Fall Term 2003
Plasma Transport Theory, 22.616
Problem Set #6
Prof. Molvig

Passed Out: Nov. 18, 2003 DUE: Nov. 25, 2003

1. **Ambipolar Potential in a Magnetized Plasma Column:**

   Compute the electrostatic floating potential, Φ, of a strongely magnetized plasma column, $\nu_{ii} \ll \Omega_i$, in steady state as required by ambipolarity. What is the sign of this potential? How does the potential energy, $e\Phi$, compare to the thermal energies, T_e, and, T_i? What is the net rotation of the plasma in this steady state?

2. **Self-Adjoint Property of Collision Operator:**

   Show that the linearized collision operator (for like-particle collisions),

   $$\mathcal{C}_l\left(\widehat{f}\right) = \frac{\Gamma}{2}\frac{\partial}{\partial \mathbf{v}} \cdot \int d^3v' f_0 f_0' \mathbf{U} \cdot \left(\frac{\partial}{\partial \mathbf{v}}\widehat{f} - \frac{\partial}{\partial \mathbf{v}'}\widehat{f}'\right)$$

   with the usual definitions,

   $$\Gamma \equiv \frac{4\pi e^4 \ln \Lambda}{m^2}$$
   $$\mathbf{U}\left(\mathbf{v}, \mathbf{v}'\right) \equiv \frac{1}{|\mathbf{v}-\mathbf{v}'|}\left(\mathbf{I} - \frac{\left(\mathbf{v}-\mathbf{v}'\right)\left(\mathbf{v}-\mathbf{v}'\right)}{|\mathbf{v}-\mathbf{v}'|^2}\right)$$

   is self-adjoint,

   $$\int d^3v \widehat{g} \mathcal{C}_l\left(\widehat{f}\right) = \int d^3v \widehat{f} \mathcal{C}_l\left(\widehat{g}\right)$$

   for any two functions, $\widehat{f}$, and, $\widehat{g}$, that are reasonably well behaved and don't diverge exponentially at infinity. The distribution, f_0, is the local Maxwellian.

3. **Conservation Laws for Linearized Collision Operator:**

   Show that the linearized collision operator automatically satisfies the conservation laws for *any* $\widehat{f}$,

   $$\int d^3v \begin{bmatrix} 1 \\ m\mathbf{v} \\ \frac{1}{2}mv^2 \end{bmatrix} \mathcal{C}_l\left(\widehat{f}\right) = 0$$

4. **Ambipolarity and Impurity Diffusion:**

   We have seen that reasonable impurity levels can lead to a situation in which particle transport is dominated by ion-impurity collisions rather than ion-electron collisions. In this case, classical transport is still ambipolar to leading order in the gyroradius, but this comes about by opposite flows in the ion and impurity channels. Conventional wisdom has it that impurities flow IN and ions OUT, leading to the deliterious accumulation of impurities in the core – a decidedly *bad* situation.

It is, however, possible that a very *good* situation could result wherein ions flow IN and impurities OUT, leading to an ideal fueling situation for main ions and an impurity and/or helium ash removal scheme (if impurities are alphas). Take what you know about ambipolarity and density and temperature profiles to either: 1) design a system where this *good* arrangement happens or, 2) show that some basic constraints prohibits it and that the *bad* situation invariably prevails if classical impurity transport dominates.

5. **Diamagnetic Fluxes:**

   The first order, in ρ, distribution function takes the form,

$$f_1 = \frac{v_y}{\Omega}\left(A_1 + A_2\left(\frac{mv^2}{2T} - \frac{5}{2}\right)\right) f_M$$

   where the thermodynamic forces are defined in terms of pressure gradient and temperature gradient,

$$\begin{aligned} A_1 &\equiv d\ln P/dx \\ A_2 &\equiv d\ln T/dx \end{aligned}$$

   Show that the diamagnetic flow is proportional to the pressure gradient alone,

$$nmV_{*y} = \int d^3v m v_y f_1 \propto A_1$$

   with no separate temperature gradient driven flow. Write the complete expression for nmV_{*y}.

