Plasma Transport theory

problem Set #4 Solutions

1. Collisional Guiding Center Scattering

$$C_{ei}^{R}(f_e) = \left\langle \frac{1}{\Omega_e} \underline{b}\times\nabla_R \cdot \frac{\nu_{ei}}{2} v_{Te}^3 \underline{\underline{U}} \cdot \frac{1}{\Omega_e} \underline{b}\times\nabla_R f_e \right\rangle_\phi$$

(Assume $f_e = f_e(\underline{R}, \underline{v})$)

$$= \cancel{\frac{\nu_{ei} v_{Te}^2}{2\Omega_e^3} \frac{1}{2\pi}\oint d\phi} \nabla_R$$

$$= \nabla_R D(v_\parallel, v_\perp) \nabla_R f_e$$

with $$D(v_\parallel, v_\perp) = \left\langle \nu_{ei} \frac{v_{Te}^3}{2\Omega_e^2} \underline{e}_y \cdot \underline{\underline{U}} \cdot \underline{e}_y \right\rangle_\varphi$$

(An simplified version is considered here. Assume y-direction homogeneous)

$$= \frac{1}{2\pi} \frac{\nu_{ei} v_{Te}^3}{2\Omega_e^2} \oint d\varphi \frac{\underline{e}_y}{v} \left(\underline{\underline{I}} - \frac{\underline{v}\,\underline{v}}{v^2}\right) \cdot \underline{e}_y$$

$$= \frac{\nu_{ei} v_{Te}^3}{2\Omega_e^2} \frac{1}{2\pi} \oint d\varphi \frac{1}{v} \left(1 - \frac{v_y v_y}{v^2}\right)$$

$$= \frac{\nu_{ei} v_{Te}^3}{2\Omega_e^2} \frac{1}{2\pi} \oint d\varphi \frac{1}{v^3} \left(v^2 - v_\perp^2 \sin^2\varphi\right)$$

$$= \frac{\nu_{ei} v_{Te}^3}{2\Omega_e} \frac{1}{v^3} \left(v^2 - \frac{1}{2} v_\perp^2\right)$$

$$\begin{cases} v^2 = v_\perp^2 + v_\parallel^2 \\ v_x = v_\perp \cos\varphi \\ v_y = v_\perp \sin\varphi \end{cases}$$

$$= \underline{\frac{\nu_{ei} v_{Te}^3}{2\Omega_e} \frac{1}{v^3} \left(v_\parallel^2 + \frac{1}{2} v_\perp^2\right)}$$

$$\frac{\partial}{\partial \underline{v}}\cdot\underline{\underline{U}}\cdot\frac{\partial}{\partial \underline{v}} = \frac{\partial}{\partial \underline{v}}\cdot\frac{1}{v}\left(\underline{\underline{I}}-\frac{\underline{v}\underline{v}}{v^2}\right)\cdot\frac{\partial}{\partial \underline{v}}$$

$$= \frac{\partial}{\partial \underline{v}}\cdot\left(\frac{1}{v}\frac{\partial}{\partial \underline{v}} - \frac{\underline{v}\,\underline{v}}{v^3}\cdot\frac{\partial}{\partial \underline{v}}\right) \qquad \text{(spherical coordinates)}$$

where $\frac{\underline{v}\underline{v}}{v^3}\cdot\frac{\partial}{\partial \underline{v}} = \frac{1}{v}\underline{e}_v\underline{e}_v\cdot\left(\underline{e}_v\frac{\partial}{\partial v} + \underline{e}_\theta\frac{1}{v}\frac{\partial}{\partial\theta} + \underline{e}_\varphi\frac{1}{v\sin\theta}\frac{\partial}{\partial\varphi}\right)$

$$= \frac{1}{v}\underline{e}_v\frac{\partial}{\partial v}$$

$$\frac{1}{v}\frac{\partial}{\partial \underline{v}} = \frac{1}{v}\underline{e}_v\frac{\partial}{\partial v} + \frac{1}{v}\left(\underline{e}_\theta\frac{1}{v}\frac{\partial}{\partial\theta} + \underline{e}_\varphi\frac{1}{v\sin\theta}\frac{\partial}{\partial\varphi}\right)$$

