_i V_j \frac{\partial}{\partial x_j}V_i = V_j\frac{\partial}{\partial x_j}\left(\frac{1}{2}V_iV_i\right)$$

$$= \underline{V}\cdot\nabla\frac{1}{2}V^2 = \frac{1}{2}\underline{V}\cdot\nabla V^2.$$

So Eq(23) is

$$nm\underline{V}\cdot\frac{\partial \underline{V}}{\partial t} + \frac{1}{2}nm\underline{V}\cdot\nabla V^2 = -\underline{V}\cdot\nabla p + qn\underline{V}\cdot\underline{E} - \underline{V}\cdot(\nabla\cdot\underline{\underline{\Pi}})$$

$$\text{---(24)}$$

combine Eq.(22) & Eq.(24), we finally get

$$\boxed{\frac{\partial}{\partial t}\left(\frac{3}{2}nT\right) + \nabla\cdot\left(\frac{5}{2}nT\underline{V} + \underline{q}\right) = \underline{V}\cdot\nabla p - \underline{\underline{\Pi}} : \nabla\underline{V}} \qquad ---(25)$$

Notice

$$\nabla\cdot\left(\frac{5}{2}nT\underline{V}\right) = \nabla\cdot(p\underline{V}) + \frac{3}{2}\nabla\cdot(nT\underline{V})$$

$$= \underline{V}\cdot\nabla p + p\nabla\cdot\underline{V} + \frac{3}{2}n\underline{V}\cdot\nabla T + \frac{3}{2}T\nabla\cdot(n\underline{V})$$

$$\frac{\partial}{\partial t}\left(\frac{3}{2}nT\right) = \frac{3}{2}n\frac{\partial T}{\partial t} + \frac{3}{2}T\frac{\partial n}{\partial t}$$

So Eq.(25) $\Rightarrow$

$$\frac{3}{2}n\left(\frac{\partial}{\partial t} + \underline{V}\cdot\nabla\right)T + \frac{3}{2}T\,\overbrace{\left(\frac{\partial n}{\partial t} + \nabla\cdot(n\underline{V})\right)}^{=0}$$

$$+ p\nabla\cdot\underline{V} = -\nabla\cdot\underline{q} - \underline{\underline{\Pi}} : \nabla\underline{V}$$

So we obtain

$$\frac{3}{2}n\left(\frac{\partial}{\partial t} + \underline{V}\cdot\nabla\right)T + p\nabla\cdot\underline{V} = -\nabla\cdot\underline{q} - \underline{\underline{\Pi}}\cdot\nabla\underline{V} \qquad ---(26)$$

If $\underline{\underline{\Pi}} = 0$ (zero viscous stress), $\underline{q} = 0$ (no heat flux)

Eq(26) can be written as

$$\frac{3}{2} n \left(\frac{\partial}{\partial t} + \underline{V} \cdot \nabla \right) T + P \nabla \cdot \underline{V} = 0 \qquad \text{---(27)}$$

from the density moment eq.

$$n \nabla \cdot \underline{V} = - \frac{Dn}{Dt}$$

So Eq(27) change to

$$\frac{3}{2} n \frac{DT}{Dt} - T \frac{Dn}{Dt} = 0$$

$$\Rightarrow \quad \frac{1}{T} \frac{DT}{Dt} - \frac{2}{3} \frac{1}{n} \frac{Dn}{Dt} = 0$$

i.e. $$\frac{D}{Dt} \left(\ln (T n^{-2/3}) \right) = 0$$

$$\Rightarrow \quad \frac{D}{Dt} \left(\ln p\, n^{-5/3} \right) = 0$$

i.e. $$\boxed{\frac{D}{Dt} (p n^{-5/3}) = 0} \qquad \text{—— (28)}$$