Peano Arithmetic

Peano Arithmetic[1] or *PA* is the system we get from Robinson's Arithmetic by adding the *induction axiom schema*:

$$((R(0) \wedge (\forall x)(R(x) \rightarrow R(sx))) \rightarrow (\forall x)R(x)).$$

What this means is that any sentence of the language of arithmetic that you get from the schema by replacing the schematic letter "R" with a formula, then prefixing universal quantifiers to bind all the free variables is an axiom of PA. Thus PA consist of the axioms (Q1) through (Q11), together with infinitely many *induction axioms.*

The induction axiom schema formalizes a familiar method of reasoning about the natural numbers. To show that every natural number has the property expressed by the formula we substitute for "R," we begin by showing that 0 has the property; this is the *base case*. Next we derive, by conditional proof, the conditional

$$(\forall x)(R(x) \rightarrow R(sx));$$

we assume R(x) as *inductive hypothesis*, then derive R(sx). The rule of mathematical induction permits us to infer (∀x)R(x).

Virtually all of our ordinary mathematical reasoning about the natural numbers can be formalized in PA. Indeed, after some initial awkwardness, in which we produce proofs of facts of elementary arithmetic that we've taken for granted since childhood, reasoning in PA is nearly indistinguishable from ordinary arithmetical thinking.

I'll do a couple of these early proofs informally here, just to get an idea of what's going on.

---

1 The so-called Peano axioms were first formulated by Richard Dedekind. Peano said as much in a footnote, but somehow "Peano Arithmetic" was the name that stuck.

Proposition 1. PA $\vdash (\forall x)(x = 0 \lor (\exists y)x = sy)$.

Proof: Use the following induction axiom:

$$[[(0 = 0 \lor (\exists y)0 = sy) \land (\forall x)((x = 0 \lor (\exists y)x = sy) \rightarrow (sx = 0 \lor (\exists y)sx = sy))]$$

$$\rightarrow (\forall x)(x = 0 \lor (\exists y)x = sy)]$$

The antecedent is a theorem of pure logic.⊠

Proposition 2. PA $\vdash (\forall x)(0 + x) = x$.

Proof: Use the following induction axiom:

$$[[(0 + 0) = 0 \land (\forall x)((0 + x) = x \rightarrow (0 + sx) = sx)] \rightarrow (\forall x)(0 + x) = x]$$

The base clause, "$(0 + 0) = 0$," follows from (Q3). To get the induction step, assume, as inductive hypothesis (IH) that $(0 + x) = x$. We have

$$(0 + sx) = s(0 + x) \qquad \text{[by (Q4)]}$$

$$= sx \qquad \text{[by IH]}.$$⊠

Proposition 3. PA $\vdash (\forall x)(\forall y)(sx + y) = s(x + y)$.

Proof: We use the following induction axiom:

$$(\forall x)[[(sx + 0) = s(x + 0) \land (\forall y)((sx + y) = s(x + y) \rightarrow (sx + sy) = s(x + sy))]$$

$$\rightarrow (\forall y)(sx + y) = s(x + y)].$$

The base clause is easy. Two applications of (Q3) yield

$$(sx + 0) = sx$$

$$= s(x + 0)$$

To get the induction step, assume, as IH,

$$(sx + y) = s(x + y).$$

We have:

$(sx + sy) = s(sx + y)$ [by (Q4)]

$= ss(x + y)$ [by IH]

$= s(x + sy)$ [by (Q4) again].⊠

Proposition 4 (Commutative law of addition). PA $\vdash (\forall x)(\forall y)(x + y) = (y + x)$.

Proof: We use this induction axiom:

$$(\forall x)[[(x + 0) = (0 + x) \wedge (\forall y)((x + y) = (y + x) \rightarrow (x + sy) = (sy + x))]$$

$$\rightarrow (\forall y)(x + y) = (y + x)].$$

(Q3) gives us "$(x + 0) = x$," and Proposition 2 gives us "$(0 + x) = x$"; these together yield the base clause, "$(x + 0) = (0 + x)$." To get the induction step, assume as IH:

$$(x + y) = (y + x).$$

We have:

$(x + sy) = s(x + y)$ [by (Q4)]

$= s(y + x)$ [by IH]

$= (sy + x)$ [by Proposition 3]⊠

Proposition 5 (Associative law of addition). PA $\vdash (\forall x)(\forall y)(\forall z)((x + y) + z)$ $= (x + (y + z))$.