So $\frac{1}{v}\frac{\partial}{\partial \underline{v}} - \frac{\underline{v}\underline{v}}{v^3}\cdot\frac{\partial}{\partial \underline{v}} = \frac{1}{v}\left(\underline{e}_\theta\frac{1}{v}\frac{\partial}{\partial\theta} + \underline{e}_\varphi\frac{1}{v\sin\theta}\frac{\partial}{\partial\varphi}\right)$

Then $\frac{\partial}{\partial \underline{v}}\cdot\underline{\underline{U}}\cdot\frac{\partial}{\partial \underline{v}} = \frac{\partial}{\partial \underline{v}}\cdot\frac{1}{v^2}\left(\underline{e}_\theta\frac{\partial}{\partial\theta} + \frac{\underline{e}_\varphi}{\sin\theta}\frac{\partial}{\partial\varphi}\right)$

$$= \frac{1}{v\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{1}{v^2}\frac{\partial}{\partial\theta} + \frac{1}{v\sin\theta}\frac{1}{\sin\theta}\frac{\partial^2}{\partial\varphi^2}\frac{1}{v^2}$$

$$= \frac{1}{v^3}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)$$

Therefore

$$C_{ei}^{v}(f_e) = \left\langle \frac{\partial}{\partial \underline{v}}\cdot\nu_{ei}\frac{v_{Te}^3}{2}\underline{\underline{U}}\cdot\frac{\partial}{\partial \underline{v}}\right\rangle_\varphi f_e$$

$$= \nu_{ei}\frac{v_{Te}^3}{2}\frac{1}{v^3}\frac{1}{2\pi}\oint d\varphi\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)f_e$$

$$= \nu_{ei}\frac{v_{Te}^3}{2}\frac{1}{v^3}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}f_e$$

$$\left(\oint d\varphi\frac{\partial^2}{\partial\varphi^2}f_e = \left.\frac{\partial f_e}{\partial\varphi}\right|_{\varphi=0}^{\varphi=2\pi} = 0\right)$$

Let $\mu = \cos\theta$, then

$$\underline{C_{ei}^{v}(f_e) = \nu_{ei}\frac{v_{Te}^3}{v^3}\frac{1}{2}\frac{\partial}{\partial\mu}(1-\mu^2)\frac{\partial}{\partial\mu}f_e}$$

2. Diamagnetic Flow

$$f_M(\underline{R}) = f_M(\underline{r}-\underline{\rho}) \simeq f_M(\underline{r}) - \underline{\rho}\cdot\nabla f(\underline{r})$$

First order approximation.

Then

$$nu^{*} = \int d^3v\, \underline{v}\, f_M(\underline{R})$$

$$= \int d^3v\, \underline{v}\,(f_M(\underline{r}) - \underline{\rho}\cdot\nabla f_M(\underline{r}))$$

$$= -\int d^3v\, \underline{\rho}\cdot\nabla f_M(\underline{r})$$

$$= -\int d^3v\, \frac{\underline{b}\times\underline{v}}{\Omega_e}\cdot\left(\nabla \ln p_e + \left(\frac{mv^2}{2T_e}-\frac{5}{2}\right)\nabla\ln T_e\right) f_M\, \underline{v}$$

$$\cancel{= \frac{1}{\Omega_e}\, b\times(\nabla \ln p_e \mp (}$$

$$= \frac{1}{\Omega_e}\,\underline{b}\times\nabla\ln p_e\cdot\int d^3v\,\underline{v}\underline{v}\, f_M + \frac{1}{\Omega_e}\,\underline{b}\times\nabla\ln T_e\cdot\int d^3v\,\underline{v}\underline{v}\left(\frac{mv^2}{2T_e}-\frac{5}{2}\right)f_M$$

where

$$\left(\int d^3v\,\underline{v}\underline{v}\, f_M\right)_{ij} = \int d^3v\, v_i v_j\, f_M$$

$$= \delta_{ij} \int d^3v\, v_i^2 f_M$$

$$= \delta_{ij} \frac{T_e n_e}{m_e}$$

So $\displaystyle \int d^3v\, \underline{v}\,\underline{v}\, f_M = \frac{n_e T_e}{m_e} \underline{\underline{I}}$

