

Nonstandard Models of True Arithmetic

We have talked a lot about the standard model of the language of arithmetic, but there are other models of true arithmetic (the set of sentences true in the standard model) that aren't isomorphic to the standard model. Indeed, consider the theory Γ consisting of true arithmetic, together with all sentences " $[n] < c$," where " c " is a new constant. If Δ is a finite subset of Γ , then we can model Δ by expanding the standard model by letting " c " denote a number larger than any of the numbers n with " $[n] < c$ " $\in \Delta$. It follows by the compactness theorem that there is a model of Γ . A model of Γ will be a nonstandard model of true arithmetic, that is, a model of true arithmetic that isn't isomorphic¹ to the standard model.

An *initial segment* of a nonstandard model \mathcal{Q} of true arithmetic is defined just as it is for the standard model: $S \subseteq |\mathcal{Q}|$ is an initial segment iff, for any $y \in S$, any element of $|\mathcal{Q}|$ that is $<^{\mathcal{Q}}$ y is an element of S . The map taking n to $[n]^{\mathcal{Q}}$ is an isomorphism from the standard models onto an initial segment of \mathcal{Q} . The elements of $|\mathcal{Q}|$ that aren't in the range of this isomorphism are the *nonstandard* elements. To see that the range of the isomorphism is an initial segment, note first that \mathcal{Q} can't hide any nonstandard elements below $0^{\mathcal{Q}}$, because " $(\forall x)\neg x < 0$ " is part of true arithmetic. It can't sneak any nonstandard elements below $[n+1]^{\mathcal{Q}}$, because $(\forall x)(x < [n+1] \leftrightarrow (x =$

1 Where \mathcal{Q} and \mathcal{B} are models of the language of arithmetic, an *isomorphism* from \mathcal{Q} to \mathcal{B} is a bijection f from $|\mathcal{Q}|$ to $|\mathcal{B}|$ with the property that $f(0^{\mathcal{Q}}) = 0^{\mathcal{B}}$, $f(x +^{\mathcal{Q}} y) = f(x) +^{\mathcal{B}} f(y)$, $x <^{\mathcal{Q}} y$ iff $f(x) <^{\mathcal{B}} f(y)$, and so on. If σ is a variable assignment for \mathcal{Q} , then σ satisfies the same formulas in \mathcal{Q} that $f \circ \sigma$ satisfies in \mathcal{B} . (Here $f \circ \sigma$ is the variable assignment for \mathcal{B} given by $f \circ \sigma(v) = f(\sigma(v))$.) It follows that the same sentences are true in \mathcal{Q} and in \mathcal{B} .

$[0] \vee x = [1] \vee \dots \vee x = [n])$ ” is in true arithmetic. Because we have “ $(\forall x)(\forall y)(x < y \vee x = y \vee y < x)$,” all the nonstandard elements are greater than all the standard elements.

Let a be a nonstandard element. Because true arithmetic assures us that every number has an immediate successor and that every number other than 0 has an immediate predecessor, the immediate neighborhood of a looks just like the (positive and negative) integers. $[2]^{qt} \bullet^{qt} a$ is bigger than all the $a +^{qt} [n]^{qt}$ s, and the immediate neighborhood of $[2]^{qt} \bullet^{qt} a$ looks just like the integers. Similarly, $[3]^{qt} \bullet^{qt} a$ is bigger than all the $([2]^{qt} \bullet^{qt} a) +^{qt} [n]^{qt}$ s, and the immediate neighborhood of $[3]^{qt} \bullet^{qt} a$ looks just like the integers. $a \bullet^{qt} a$ is bigger than all the $[n]^{qt} \bullet^{qt} a$ s, and the immediate neighborhood of $a \bullet^{qt} a$ looks just like the integers.

a satisfies either “ x is even” or “ x is odd.”² If the former, then there is a nonstandard element that when doubled yields a . If the latter, there is a nonstandard element that, when doubled yields $a +^{qt} [1]^{qt}$. Either way, we get a nonstandard number that is approximately one-half of a . The immediate neighborhood of that nonstandard model looks just like the integers. Similarly, there is a nonstandard number that is approximately one-third of a , one that is approximately two-thirds of a , and so on. For any positive integers p and q , there is a nonstandard number b such that $[q]^{qt} \bullet^{qt} b$ and $[p]^{qt} \bullet^{qt} a$ differ by a standard number. In other words, there is a nonstandard number that is approximately equal to p/q times a .