Proof: Two applications of (Q3) give us the basis clause, "$((x + y) + 0) = (x + (y + 0))$." To get the induction step, assume as IH:

$$((x + y) + z) = (x + (y + z)).$$

We have:

$((x + y) + sz) = s((x + y) + z)$ [by (Q4)]

$= s(x + (y + z))$ [by IH]

$= (x + s(y + z))$ [by (Q4)]

$= (x + (y + sz))$ [by (Q4)].⊠

Proposition 6. PA $\vdash (\forall x)(0 \bullet x) = 0$.

Proof: The base clause, "$(0 \bullet 0) = 0$," comes from (Q5). To get the induction, assume as IH:

$(0 \bullet x) = 0$.

We have:

$(0 \bullet sx) = ((0 \bullet x) + 0)$ [by (Q6)]

$= (0 \bullet x)$ [by (Q4)]

$= 0$ [by IH].⊠

Proposition 7. PA $\vdash (\forall x)(\forall y)(sx \bullet y) = ((x \bullet y) + y)$.

Proof: We derive the base clause as follows:

$(sx \bullet 0) = 0$ [by (Q5)

$= (x \bullet 0)$ [by (Q5) again]

$= ((x \bullet y) + 0)$ [by (Q3)].

Assuming, as IH,

$(sx \bullet y) = ((x \bullet y) + y)$,

we compute:

$(sx \bullet sy) = ((sx \bullet y) + sx)$ [by (Q6)]

$= (((x \bullet y) + y) + sx)$ [by IH]

$=((x \bullet y) + (y + sx))$ [by Proposition 5]

$= ((x \bullet y) + s(y + x))$ [by (Q4)]

$=((x \bullet y) + s(x + y))$ [by Proposition 4]

$= ((x \bullet y) + (x + sy))$ [by (Q4)]

$= (((x \bullet y) + x) + sy)$ [by Proposition 5]

$= ((x \bullet sy) + sy)$ [by (Q6)].⊠

Proposition 8 (Commutative Law of Multiplication). PA $\vdash (x \bullet y) = (y \bullet x)$.

Proof: The base clause, "$(x \bullet 0) = (0 \bullet x)$," uses (Q5) and Proposition 6. As inductive hypothesis, assume:

$$(x \bullet y) = (y \bullet x).$$

We compute:

$(x \bullet sy) = ((x \bullet y) + x)$ [by Q6)]

$= ((y \bullet x) + x)$ [by IH]

$= (sy \bullet x)$ [by Proposition 7].⊠

Proposition 9 (Distributive law). PA $\vdash (\forall x)(\forall y)(\forall z)(x \bullet (y + z)) = ((x \bullet y) + (x \bullet z))$.

Proof: We prove this equivalent formula:

$$(\forall y)(\forall z)(\forall x)(x \bullet (y + z)) = ((x \bullet y) + (x \bullet z)),$$

by using this induction axiom:

$$(\forall y)(\forall z)[[(0 \bullet (y + z)) = ((0 \bullet y) + (0 \bullet z)) \wedge (\forall x)((x \bullet (y + z)) = ((x \bullet y) + (x \bullet z)) \rightarrow$$

$$(sx \bullet (y + z)) = ((sx \bullet y) + (sx \bullet z)))] \rightarrow (\forall x)(x \bullet (y + z)) = ((x \bullet y) + (x \bullet z))]$$

To get the base clause, we compute:

$(0 \bullet (y + z)) = 0$ [by Proposition 6]

$= (0 + 0)$ [by (Q3)]

$= ((0 \bullet y) + (0 \bullet z))$ [by Proposition 6 again].