6. **Generalized Flux-Friction Relations:**

   Carry out the integrals and linearization of the collision operator to show that the energy flux moment, $\int d^3v \frac{1}{2}mv^2\mathbf{v}$, of the leading order kinetic equation,

$$\Omega \mathbf{v} \times \mathbf{b} \cdot \frac{\partial}{\partial \mathbf{v}} f \simeq \mathcal{C}(f, f)$$

   leads to the heat-flux, heat-friction relation,

$$\Omega Q_x = \int d^3v \frac{1}{2} m v^2 v_y \mathcal{C}_l\left(\widehat{f}\right)$$

   where, Q_x, is the energy flux,

$$Q_x \equiv \int d^3v \frac{1}{2} m v^2 v_x f$$

   and, $\widehat{f}$, is the first order in ρ/a piece of the distribution, defined as,

$$f = f_0\left(1 + \widehat{f}\right)$$

7. **Like-Particle (Ion) Collision Fluxes:**

   Assume that the kinetic equation for the distribution of ion guiding centers can be written in the form,

$$\frac{\partial f}{\partial t} = \mathcal{C}_0(f, f) + \mathcal{C}_2(f, f)$$

where $\mathcal{C}_0$ is the "velocity" operator,

$$\mathcal{C}_0\left(f,f\right) \equiv \frac{\Gamma}{2}\frac{\partial}{\partial \mathbf{v}} \cdot \int d^3v'\mathbf{U} \cdot \left(\frac{\partial}{\partial \mathbf{v}} - \frac{\partial}{\partial \mathbf{v}'}\right) ff'$$

and $\mathcal{C}_2$ is the spatial diffusion operator (including the back reaction effect of the field particles),

$$\begin{aligned} \mathcal{C}_2\left(f,f\right) &= \frac{\partial}{\partial X}\int d^3v' D\left(\mathbf{v},\mathbf{v}'\right)\left(\frac{\partial f}{\partial X}f' - f\frac{\partial f'}{\partial X}\right) \\ D\left(\mathbf{v},\mathbf{v}'\right) &\equiv \frac{\Gamma}{\Omega^2}\mathbf{e}_y \cdot \mathbf{U} \cdot \mathbf{e}_y \end{aligned}$$

Show that the transport equations result from the particle and energy moments of the second order kinetic equation and are given by,

$$\begin{aligned} \frac{\partial n}{\partial t} + \frac{\partial}{\partial X}\Gamma_i &= 0 \\ \Gamma_i &= -\int d^3v d^3v' D\left(\mathbf{v},\mathbf{v}'\right)\left(\frac{\partial f_0}{\partial X}f_0' - f_0\frac{\partial f_0'}{\partial X}\right) \end{aligned}$$

and,

$$\begin{aligned} \frac{\partial}{\partial t}\frac{3}{2}nT_i + \frac{\partial}{\partial X}\left[\frac{5}{2}T_i\Gamma_i + q_i\right] &= 0 \\ q_i &= -T_i\int d^3v d^3v'\left(\frac{mv^2}{2T_i} - \frac{5}{2}\right) D\left(\mathbf{v},\mathbf{v}'\right)\left(\frac{\partial f_0}{\partial X}f_0' - f_0\frac{\partial f_0'}{\partial X}\right) \end{aligned}$$

Finally carry out the calculations to show that, $\Gamma_i = 0$, for any values of, $A_1 = d\ln P_i/dx$, and, $A_2 = d\ln T_i/dx$, and that the heat flux can be expressed as,

$$\frac{q_i}{T_i} = -A_2\int d^3v d^3v' \frac{\left(\frac{1}{2}mv^2 - \frac{1}{2}mv'^2\right)^2}{2T_i^2} D\left(\mathbf{v},\mathbf{v}'\right) f_0 f_0'$$

In terms of the overall transport matrix,

$$\begin{bmatrix} \Gamma_i \\ q_i/T_i \end{bmatrix} = \frac{1}{n}\begin{bmatrix} -D & T_{12} \\ T_{21} & -\chi_i \end{bmatrix}\begin{bmatrix} d\ln P_i/dx \\ d\ln T_i/dx \end{bmatrix}$$

you have just shown that for ion-ion collisions that, $D = T_{12} = T_{21} = 0$, with only, χ_i non-zero.

Extra Credit: Do the integrals to recover result from class!