Just looking at the order, we can say precisely what the countable nonstandard models look like. There is an initial segment that looks like the natural numbers, followed by a bunch of copies of the integers. The copies of the integers are ordered; we can say that one copy is less than another iff the members of the first are $<^{qt}$ the members of the second. The order on the

2 That is, a satisfies either “ $(\exists y)([2] \bullet y) = x$ ” or “ $(\exists y)(([2] \bullet y) + [1]) = x$.”

copies is isomorphic to the order on the rational numbers. This description characterizes the order relation on countable, nonstandard models of true arithmetic, uniquely up to isomorphism. (For uncountable models, the picture is similar, but harder to make precise, since uncountable nonstandard models aren't mutually isomorphic.) I won't go through the proof here, but you could look it up in Chapter 17 of Boolos and Jeffrey.³

Nonstandard models are troubling, epistemologically. The name “Fido” refers to something we've scratched behind the ears, and the causal connection between our usage of the word “Fido” and the dog Fido is part of the explanation of how it came to pass that the word refers as it does. For theoretical terms and other terms that don't refer to things to which we are causally connected, there isn't a direct causal explanation, but there may be an indirect causal explanation. We have a, probably informal, theory that tells how the theoretical entities are related to the entities to which we are directly causally connected, and the theoretical terms refer to whatever entities come the closest to playing the role the theory ascribes to them. (“Comes the closest to playing the role” rather than “plays the role,” since it would be silly to pretend that our theories are completely accurate.) The closer the theoretical entity is to the objects of experience, the more prominent the role that causal connections will play in pinning down reference. When we get to things that are very far removed from the objects of experience, like numbers, the causal connections have nearly dropped out of the picture. To be sure, we use numbers to count everyday objects, but for counting purposes, we use only numbers at the very beginning of the natural number system, so the way we use those numbers doesn't go far at all in telling us about

3 George Boolos and Richard Jeffrey, *Computability and Logic*, 3rd ed. (New York and Cambridge: Cambridge University Press, 1989).

the structure of the natural numbers as a whole. It is only a slight exaggeration to say that the meaning of our mathematical terms is given by our mathematical theories.

Now we have a problem. As we shall see in detail later on, our arithmetical theory – the set of arithmetical sentences we can recognize as true – stops far short of true arithmetic. But even if we were able to help ourselves to true arithmetic, that isn't enough to pin down the meanings of the arithmetical terms. Even if our arithmetical theory were true arithmetic, that wouldn't be enough to pin down the structure of the natural numbers, because the theory has nonstandard models.

An easy response would be to say that our arithmetical theory isn't isolated. Our beliefs about natural numbers are embedded in a larger system of beliefs that include our beliefs about real numbers and our beliefs about sets. We should be looking at the role of arithmetical terms within that larger theoretical system.

This is an easy response, but not a useful one, since we can apply the same argument to a larger language that includes the language of set theory and the language of real analysis. The compactness theorem still applies, so the set of true sentences of the larger language will still have models with nonstandard natural numbers. We are still left with a deeply disturbing cause for skepticism about arithmetic.

A different response is that we shouldn't be formalizing our mathematical theory within the predicate calculus, but within some more robust logic to which the compactness theorem does not apply. (The alternative that is usually proposed is the second-order predicate calculus, which we'll describe briefly presently.) This response is potentially more helpful, but it's not as easy. It arouses the suspicion that we have "solved" our problem by Enron-style accounting

practices. Some ideas that we used to refer to as “mathematical” notions have now been rechristened “logical” notions, so that what used to be a problem about the foundations of mathematics is not a problem about the foundations of logic. We’ve relabeled a problem, but we haven’t solved anything, since the old difficulty about fixing the meanings of mathematical terms has reemerged as a problem about fixing the meaning of the new logical terms we introduce when we move beyond the first-order predicate calculus. Or so one suspects. The issue remains highly controversial, and we’ve no hope of resolving it here.