In proving the induction step, we assume the IH:

$(x \bullet (y + z)) = ((x \bullet y) + (x \bullet z))$.

Now we calculate:

$$
\begin{aligned}
(sx \bullet (y + z)) &= ((x \bullet (y + z)) + (y + z)) && \text{[by Proposition 7]} \\
&= (((x \bullet y) + (x \bullet z)) + (y + z)) && \text{[by IH]} \\
&= ((x \bullet y) + ((x \bullet z) + (y + z))) && \text{[by Proposition 5]} \\
&= ((x \bullet y) + (((x \bullet z) + y) + z)) && \text{[by Proposition 5]} \\
&= ((x \bullet y) + ((y + (x \bullet z)) + z)) && \text{[by Proposition 4]} \\
&= ((x \bullet y) + (y + ((x \bullet z) + z))) && \text{[by Proposition 5]} \\
&= (((x \bullet y) + y) + ((x \bullet z) + z)) && \text{[by Proposition 5]} \\
&= ((sx \bullet y) + (sx \bullet z)) && \text{[by Proposition 7].} \boxtimes
\end{aligned}
$$

Proposition 10 (Associative law of multiplication). PA $\vdash (\forall x)(\forall y)(\forall z)((x \bullet y) \bullet z) = (x \bullet (y \bullet z))$.

Proof: The induction axiom we intend to employ is this:

$$(\forall x)(\forall y)[[((x \bullet y) \bullet 0) = (x \bullet (y \bullet 0)) \wedge (\forall z)(((x \bullet y) \bullet z) = (x \bullet (y \bullet z)) \rightarrow ((x \bullet y) \bullet sz) = (x \bullet (y \bullet sz)))] \rightarrow (\forall z)((x \bullet y) \bullet z) = (x \bullet (y \bullet z))].$$

We get the base clause thus:

$$
\begin{aligned}
((x \bullet y) \bullet 0) &= 0 && \text{[by (Q6)]} \\
&= (x \bullet 0) && \text{[by (Q6)]} \\
&= (x \bullet (y \bullet 0)) && \text{[by (Q6)].}
\end{aligned}
$$

To get the induction step, we assume this IH:

$((x \bullet y) \bullet z) = (x \bullet (y \bullet z))$.

We compute:

$$((x \bullet y) \bullet sz) = (((x \bullet y) \bullet z) + (x \bullet y)) \qquad \text{[by (Q6)]}$$

$$= ((x \bullet (y \bullet z)) + (x \bullet y)) \qquad \text{[by IH]}$$

$$= (x \bullet ((y \bullet z) + y)) \qquad \text{[by Proposition 9]}$$

$$= (x \bullet (y \bullet sz)) \qquad \text{[by (Q6)].} \boxtimes$$

We could keep going like this for a very long time.

The induction axiom schema we have been using is sometimes called the "weak induction schema," to distinguish it form the following *strong induction schema*:

$$((\forall x)((\forall y < x)Sy \rightarrow Sx) \rightarrow (\forall x)Sx).$$

In applying this schema, we assume as inductive hypothesis that every number less than x has the property represented by Sx, then try to show that x has the property. If we succeed, we conclude that every number has the property. We don't need to assume the instances of the strong induction schema as additional axioms, because we can derive them using the regular induction schema. Specifically, the induction axiom we use is this:

$$[[(\forall y < 0)Sy \wedge (\forall x)((\forall y < x)Sy \rightarrow (\forall y < sx)Sy)] \rightarrow (\forall x)(\forall y < x)Sy].$$

The inductive hypothesis, "$(\forall y < 0)Sy$," is a consequence of (Q9). (Q10) tells us that the induction clause, "$(\forall x)((\forall y < x)Sy \rightarrow (\forall y < sx)Sy)$," is equivalent to this:

$$(\forall x)((\forall y < x)Sy \rightarrow (\forall y)((y < x \vee y = x) \rightarrow Sy)),$$

which, in turn is equivalent to this:

$$(\forall x)((\forall y < x)Sy \rightarrow ((\forall y < x)Sy \wedge Sx)),$$

which is equivalent to

$$(\forall x)((\forall y < x)Sy \rightarrow Sx).$$

Thus we have this:

$$((\forall x)((\forall y < x)Sy \rightarrow Sx) \rightarrow (\forall x)(\forall y < x)Sy).$$