Similarily.

$$\int d^3v\, \underline{v}\,\underline{v} \left(\frac{m_e v^2}{2T_e} - \frac{5}{2} \right) f_M$$

$$= \underline{\underline{I}} \int d^3v\, \frac{v^2}{3} \left(\frac{v^2}{v_{Te}^2} - \frac{5}{2} \right) \frac{n_e}{(\pi v_{Te}^2)^{3/2}} e^{-\frac{v^2}{v_{Te}^2}} \quad ; \quad v_{Te} \equiv \sqrt{\frac{2T_e}{m_e}}$$

$$= n_e \underline{\underline{I}} \frac{4\pi}{3} \int_0^\infty dv\, v^4 \left(\frac{v^2}{v_{Te}^2} - \frac{5}{2} \right) \frac{1}{v_{Te}^3} e^{-\frac{v^2}{v_{Te}^2}}$$

$$= n_e \underline{\underline{I}} \frac{4\pi}{3} v_{Te}^2 \int_0^\infty dx\, x^4 \left(x^2 - \frac{5}{2} \right) e^{-x^2} \qquad \left(x \equiv \frac{v}{v_{Te}} \right)$$

$$= n_e \underline{\underline{I}} \frac{T_e}{m_e} \frac{4\pi}{3} \int_0^\infty dy\, y^{3/2} \left(y - \frac{5}{2} \right) e^{-x^2} \qquad (y \equiv x^2)$$

$$= \frac{n_e T_e}{m_e} \frac{4\pi}{3} \underline{\underline{I}} \left(\Gamma\left(\frac{5}{2}\right) - \frac{5}{2} \Gamma\left(\frac{3}{2}\right) \right)$$

$$= 0$$

Therefore we get

$$n u^* = \frac{1}{\Omega_e} \underline{b} \times \ln P_e \cdot \frac{n_e T_e}{m_e} \underline{\underline{I}}$$

$$= \frac{1}{m_e \Omega_e} \underline{b} \times \nabla P_e$$

3. Electron-Ion Temperature Equilibrium

The electron-ion Energy exchange operator is in the form

$$C_{ei}^{E}(f_e) = \frac{m_e}{2m_i} V_{Te}^3 \frac{1}{v^2} \frac{\partial}{\partial v}\left(\nu_{ei}(v)\left(f_e + \frac{T_i}{m_e}\frac{1}{v}\frac{\partial}{\partial v} f_e\right)\right)$$

Compare the expression in Eq 3.40 text book. we easily get

$$\nu_{ei}(v) = \hat{\nu}_{ei} \left(\frac{v}{V_{Ti}}\right)^2 4 G\left(\frac{v}{V_{Ti}}\right)$$

with $\qquad \hat{\nu}_{ei} = \dfrac{4\pi n_i Z_i^2 e^4 \ln\Lambda}{m_e^2 V_{Te}^3}$

If we make the ordering $\frac{m_e}{m_i} \sim \frac{\rho_e}{L_\perp} \sim \frac{E}{E_R}$, the zeroth order kinetic equation gives $f_e^0 = f_{max} = \frac{n_e}{(\pi V_{Te}^2)^{3/2}} e^{-\left(\frac{v}{V_{Te}}\right)^2}$

The first order kinetic equation gives

$$v_\parallel \underline{b} \cdot \nabla f_e^0 - \frac{e}{m_e} E_\parallel \frac{\partial}{\partial v_\parallel} f_e^0 = C_e^v(f_e^1) + C_{ei}^E(f_e^0)$$