We also have this:

$$((\forall x)(\forall y < x)Sy \rightarrow (\forall x)Sx),$$

which we obtain by the following derivation:

1	1. $(\forall x)(\forall y)(y < x \rightarrow Sy)$	PI
1	2. $(\forall y)(y < sa \rightarrow Sy)$	US, 1
1	3. $(a < sa \rightarrow Sa)$	US, 2
(Q10)	4. $(\forall x)(\forall y)(x < sy \leftrightarrow (x < y \vee x = y))$	
(Q10)	5. $(\forall y)(a < sy \leftrightarrow (a < y \vee a = y))$	US, 4
(Q10)	6. $(a < sa \rightarrow (a < a \vee a = a))$	US, 5
	7. $a = a$	IR
(Q10)	8. $a < sa$	TC, 6, 7
1, (Q10)	9. Sa	TC, 3, 8
1, (Q10)	10. $(\forall x)Sx$	UG, 9
(Q10)	11. $((\forall x)(\forall y < x)Sy \rightarrow (\forall x)Sx)$	CP, 1, 10

Combining results, we obtain:

$$((\forall x)((\forall y < x)Sy \rightarrow Sx) \rightarrow (\forall x)Sx).$$

What we'd like to do now is reverse the process, showing how we could, if we had chosen, have taken the strong induction schema as axiomatic, and derived the weak induction schema. However, our attempt to do so runs into a glitch. We used weak induction to derive Proposition 1, the statement that every number is either 0 or a successor. If we replace weak by strong induction, we can't derive Proposition 1. Indeed, it's possible to put together a model of Q

+ the strong induction schema in which Proposition 1 is false (though we won't do so here). What we can show, however, is that Q + Proposition 1 + the strong induction schema entails the weak induction schema. Thus, what we want to show is this:

$$((R0 \wedge (\forall x)(Rx \rightarrow Rsx)) \rightarrow (\forall x)Rx).$$

Strong induction gives us this:

$$((\forall x)((\forall y < x)Ry \rightarrow Rx) \rightarrow (\forall x)Rx).$$

So what we need to show is this:

$$((R0 \wedge (\forall x)(Rx \rightarrow Rsx)) \rightarrow (\forall x)((\forall y < x)Ry \rightarrow Rx)).$$

Assume

$$R0$$

and

$$(\forall x)(Rx \rightarrow Rsx)$$

Take any y. What we want to show is this:

$$((\forall y < x)Ry \rightarrow Rx).$$

If x = 0, this follows immediately from our assumption that R0. So we may assume (using Proposition 1) that x is a successor; say x = sz. So what we have to show is this:

$$((\forall y < sz)Ry \rightarrow Rsz.$$

We assumed $(\forall x)(Rx \rightarrow Rsx)$, which gives us this:

$$(Rz \rightarrow Rsz).$$

So what we need is this:

$$((\forall y < sz)Ry \rightarrow Rz).$$

In other words,

(†) $((\forall y)(y < sz \rightarrow Ry) \rightarrow Rz)$.

We have

$$((\forall y)(y < sz \rightarrow Ry) \rightarrow (z < sz \rightarrow Rz)).$$

Since "z < sz" is a consequence of (Q11), (†) follows immediately.

Plug in "¬ Qx" in place of "Sx" in the strong induction schema, and you get a schema logically equivalent to the following:

$$((\exists x)Qx \rightarrow (\exists x)(Qx \wedge (\forall y <)\neg Qy)).$$

This schema is a formalized version of the *well-ordering principle*: Every nonempty collection of natural numbers has a least element.

The induction axiom schema is a formalized version of the

> **Principle of Mathematical Induction.** Any collection that contains 0 and contains the successor of any natural number it contains contains every natural number.

This principle is central to out reasoning about the natural numbers. A reason for this centrality is singled out in the following:

> **Theorem** (Richard Dedekind). Any two models of Q that both satisfy the principle of mathematical induction are isomorphic.[2]

---

2 An *isomorphism* from a model $\mathfrak{A}$ to a model $\mathfrak{B}$ of the language of arithmetic is a bijection f from $|\mathfrak{A}|$ to $|\mathfrak{B}|$ that satisfies the following conditions:

$f(0^{\mathfrak{A}}) = 0^{\mathfrak{B}}$.

$f(s^{\mathfrak{A}}(x)) = s^{\mathfrak{B}}(f(x))$.

$f(x +^{\mathfrak{A}} y) = f(x) +^{\mathfrak{B}} f(y)$.

Proof: Let f be the smallest subcollection of $|\mathfrak{A}| \times |\mathfrak{B}|$ that meets these conditions:

(D1) $<0^{\mathfrak{A}},0^{\mathfrak{B}}>$ is in the collection.

(D2) If $<x,y>$ is in the collection, so is $<s^{\mathfrak{A}}(x), s^{\mathfrak{B}}(y)>$.

That is, f is the intersection of all subcollections of $|\mathfrak{A}| \times |\mathfrak{B}|$ that satisfy (D1) and (D2).

f is a function from $|\mathfrak{A}|$ to $|\mathfrak{B}|$. To see this, note, first, that f pairs $0^{\mathfrak{A}}$ with one and only one element of $|\mathfrak{B}|$: $<0^{\mathfrak{A}}, 0^{\mathfrak{B}}> \in$ f by (D1). If $y \neq 0^{\mathfrak{B}}$, f ~ $\{<0^{\mathfrak{A}},y>\}$ satisfies (D1) and (D2), which implies, since f is smallest, that f ~ $\{<0^{\mathfrak{A}},y>\}$ = f and $<0^{\mathfrak{A}},y> \notin$ f.

Next, assume that f pairs x with one and only one element y of $|\mathfrak{B}|$. Because f satisfies (D2), the pair $<s^{\mathfrak{A}}(x), s^{\mathfrak{B}}(y)>$ is in f. Suppose that $z \neq s^{\mathfrak{B}}(y)$. Let g = f ~ $\{<s^{\mathfrak{A}}(x),z>\}$. Because $\mathfrak{A}$ satisfies (Q1), $s^{\mathfrak{A}}(x) \neq 0^{\mathfrak{A}}$, and so g satisfies (D1). To see that g also satisfies (D2), take $<a,b> \in$ g. If $s^{\mathfrak{A}}(a) \neq s^{\mathfrak{A}}(x)$, $<s^{\mathfrak{A}}(a), s^{\mathfrak{B}}(b)>$ will be in g because it's in f. If $s^{\mathfrak{A}}(a) = s^{\mathfrak{A}}(x)$, then, because $\mathfrak{A}$ satisfies (Q2), a = x. Because f pairs x with only one element of $|\mathfrak{A}|$, b must be equal to y, and so $s^{\mathfrak{B}}(b) \neq z$; hence, again, $<s^{\mathfrak{A}}(a), s^{\mathfrak{B}}(b)>$ is in g. Thus g satisfies (D1) and (D2). Because f is the smallest class that satisfies (D1) and (D2), g must be equal to f, which means that $<s^{\mathfrak{A}}(x), z>$ isn't in f. Consequently, f pairs $s^{\mathfrak{A}}(x)$ with $s^{\mathfrak{B}}(y)$, and with nothing else.

$f(x \bullet^{\mathfrak{A}} y) = f(x) \bullet^{\mathfrak{B}} f(y)$.

$f(x \, E^{\mathfrak{A}} \, y) = f(x) \, E^{\mathfrak{B}} \, f(y)$.

$x <^{\mathfrak{A}} y$ iff $f(x) <^{\mathfrak{B}} f(y)$.