--- <1>

Use the properties of C_e^V : Conserve particles and Energy

Take the particle and Energy momentes of above equation. Notice the left handside vanishes. the we ~~req~~ have

$$\int d^3v\, C_{ei}^E(f_e^0) = 0 \qquad --- \ 1^\circ$$

$$\int d^3v\, \frac{1}{2}mv^2\, C_{ei}^E(f_e^0) = 0 \qquad --- \ 2^\circ$$

Eq 1° Can be verified as:

$$\int d^3v\, C_{ei}^E(f_e^0) = \frac{m_e}{2m_i} v_{Te}^3\, 4\pi \int_0^\infty dv\, v^2 \frac{1}{v^2}\frac{\partial}{\partial v}\left[\nu_{ei}(v)\left(f_e^0 + \frac{T_i}{m_e}\frac{1}{v}\frac{\partial f_e^0}{\partial v}\right)\right]$$

$$= 2\pi \frac{m_e}{m_i} v_{Te}^3 \left[\nu_{ei}\left(f_e^0 + \frac{T_i}{m_e}\left(\frac{-m_e}{T_e}\right) f_e^0\right)\right]_0^\infty$$

$\nu_{ei}(v) f_e^0 \to 0$, as $v \to \infty$

$\nu_{ei}(v) \to 0$, as $v \to 0$.

So $\int d^3v\, C_{ei}^E(f_e^0) = 0$

Eq 2° ~~turns out~~ can be evaluated as

$$\int_0^\infty d^3v \, \tfrac{1}{2} m_e v^2 \, C_{ei}^E (f_e^0)$$

$$= \frac{m_e}{4m_i} m_e V_{Te}^3 \, 4\pi \int_0^\infty dv \, v^2 v^2 \frac{1}{v^2} \frac{\partial}{\partial v} \left[\nu_{ei}(v) \left(f_e^0 + \frac{T_i}{m_e} \frac{1}{v} \frac{\partial}{\partial v} f_e^0 \right) \right]$$

$$= -\pi \frac{m_e}{m_i} m_e V_{Te}^3 \int_0^\infty dv \; 2v \, \nu_{ei}(v) \, f_e^0 \left(1 - \frac{T_i}{T_e} \right)$$

$$= 2\pi \frac{m_e}{m_i} m_e V_{Te}^3 \frac{T_i - T_e}{T_e} \int_0^\infty dv \; v \, \nu_{ei}(v) f_e^0$$

$$= 4\pi (T_i - T_e) \frac{m_e}{m_i} V_{Te} \underbrace{\int_0^\infty dv \; v \, \nu_{ei}(v) f_e^0}_{\sim \frac{\hat{\nu}_{ei}}{V_{Te}} n_e}$$

$$\sim 4\pi \, \hat{\nu}_{ei} \, (T_i - T_e) \frac{m_e}{m_i}$$

Usually ~~$T_i = T_e$~~, $T_i \neq T_e$, So ~~the~~ $\int_0^\infty d^3v \, \tfrac{1}{2} m_e v^2 C_{ei}^E (f_e^0) \neq 0$

i.e., The Integratability of First order kinetic Equation <1> requires no energy exchange term $C_{ei}^E(f_e^0)$. So $C_{ei}^E(f_e^0) \sim \frac{m_e}{m_i}$ is a higher order (>1) process. If we make $\frac{m_e}{m_i} \sim \frac{\rho_e}{L_L} \sim \frac{E}{E_R}$ will leads to an ill-posed transport theory

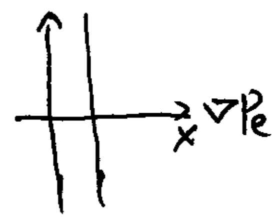
The physical basis is that in the first order kinetic equation.

we require the collision operator to conserve Energy for electrons.

But the $C_{ei}^E(f_e)$ stands for an energy exchange process between Ions and electrons. It can not make the energy balance.

4. Flux-Friction Calculation of Radial Flux:

Momentum balance equation:



$$0 \simeq -\nabla P_e + m_e \Omega_e n \underline{V} \times \underline{b} + \underline{F}_{ei}$$

1° The x-component gives $0 \simeq -\underline{e}_x \cdot \nabla P_e + m \Omega_e n \underline{e}_x \cdot \underline{V} \times \underline{b}$

$$nV_x = \frac{\cancel{F}}{m_e \Omega_e} \cancel{V_y}$$

$$\Rightarrow \quad V_y = \frac{1}{n m_e \Omega_e} \frac{\partial P_e}{\partial x}$$