If σ is a variable assignment for $\mathfrak{A}$, then, for any formula ϕ, σ satisfies ϕ in $\mathfrak{A}$ iff f∘σ satisfies ϕ in $\mathfrak{B}$. (f∘σ is defined by setting f∘σ(v) equal to f(σ(v)).) It follows that the same sentences are true in $\mathfrak{A}$ and in $\mathfrak{B}$.

Let C be the set of elements of $|\mathfrak{A}|$ that are paired by f with exactly one element of $|\mathfrak{A}|$. We see that $0^{\mathfrak{A}}$ is in C and that, whenever x is in C, $s^{\mathfrak{A}}(x)$ is in C. Because $\mathfrak{A}$ satisfies the principle of mathematical induction, C must be equal to $|\mathfrak{A}|$, which means that f is a function from $|\mathfrak{A}|$ to $|\mathfrak{B}|$.

A similar argument, this time using the fact that $\mathfrak{B}$ satisfies the principle of mathematical induction, shows that f is a bijection.

To complete the proof that f is an isomorphism, we have to show several things. We have to show that $f(0^{\mathfrak{A}}) = 0^{\mathfrak{B}}$; this follows immediately from the way f was defined. For each of the function signs of the language, we have to show that f respects the operation of the function sign; for example, we have to show that $f(x +^{\mathfrak{A}} y) = f(x) +^{\mathfrak{B}} f(y)$. Finally, we have to show that f preserves the "<" relation, that is, that $x <^{\mathfrak{A}} y$ iff $f(x) <^{\mathfrak{B}} f(y)$. Of these, we'll only write out the proofs for "s" and "+" here.

(D1) tells us that, if $<x,y> \in f$, $<s^{\mathfrak{A}}(x), s^{\mathfrak{B}}(y)> \in f$. Consequently, for $x \in |\mathfrak{A}|$, since $<s,f(x)> \in f$, $<s^{\mathfrak{A}}(x), s^{\mathfrak{B}}(f(x))> \in f$, that is, $f(s^{\mathfrak{A}}(x)) = s^{\mathfrak{B}}(f(x))$.

To get the clause for "+," pick $x \in |\mathfrak{A}||$. Let $E = \{y \in |\mathfrak{A}|: f(x +^{\mathfrak{A}} y) = f(x) +^{\mathfrak{B}} f(y)\}$. We want to show that $0^{\mathfrak{A}}$ is in E, and also to show that, if y is in E, so is $s^{\mathfrak{A}}(y)$. Because $\mathfrak{A}$ satisfies the principle of mathematical induction, this will suffice to show that every member of $|\mathfrak{A}|$ is in E.

Because $\mathfrak{A}$ satisfies (Q3), $x +^{\mathfrak{A}} 0^{\mathfrak{A}} = x$. Because $\mathfrak{B}$ satisfies (Q3), $f(x) +^{\mathfrak{B}} 0^{\mathfrak{B}} = f(x)$. Consequently, $f(x +^{\mathfrak{A}} 0^{\mathfrak{A}}) = f(x) = f(x) +^{\mathfrak{B}} 0^{\mathfrak{B}} = f(x) +^{\mathfrak{B}} f(0^{\mathfrak{A}})$, and $0^{\mathfrak{A}}$ is in E.

Suppose that y is in E. We compute

$$
\begin{aligned}
f(x +^{\mathfrak{A}} s^{\mathfrak{A}}(y)) &= f(s^{\mathfrak{A}}(x +^{\mathfrak{A}} y)) && \text{[because } \mathfrak{A} \text{ satisfies (Q4)]} \\
&= s^{\mathfrak{B}}(f(x +^{\mathfrak{A}} y)) && \text{[because f respects "s"]} \\
&= s^{\mathfrak{B}}(f(x) +^{\mathfrak{B}} f(y)) && \text{[because } y \in E\text{]}
\end{aligned}
$$

$= f(x) +^{\mathfrak{B}} s^{\mathfrak{B}}(f(y))$ [because $\mathfrak{A}$ satisfies (Q4)]

$= f(x) +^{\mathfrak{B}} f(s^{\mathfrak{A}}(y))$ [because f respects "s"].