2° The y-component gives $0 \simeq m_e \Omega_e n \underline{V} \times \underline{b} \cdot \underline{e}_y + \underline{F}_e \cdot \underline{e}_y$

$$nV_x = \frac{1}{m_e \Omega_e} \underline{e}_y \cdot F_{ei}$$

$$= \frac{1}{\Omega_e} \underline{e}_y \cdot \int d^3v \, \underline{v} \, C_{ei}^L(f_e^1)$$

Assme f_e^0 to be a drift Maxwellian

$$f_e^0 = \frac{n}{(\pi v_{Te}^2)^{3/2}}\, e^{-\frac{|\underline{v}-\underline{V}_y|^2}{v_{Te}^2}}$$

Let's first calculate

$$\int d^3v\, \underline{v}\, C_{ei}^L(f_e^0)$$

$$= \int d^3v\, \underline{v}\, \frac{\partial}{\partial \underline{v}} \cdot \hat{\nu}_{ei} \frac{v_{Te}^3}{2}\, \underline{\underline{U}} \cdot \frac{\partial}{\partial \underline{v}} f_e^0$$

Integrate by parts

$$= -\hat{\nu}_{ei} \frac{v_{Te}^3}{2} \int d^3v\, \underline{\underline{U}} \cdot \frac{\partial}{\partial \underline{v}} f_e^0$$

$$= +\hat{\nu}_{ei} \frac{v_{Te}^3}{2} \int d^3v\, \frac{1}{v}\left(\underline{\underline{I}} - \frac{\underline{v}\underline{v}}{v^2}\right) \cdot \frac{(\underline{v} - \underline{V}_y)\,2}{v_{Te}^2} f_e^0$$

$$= -\hat{\nu}_{ei}\, v_{Te} \int d^3v\, \frac{1}{v}\left(\underline{\underline{I}} - \frac{\underline{v}\underline{v}}{v^2}\right) \cdot \underline{V}_y\, f_e^0$$

Notice $V_y = \frac{1}{n m_e \Omega_e}\frac{\partial p_e}{\partial x} \sim \frac{p_e}{n m_e \Omega_e}\frac{1}{L_\perp} \sim v_{Te}\frac{\rho_e}{L_\perp} \ll v_{Te}$

So the zeroth order approximation. which is required for this calculation. we can take

$$f_e^0 \simeq \frac{n}{(\pi v_{Te}^2)^{3/2}}\, e^{-\frac{v^2}{v_{Te}^2}}$$

Therefore we have

$$nV_x = \frac{-1}{\Omega_e}\, \underline{e}_y \cdot \int d^3v \, \frac{\hat{\nu}_{ei} v_{Te}}{v} \left(\underline{\underline{I}} - \frac{\underline{v}\underline{v}}{v^2} \right) \cdot \underline{V}_y \, f_e^0$$

$$= -\frac{\hat{\nu}_{ei}}{\Omega_e}\, \underline{e}_y \cdot \int d^3v \, \frac{a\, v_{Te}}{v} \left(\underline{\underline{I}} - \frac{\underline{v}\underline{v}}{v^2} \right) f_e^0 \cdot \underline{V}_y$$

$$= -\frac{\hat{\nu}_{ei}}{\Omega_e}\, \underline{e}_y \cdot \underline{\underline{I}}\; 4\pi \int dv\, v^2 \frac{v_{Te}}{v} \left(1 - \frac{1}{3}\frac{v^2}{v^2} \right) f_e^0 \cdot \underline{V}_y$$

$$= -\frac{\hat{\nu}_{ei}}{\Omega_e} V_y \,\frac{4\pi}{3}\, \frac{2n}{\pi^{3/2} v_{Te}^2} \int_0^\infty dv\, v\, e^{-\left(\frac{v}{v_{Te}}\right)^2}$$

$$= -n \frac{\hat{\nu}_{ei}}{\Omega_e} V_y \frac{4}{3\pi^{1/2}}$$

$$= -\frac{nV_y}{\Omega_e \tau_{ei}}$$

$$= -\frac{1}{m_e \Omega_e^2 \tau_{ei}} \frac{\partial p}{\partial x}$$

This is consistent with (Eq 4.17) In textbook.