Therefore, $s^{\mathfrak{A}}(y)$ is in E.⊠

Now we have a puzzle. Dedekind's theorem tells us that any model of Q that satisfies the principle of mathematical induction is isomorphic to the standard model. In particular, since true arithmetic includes Q and it also includes all the instances of the induction axiom schema, all models of true arithmetic ought to be isomorphic to the standard model. But they aren't. The Compactness Theorem tells us that there are nonstandard models of true arithmetic, that is, models of true arithmetic that aren't isomorphic to the standard model.

The solution to this puzzle is to realize that the induction axiom schema doesn't fully succeed in expressing the content of the principle of mathematical induction. What the induction axiom schema tells us is that the principle of mathematical induction is satisfied by every collection that is named by some predicate of the language.[3] There's no way the schema could tell us about collections that aren't named by predicates of the language. The collections that appear in the proof of Dedekind's theorem – the domain of the function f, and so on – aren't named by predicates of the language.

---

3 To put the matter a little more precisely, let $\mathfrak{A}$ be a model of the language of arithmetic. Extend the language of arithemetic by adding a new constant to serve as a standard name of each element of the universe of $\mathfrak{A}$. If $\mathfrak{A}$ satisfies all the induction axioms, we are assured that the priniciple of mathematical induction holds for every subcollection of $|\mathfrak{A}|$ t hat is the extension of some predicate of the extended language. The slogan is that the priniciple holds for collections that are named by some "predicate with parameters" in $\mathfrak{A}$.

To realize the full strength of the principle of mathematical induction, we have to go beyond the familiar language of arithmetic to the language of *second-order* arithmetic. In addition to the familiar symbols of the language of arithmetic, this new language includes the second-order variables, "X_0," "X_1," "X_2," "X_3," and so on.[4] The definition of "formula" is changed in two ways: For any term τ, $X_m\tau$ is an atomic formula. Also, if ϕ is a formula, so are $(\exists X_m)\phi$ and $(\forall X_m)\phi$. The distinction of "free" and "bound" occurrences of second-order variables, and the distinction between sentences and other formulas, works exactly the way it did for the first-order language.

The definition of "model" is unchanged, but there are small changes in the semantics. A variable assignment for a model $\mathfrak{A}$ assigns an element of $|\mathfrak{A}|$ to each ordinary variable (or each *individual* variable, as they're called in this context), and it assigns a subset of $|\mathfrak{A}|$ to each second-order variable. σ satisfies $X_m\tau$ iff the individual τ denotes with respect to σ is an element of $\sigma(X_m)$. An X_m-variant of a variable assignment σ agrees with σ except perhaps in what it assigns to X_m. σ satisfies $(\exists X_m)\phi$ in $\mathfrak{A}$ iff some X_m-variant of σ satisfies ϕ in $\mathfrak{A}$. σ satisfies $(\forall X_m)\phi$ in $\mathfrak{A}$ iff every X_m-variant of σ satisfies ϕ in $\mathfrak{A}$.

Second-order PA consists of axioms (Q1) through (Q11), together with the following *second-order induction axiom*:

$$(\forall X_0)((X_0 0 \wedge (\forall y)(X_0 y \rightarrow X_0 sy)) \rightarrow (\forall y)X_0 y).$$

---

4 We are allowing second-order variables to take the place of unary predicates. We could also, if we wanted, allow second-order variables that take the place of predicates of more than one arguments. As far as what what we're doing here goes, this wouldn't make any difference.

Thus we can write down second-order PA as a single sentence of the second-order language of arithmetic.

Because the second-order variables range over all subcollections of the universe of discourse, not just those subcollections that happen to be named by some formula or other, the second-order induction axiom expresses the full strength of the principle of mathematical induction. Dedekind's theorem amounts to the following:

> **Corollary.** Second-order PA is *categorical*; that is, any two models of the theory are isomorphic.